F-REGULAR AND F-PURE RINGS
VS.
LOG TERMINAL AND
LOG CANONICAL SINGULARITIES

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Introduction

The notions of F-regular and F-pure rings are defined by Hochster and Huneke [HH1] and Hochster and Roberts [HR], respectively, by using the Frobenius map in characteristic $p > 0$. These notions have some similarity to the notion of rational singularities defined for singularities of characteristic zero. But if we look more closely, we find that F-regularity is strictly stronger than rational singularity and that there is no implication between F-purity and rational singularity. (There are several variants of the concept of "F-regular" rings which are expected and in some cases proved to be equivalent to each other. In this paper, we always concern "strongly F-regular" rings as in Definition 1.1 (2).)

In dimension two, F-regular rings and F-pure rings are well investigated and we find very strong connection between F-regular rings and "quotient singularities" [W2], [Ha2]. To generalize this result to higher dimension, we notice that there is a notion called log terminal singularities for rings of characteristic zero, which is equivalent to quotient singularities in dimension two. Also, the notion of log canonical singularities is defined similarly as log terminal singularities. Looking more closely, we find that F-regular (resp. F-pure) rings and log terminal (resp. log-canonical) singularities have very similar properties.

The notions of F-regular and F-pure rings are also defined for rings of "characteristic zero." Namely, we say that a ring essentially of finite type over a field of characteristic zero is of F-regular (resp. F-pure) type if its reduction to characteristic p is F-regular (resp. F-pure) for infinitely many p.

In this paper, we will show that a $\mathbb{Q}$-Gorenstein ring of F-regular (resp. F-pure) type has log terminal (resp. log canonical) singularities. Actually, we have the following result in characteristic $p > 0$, as a special case of our main theorem (Theorem 3.3).

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Theorem. *Let $(A, \mathfrak{m})$ be a $\mathbb{Q}$-Gorenstein normal local ring of characteristic $p > 0$ and $f \colon X \to Y = \operatorname{Spec} A$ be a proper birational morphism with X normal. Let $E = \bigcup_{i=1}^{s} E_i$ be the exceptional divisor of f and let*

$$K_X = f^* K_Y + \sum_{i=1}^{s} a_i E_i$$

as in (3.1.1). If A is F-pure (resp. strongly F-regular), then $a_i \geq -1$ (resp. $a_i > -1$) for every i.

If A is a normal local ring essentially of finite type over a field of characteristic zero and if $f \colon X \to Y = \operatorname{Spec} A$ is a resolution of singularities of A, then we can apply the above theorem to "reduction modulo p" of this map. Also, this result is applicable to strongly F-regular (resp. normal F-pure) rings of fixed characteristic p. It is crucially used in [Ha2] to classify two-dimensional F-regular and normal F-pure rings in every characteristic p.

The technique we use in our proof is that of Frobenius splitting used by [MR]. If a ring has some splitting of Frobenius, we can lift it to an open set of its resolution. Then the splitting affects "discrepancy" of the exceptional divisors.

The converse of the "F-regular" part of the above theorem has been established as well [Ha3], and we have a Frobenius characterization of log terminal singularities. Namely, a ring in characteristic zero has log terminal singularities if and only if it is of F-regular type and $\mathbb{Q}$-Gorenstein. See also [MS2], [S1,3].

The other new ingredient of this paper is an attempt to generalize the notions of F-regular and F-pure rings to those for "pairs." By a *pair*, we mean a pair (A, Δ) of a normal ring A and a $\mathbb{Q}$-divisor Δ on $\operatorname{Spec} A$. The notions of log terminal and log canonical singularities are defined not only for normal rings (in characteristic zero) but also for pairs (cf. [KMM]), and these "singularities of pairs" play a very important role in birational algebraic geometry. (To wit, see Kollár's lecture note [Ko].) Therefore we are tempted to define corresponding "F-singularities of pairs."

In this paper, we define a few variants of "F-singularities of pairs," namely, strong F-regularity, divisorial F-regularity and F-purity of pairs, which are expected to correspond singularities of pairs called Kawamata log terminal (KLT), purely log terminal (PLT) and log canonical (LC).

The significance of our main theorem (Theorem 3.3) is enhanced by considering F-singularities of pairs. We also prove some results on F-singularities of pairs, which are analoguous to the results proved for singularities of pairs in characteristic zero (cf. [Ko]). If we keep in mind the (expected) correspondence of F-singularities and zero characteristic singularities of pairs, we find that Theorem 4.8 is parallel to the behavior of singularities of pairs under finite covering. Also, Theorem 4.9 is an analog of the so-called "inversion of adjunction," and Proposition 2.10 corresponds to some fundamental properties of "log canonical thresholds."

Our main theorem for the no boundary case ($\Delta = 0$) was first proved by the second-named author several years ago [W3], and this result remained unpublished. Then the first-named author proposed a generalization to F-singularities of pairs, and the present paper was worked out.

In this paper, all rings will be commutative and Noetherian. Also, since we are interested only in normal local rings, *we assume all rings are integral domains.*

1. Preliminaries

Let A be an integral domain of characteristic $p > 0$ and let $F: A \to A$ be the Frobenius endomorphism given by $F(a) = a^p$. We always use the letter q for a power $q = p^e$ of p. Since A is assumed to be reduced, we can identify the following three maps:

$$F^e: A \to A, \quad A^q \hookrightarrow A, \quad A \hookrightarrow A^{1/q}.$$

The ring A is called *F-finite*, if $F: A \to A$ (or $A \hookrightarrow A^{1/p}$) is a finite map. For example, A is F-finite if it is essentially of finite type over a perfect field or it is a complete local ring with perfect residue field. *We always assume A is F-finite* throughout this paper.

Definition 1.1. (1) (Hochster and Roberts [HR]) A is *F-pure* if the map $A \hookrightarrow A^{1/p}$ (hence $A \hookrightarrow A^{1/q}$ for every $q = p^e$, $e \geq 1$) splits as an A-module homomorphism.

(2) (Hochster and Huneke [HH2]) A is *strongly F-regular* if for every $c \in A$ which is not in any minimal prime of A (or equivalently, $c \neq 0$ in our case), there exists $q = p^e$ such that the A-module homomorphism $A \to A^{1/q}$ sending 1 to $c^{1/q}$ splits.

F-pure and (strongly) F-regular rings have many nice properties. But here, we will only list the following. Interested readers could refer [HH1], [HH2], [HR], [FW] or [W2] (in the last reference, "F-regular" should read "strongly F-regular").

Remark 1.2. (1) Let $(A, \mathfrak{m})$ be a local ring and $E = E_A(A/\mathfrak{m})$ be the injective envelope of the A-module $A/\mathfrak{m}$. Then the splitting of the A-homomorphism $A \to A^{1/q}$ sending 1 to $c^{1/q}$ is equivalent to the injectivity of the map $E \to E \otimes_A A^{1/q}$ sending $\xi \in E$ to $\xi \otimes c^{1/q} \in E \otimes_A A^{1/q}$ (cf. Proposition 2.4).

(2) There are notions of F-regular and weakly F-regular rings also defined by Hochster and Huneke [HH1], which we do not define here. On the other hand, a local ring $(A, \mathfrak{m})$ of dimension d is called F-rational [FW], if A is Cohen–Macaulay and if for every $c \neq 0 \in A$, there exists $q = p^e$ such that the map $cF^e: H^d_{\mathfrak{m}}(A) \to H^d_{\mathfrak{m}}(A)$, or equivalently, the mapping

$$H^d_{\mathfrak{m}}(A) = H^d_{\mathfrak{m}}(A) \otimes_A A \to H^d_{\mathfrak{m}}(A) \otimes_A A^{1/q} \cong H^d_{\mathfrak{m}}(A^{1/q}),$$

sending $\xi \in H^d_{\mathfrak{m}}(A)$ to $\xi \otimes c^{1/q} = (cF^e(\xi))^{1/q}$, is injective.

Then the following implications hold:

strongly F-regular $\Rightarrow$ F-regular $\Rightarrow$ weakly F-regular $\Rightarrow$ F-rational $\Rightarrow$ normal.

Also, for a Gorenstein local ring, strongly F-regular is equivalent to F-rational.

(3) ([Mc], [Wi]) For $\mathbb{Q}$-Gorenstein local rings, strongly F-regular and weakly F-regular are equivalent. (A normal local ring is $\mathbb{Q}$-Gorenstein if its canonical module has a finite order in the divisor class group.) In fact, this remains true if the ring has an isolated non-$\mathbb{Q}$-Gorenstein point.

(4) ([HR], [HH1], [HH2]) If a ring is (strongly or weakly) F-regular (resp. F-pure), so are its pure subrings. (A ring homomorphism $A \to B$ is pure if for every A-module M, the map $M \to M \otimes_A B$ is injective).

(5) ([MS1], [W2]) Conversely, if $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a finite local homomorphism of normal local rings which is étale in codimension 1 and if A is strongly F-regular (resp. F-pure), so is B.

Notation 1.3. Let A be a normal domain with quotient field L. A $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$ is a linear combination $D = \sum_{i=1}^{r} \alpha_i D_i$ of irreducible reduced subschemes $D_i \subset Y$ of codimension 1 with coefficients $\alpha_i \in \mathbb{Q}$. The round-down and round-up of D is defined by $\lfloor D \rfloor = \sum_{i=1}^{r} \lfloor \alpha_i \rfloor D_i$ and $\lceil D \rceil = \sum_{i=1}^{r} \lceil \alpha_i \rceil D_i$, respectively. We also denote

$$A(D) = \{ f \in L \mid \operatorname{div}_Y(f) + D \geq 0 \}.$$

Clearly $A(D) = A(\lfloor D \rfloor)$, and this is a divisorial (i.e., finitely generated reflexive) submodule of L. Conversely, any divisorial submodule I of L is written as $I = A(D)$ for some unique *integer coefficient* Weil divisor D. We denote by $I^{(m)}$ the reflexive hull of I^m. If $I = A(D) = A(\lfloor D \rfloor)$, then $I^{(m)} = A(m \lfloor D \rfloor)$.

For any ideal $I \subset A$ and $q = p^e$, we denote by $I^{[q]}$ the ideal of A generated by the qth powers of elements of I. Also, the notation $(\ )^{1/q}$ will show that the module under consideration is an $A^{1/q}$-submodule of $L^{1/q}$. For example, $(I^{[q]})^{1/q} = I \cdot A^{1/q}$.

1.4. Frobenius map and its splitting. Let A be a normal domain of characteristic $p > 0$ and let D be an effective Weil divisor on $Y = \operatorname{Spec} A$. Then for any $q = p^e$, we have a natural inclusion map $\imath \colon A \hookrightarrow A(D)^{1/q}$, which is identified with the e-times Frobenius $F^e \colon A \to A$ followed by the inclusion map $A \hookrightarrow A(D)$. If $D \geq q D_0$ for some effective divisor D_0, then $\imath$ is factorized as

$$A \hookrightarrow A(D_0) = A(D_0) \otimes_A A \to A(D_0) \otimes_A A^{1/q} \to A(q D_0)^{1/q} \hookrightarrow A(D)^{1/q}.$$

Hence, if $\imath$ splits as an A-module homomorphism, then $A \hookrightarrow A(D_0)$ also splits, and this implies $D_0 = 0$. This kind of argument is used frequently in the sequel. Note also that a splitting is preserved under localization, in particular, a localization of a strongly F-regular ring is again strongly F-regular.

2. F-regularity and F-purity of Pairs

Definition 2.1. Let A be an F-finite normal domain of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$.

(1) We say that the pair (A, Δ) is *F-pure* if the inclusion map $A \hookrightarrow A((q-1)\Delta)^{1/q}$ splits as an A-module homomorphism for every $q = p^e$.

(2) (A, Δ) is *strongly F-regular* if for every nonzero element $c \in A$, there exists $q = p^e$ such that $c^{1/q} A \hookrightarrow A((q-1)\Delta)^{1/q}$ splits as an A-module homomorphism.

(3) (A, Δ) is *divisorially F-regular* if for every nonzero element $c \in A$ which is not in any minimal prime ideal of $A(-\lfloor \Delta \rfloor) \subseteq A$, there exists $q = p^e$ such that $c^{1/q} A \hookrightarrow A((q-1)\Delta)^{1/q}$ splits as an A-module homomorphism.

Definition 2.1 includes Definition 1.1 as the special case $\Delta = 0$. Namely, A is F-pure if and only if $(A, 0)$ is F-pure, and A is strongly F-regular if and only if $(A, 0)$ is strongly F-regular, or equivalently, $(A, 0)$ is divisorially F-regular.

We collect some basic properties in the following.

Proposition 2.2. *Let A be an F-finite normal domain of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$.*

(1) *For a pair (A, Δ) as above, the following implications hold.*

$$\text{strongly F-regular} \Rightarrow \text{divisorially F-regular} \Rightarrow \text{F-pure.}$$

(2) *(A, Δ) is strongly F-regular if and only if for every nonzero element $c \in A$, there exists q' such that the map $c^{1/q}A \hookrightarrow A(q\Delta)^{1/q}$ splits as an A-module homomorphism for all $q = p^e \geq q'$.*

(3) *(A, Δ) is strongly F-regular if and only if (A, Δ) is divisorially F-regular and $\lfloor \Delta \rfloor = 0$.*

(4) *If (A, Δ) is F-pure, then $\lceil \Delta \rceil$ is reduced, i.e., the coefficient of $\lceil \Delta \rceil$ in every irreducible component is equal to 1.*

(5) *If (A, Δ) is F-pure (resp. strongly or divisorially F-regular), so is (A, Δ') for every effective $\mathbb{Q}$-Weil divisor $\Delta' \leq \Delta$.*

Proof. (1) It is obvious that strongly F-regular implies divisorially F-regular. Also, divisorially F-regular is equivalent to F-pure if $\dim A = 1$. If $\dim A \geq 2$, we can choose a non-unit $c \neq 0$ of A which is not in any minimal prime of $A(-\lfloor \Delta \rfloor)$. Then the map $(c^q)^{1/q'}A \hookrightarrow A((q'-1)\Delta)^{1/q'}$ does not split for every $q' \leq q$, because it factors through $cA \hookrightarrow A$ (cf. 1.4). Hence, if (A, Δ) is divisorially F-regular and if q is any power of p, then there is a power $q' > q$ such that $(c^q)^{1/q'}A \hookrightarrow A((q'-1)\Delta)^{1/q'}$ splits. This implies that $A \hookrightarrow A((q'-1)\Delta)^{1/q'}$ splits, and so does $A \hookrightarrow A((q-1)\Delta)^{1/q}$. Consequently, (A, Δ) is F-pure.

(2) The sufficiency is clear. To show the necessity, let $c \in A$ be any nonzero element, and choose $d \neq 0 \in A(-\Delta)$ so that $dA(\lceil q\Delta \rceil) \subset A((q-1)\Delta)$ for every $q = p^e$. If (A, Δ) is strongly F-regular, then there exists a power q' of p such that $A \xrightarrow{(cd)^{1/q'}} A((q'-1)\Delta)^{1/q'}$ splits. Since this map is factorized into $A \xrightarrow{c^{1/q'}} A(\lceil q'\Delta \rceil)^{1/q'} \xrightarrow{d^{1/q'}} A((q'-1)\Delta)^{1/q'}$, the map $A \xrightarrow{c^{1/q'}} A(\lceil q'\Delta \rceil)^{1/q'}$ also splits. On the other hand, since A is F-pure, the map $A^{1/q'} \hookrightarrow A^{1/qq'}$ splits for all $q = p^e$, and so does $A(\lceil q'\Delta \rceil)^{1/q'} \hookrightarrow A(q\lceil q'\Delta \rceil)^{1/qq'}$, too. Hence the map $A \xrightarrow{c^{1/q'}} A(q\lceil q'\Delta \rceil)^{1/qq'}$ splits for all $q = p^e$. Since this map is factorized into $A \xrightarrow{c^{1/qq'}} A(qq'\Delta)^{1/qq'} \xrightarrow{c^{(q-1)/qq'}} A(q\lceil q'\Delta \rceil)^{1/qq'}$, the map $A \xrightarrow{c^{1/q}} A(q\Delta)^{1/q}$ splits for all $q = p^e \geq q'$.

To prove (3) (resp. (4)), assume to the contrary that Δ has a component Δ_0 with coefficient ≥ 1 (resp. > 1). Then there is a $q = p^e$ such that the coefficient of $q\Delta$ (resp. $(q-1)\Delta$) in Δ_0 is at least q. Then the map $A \hookrightarrow A(q\Delta)^{1/q}$ (resp. $A \hookrightarrow A((q-1)\Delta)^{1/q}$) factors through $A \hookrightarrow A(\Delta_0) \hookrightarrow A(q\Delta_0)^{1/q}$, which does not split. This implies that (A, Δ) cannot be strongly F-regular (resp. F-pure).

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(5) The only being non-trivial is the assertion for divisorial F-regularity in the case $\mathrm{Supp}(\lfloor\Delta\rfloor) \setminus \mathrm{Supp}(\lfloor\Delta'\rfloor) \neq \emptyset$. To prove this, we may assume without loss of generality that there is a unique irreducible component Δ_0 of $\lfloor\Delta\rfloor$ such that $\Delta_0 \not\subseteq \mathrm{Supp}(\lfloor\Delta'\rfloor)$. Let $c \neq 0 \in A$ be any element which is in $\mathfrak{p} = A(-\Delta_0)$ but not in any minimal prime of $A(-\lfloor\Delta'\rfloor)$, and let $\nu = v_{\mathfrak{p}}(c)$, the value of c at $\mathfrak{p}$. Then by prime avoidance, we can choose $d \neq 0 \in cA(\nu\Delta_0)$ which is not in any minimal prime of $A(-\lfloor\Delta\rfloor)$. This implies that $dA((q-1)\Delta') \subseteq cA((q-1)\Delta)$ for $q \gg 0$. Now, if (A, Δ) is divisorially F-regular, then there exists $q = p^e$ such that $A \xrightarrow{d^{1/q}} A((q-1)\Delta)^{1/q}$ splits, and we can show as in (1) that this is true for infinitely many q. Since this map factors into $A \xrightarrow{c^{1/q}} A((q-1)\Delta')^{1/q} \xrightarrow{(d/c)^{1/q}} A((q-1)\Delta)^{1/q}$ if $q \gg 0$, the map $A \xrightarrow{c^{1/q}} A((q-1)\Delta')^{1/q}$ splits for some q. $\square$

Remark 2.3. In the definition of "F-pure," it seems apparently natural to refer the map $A \hookrightarrow A(q\Delta)^{1/q}$ instead of $A \hookrightarrow A((q-1)\Delta)^{1/q}$. (We have seen in Proposition 2.2 (2) that this makes no difference for strong F-regularity.) But it does make crucial difference for F-purity. Let us say (A, Δ) is *strongly F-pure* if $A \hookrightarrow A(q\Delta)^{1/q}$ splits for every $q = p^e$. Then the proof of Proposition 2.2 (3) shows that if (A, Δ) is strongly F-pure, then $\lfloor\Delta\rfloor = 0$. But this is too much stronger than what we want for "F-purity." Note also that we have the implication "strongly F-regular $\Rightarrow$ strongly F-pure $\Rightarrow$ F-pure," and that there is no implication between strong F-purity and divisorial F-regularity.

Proposition 2.4. *Let $(A, \mathfrak{m})$ be a d-dimensional F-finite normal local ring of characteristic $p > 0$ and let Δ be an effective $\mathbb{Q}$-Weil divisor on $Y = \mathrm{Spec}\, A$. Then:*

(1) *(A, Δ) is F-pure if and only for every $q = p^e$, the induced e-times Frobenius map $F^e\colon H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta))$ is injective.*

(2) *(A, Δ) is strongly F-regular if and only if for every $c \neq 0 \in A$, there exists $q = p^e$ such that $cF^e\colon H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta))$ is injective.*

(3) *(A, Δ) is divisorially F-regular if and only if for every $c \neq 0 \in A$ which is not in any minimal prime of $A(-\lfloor\Delta\rfloor)$, there exists $q = p^e$ such that $cF^e\colon H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta))$ is injective.*

Proof. The map $c^{1/q}A \hookrightarrow A((q-1)\Delta)^{1/q}$ splits as an A-module homomorphism if and only if the map

$$\mathrm{Hom}_A(A((q-1)\Delta)^{1/q}, A) \xrightarrow{c^{1/q}} \mathrm{Hom}_A(A((q-1)\Delta)^{1/q}, A) \to \mathrm{Hom}_A(A, A) = A$$

is surjective. By the local duality, the Matlis dual of $\mathrm{Hom}_A(A((q-1)\Delta)^{1/q}, A) \cong \mathrm{Hom}_A(A(qK_A + (q-1)\Delta)^{1/q}, K_A)$ is $H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta)^{1/q})$, so that the surjectivity of the above map is equivalent to the injectivity of

$$H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta)^{1/q}) \xrightarrow{c^{1/q}} H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta)^{1/q}).$$

Since this map is identified with $cF^e\colon H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta))$, we obtain the assertions (1), (2) and (3) at once. $\square$

Corollary 2.5. *If A is a normal toric ring over a perfect field k of characteristic $p > 0$ and if Δ is a reduced toric divisor, then the pair (A, Δ) is F-pure.*

Proof. Let $M = \mathbb{Z}^d$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and denote the duality pairing of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ with $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ by $\langle \, , \, \rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. Let A be the toric ring defined by a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ and let $D_1, \ldots, D_s$ be the toric divisors of $\mathrm{Spec}\, A$ corresponding to the primitive generators $n_1, \ldots, n_s \in N$ of σ, respectively. To show the F-purity of the pair (A, Δ), we may assume that $\Delta = \sum_{i=1}^{s} D_i$, by Proposition 2.2 (5). Then one has $K_A = -\Delta$, so that $A(qK_A + (q-1)\Delta) = K_A$. Therefore

$$H_{\mathfrak{m}}^d(K_A) = H_{\mathfrak{m}}^d(A(qK_A + (q-1)\Delta)) = \bigoplus_{m \in -\check{\sigma} \cap M} k \cdot x^m,$$

where $-\check{\sigma} = \{ m \in M_{\mathbb{R}} \mid \langle m, n_i \rangle \leq 0 \text{ for } i = 1, \ldots, s \}$, and the e-times Frobenius map $F^e \colon H_{\mathfrak{m}}^d(K_A) \to H_{\mathfrak{m}}^d(A(qK_A + (q-1)\Delta))$ is given by $F^e(x^m) = x^{qm}$ ($m \in -\check{\sigma} \cap M$). Since this map is injective, (A, Δ) is F-pure by Proposition 2.4 (1). $\square$

Proposition 2.6. (Fedder-type criteria [F]) *Let $(R, \mathfrak{m})$ be an F-finite regular local ring of characteristic $p > 0$ and $I \subset R$ be an ideal such that $A = R/I$ is normal. Let $g \in R \setminus I$ be an element whose image $\bar{g} \in A$ defines a reduced divisor $\mathrm{div}_Y(\bar{g})$ on $Y = \mathrm{Spec}\, A$, and consider a $\mathbb{Q}$-divisor $\Delta = t \cdot \mathrm{div}_Y(\bar{g})$ for nonnegative $\alpha \in \mathbb{Q}$. We put $r_e = \lfloor t(q-1) \rfloor$ for each $q = p^e$. Then:*

  (1) *(A, Δ) is F-pure if and only if $g^{r_e}(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$ for all $q = p^e$.*
  (2) *(A, Δ) is strongly F-regular if and only if for every $c \in R \setminus I$, there exists $q = p^e$ such that $cg^{r_e}(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$.*
  (3) *$(A, \mathrm{div}_Y(\bar{g}))$ is divisorially F-regular if and only if for every $c \in R \setminus I$ which is not in any minimal prime of $I + gR$, there exists $q = p^e$ such that $cg^{q-1}(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$.*

Proof. The proof is essentially the same as those in [F], [Gl]. First, we may assume without loss of generality that $(R, \mathfrak{m})$ is a *complete* regular local ring. We will describe the map $cF^e \colon H_{\mathfrak{m}}^d(K_A) \to H_{\mathfrak{m}}^d(A(qK_A + (q-1)\Delta))$ in Proposition 2.4. Since $A((q-1)\Delta) = \bar{g}^{-r_e}A$ and since $H_{\mathfrak{m}}^d(K_A^{(q)}) \cong E_A(A/\mathfrak{m}A) \otimes_A A^{1/q}$ by [W2, Theorem 2.5], This map is viewed as

$$(2.6.1) \qquad c\bar{g}^{r_e}F^e \colon E_A(A/\mathfrak{m}A) \to E_A(A/\mathfrak{m}A) \otimes_A A^{1/q}.$$

Let $E_R = E_R(R/\mathfrak{m})$, $E_A = E_A(A/\mathfrak{m}A)$ and let $n = \dim R$. Then $E_R \cong H_{\mathfrak{m}}^n(R)$ since R is Gorenstein, and we can identify E_R with $E_R \otimes_R R^{1/q} \cong H_{\mathfrak{m}}^n(R^{1/q})$ via the identification of R with $R^{1/q}$. Also, $E_A \cong \mathrm{Hom}_R(R/I, E_R) \cong (0 : I)_{E_R} \subset E_R$, and E_A and E_R have the 1-dimensional socle in common. Let z be a generator of the socle of E_R and let z' be the corresponding socle generator in E_A.

Now, $R \hookrightarrow R^{1/q}$ is flat since R is regular, so that via the identification $R^{1/q} \cong R$, we have

$$(2.6.2) \qquad \begin{aligned} E_A \otimes_R R^{1/q} &\cong \mathrm{Hom}_{R^{1/q}}(R/I \otimes_R R^{1/q}, E_R \otimes_R R^{1/q}) \\ &\cong \mathrm{Hom}_R(R/I^{[q]}, E_R) \cong (0 : I^{[q]})_{E_R} \end{aligned}$$

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in $E_R \otimes_R R^{1/q} \cong E_R$. Accordingly we have

$$E_A \otimes_A A^{1/q} \cong E_A \otimes_R R^{1/q} \otimes_{R^{1/q}} A^{1/q} \cong (0 : I^{[q]})_{E_R} \otimes_R A \cong \frac{(0 : I^{[q]})_{E_R}}{I(0 : I^{[q]})_{E_R}},$$

and the image of $z' \in E_A$ by the map (2.6.1) is $cg^{r_e} z^q \mod I(0 : I^{[q]})_{E_R}$, where z^q denotes the image of z by the e-times Frobenius on $E_R \cong H_{\mathfrak{m}}^n(R)$. We also know from (2.6.2) (by putting $I = \mathfrak{m}$) that $z^q \in E_R$ generates $(0 : \mathfrak{m}^{[q]})_{E_R}$.

Now the map (2.6.1) is injective if and only if $cg^{r_e} z^q \notin I(0 : I^{[q]})_{E_R}$ if and only if $cg^{r_e}(0 : \mathfrak{m}^{[q]})_{E_R} \not\subseteq I(0 : I^{[q]})_{E_R}$. Since $I(0 : I^{[q]})_{E_R} = (0 : (I^{[q]} : I)_R)_{E_R}$ by the Matlis duality, this is equivalent to saying that $cg^{r_e}(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$. $\square$

Corollary 2.7. (cf. [Ha1]) *Let $(R, \mathfrak{m})$ be an F-finite regular local ring of characteristic $p > 0$, and let $f_1, \ldots, f_s, g \in \mathfrak{m}$ be an R-regular sequence. Let $A = R/(f_1, \ldots, f_s)$ and consider an effective $\mathbb{Q}$-divisor $\Delta = t \cdot \mathrm{div}_Y(\bar{g})$ on $Y = \mathrm{Spec}\, A$. Put $r_e = \lfloor t(p^e - 1) \rfloor$. Then:*

(1) (A, Δ) is F-pure if and only if $(f_1 \cdots f_s)^{q-1} g^{r_e} \notin \mathfrak{m}^{[q]}$ for all $q = p^e$.

(2) (A, Δ) is strongly F-regular if and only if

$$\bigcap_{q = p^e, e \in \mathbb{N}} \mathfrak{m}^{[q]} : (f_1 \cdots f_s)^{q-1} g^{r_e} = (f_1, \ldots, f_s) \quad \text{in } R.$$

(3) $(A, \mathrm{div}_Y(\bar{g}))$ is F-pure (resp. divisorially F-regular) if and only if $A/\bar{g}A = R/(f_1, \ldots, f_s, g)$ is F-pure (resp. strongly F-regular).

Proof. The proof is easy, but we remark one point which might be overlooked. Put $D = \mathrm{div}_Y(\bar{g})$. Then the condition $(f_1 \cdots f_s)^{q-1} g^{r_e} \notin \mathfrak{m}^{[q]}$ holds if and only if $A \hookrightarrow A(\lfloor t(q-1) \rfloor D)^{1/q}$ splits as an A-module homomorphism by the preceding argument. So, to prove (1), we have to show the equivalence of the following two conditions.

(i) $A \hookrightarrow A((q-1)\Delta)^{1/q} = A(\lfloor (q-1)tD \rfloor)^{1/q}$ splits for all $q = p^e$.

(ii) $A \hookrightarrow A(\lfloor t(q-1) \rfloor D)^{1/q}$ splits for all $q = p^e$.

The implication (i) $\Rightarrow$ (ii) is clear since $\lfloor (q-1)tD \rfloor \geq \lfloor (q-1)t \rfloor D$. But (ii) $\Rightarrow$ (i) is not apparently clear if D is non-reduced. To show this implication, let q be any power of p, and choose q' such that $q' \lfloor (q-1)tD \rfloor \leq \lfloor t(qq' - 1) \rfloor D$. Then by (ii), the map $A \hookrightarrow A(\lfloor t(qq' - 1) \rfloor D)^{1/qq'}$ splits, and this map is factorized into $A \hookrightarrow A(\lfloor (q-1)tD \rfloor)^{1/q} \hookrightarrow A(q' \lfloor (q-1)tD \rfloor)^{1/qq'} \hookrightarrow A(\lfloor t(qq' - 1) \rfloor D)^{1/qq'}$. Hence $A \hookrightarrow A(\lfloor (q-1)tD \rfloor)^{1/q}$ splits. $\square$

Remark 2.8. We have Fedder-type criteria also for strong F-purity (see Remark 2.3 for a definition), by setting $r_e = \lfloor tp^e \rfloor$ in (1) of Proposition 2.6 and Corollary 2.7. The criteria for strong F-regularity also works if we put $r_e = \lfloor tp^e \rfloor$.

Example 2.9. (1) Let A be a regular local ring and let $\Delta = t \cdot \mathrm{div}(x_1 \cdots x_i)$ on $\mathrm{Spec}\, A$, where $x_1, \ldots, x_i$ are part of regular parameters of A. If $t \leq 1$ (resp. $t < 1$), then (A, Δ) is F-pure (resp. strongly F-regular).

(2) Let $A = k[[X, Y, Z]]/(XY - Z^2)$ and denote the images of Z in A by z. Let $\Delta = t \cdot \operatorname{div}(z)$. Corollary 2.7 tells us that if $t \leq 1$ (resp. $t < 1$), then (A, Δ) is F-pure (resp. strongly F-regular).

(3) Let $A = k[[x, y]]$ and let $\Delta = \frac{5}{6} \operatorname{div}(x^2 - y^3)$. Then in any characteristic $p > 0$, (A, Δ) is F-pure but not strongly F-regular. But (A, Δ) is strongly F-pure if and only if $p \equiv 1 \mod 3$.

The following is a variant of Fedder's result [F], see also [Ko, Lemma 8.10].

Proposition 2.10. *Let $A = k[[x_1, \ldots, x_d]]$ be a d-dimensional complete regular local ring over a perfect field k of characteristic $p > 0$ and let $f \in A$ be a nonzero element of multiplicity n, i.e., $f \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$, where $\mathfrak{m}$ is the maximal ideal of A. For nonnegative $t \in \mathbb{Q}$ we have:*

(1) *If $t \leq 1/n$, then $(A, t \cdot \operatorname{div}(f))$ is F-pure.*
(2) *If $(A, t \cdot \operatorname{div}(f))$ is F-pure (resp. strongly F-regular), then $t \leq d/n$) (resp. $t < d/n$).*
(3) *Assume that $t < \min\{1, d/n\}$ and that the initial term f_n of f defines a smooth subvariety of $\mathbb{P}^{d-1}$. Then in characteristic $p \gg 0$, $(A, t \cdot \operatorname{div}(f))$ is strongly F-pure. More precisely, if the ideal $J = (\partial f_n/\partial x_1, \ldots, \partial f_n/\partial x_d)$ contains $(x_1^{m_1}, \ldots, x_d^{m_d})$ in A, then $(A, t \cdot \operatorname{div}(f))$ is strongly F-pure, or otherwise $p < (m_1 + \cdots + m_d)/(d - nt)$.*

Remark. Note that if $n = \deg f_n$ is not divisible by p, then $f_n \in J$, so that J is equal to the Jacobian ideal (f_n, J) of $V = (f_n = 0) \subset \mathbb{P}^{d-1}$. So the smoothness of V implies that $(x_1^{m_1}, \ldots, x_d^{m_d}) \subseteq J$ for some $m_1, \ldots, m_d \in \mathbb{N}$.

Proof. The assertions (1) and (2) immediately follows from Corollary 2.7. To prove (3) we use the Fedder-type critrion for strong F-purity (2.8):

$$(A, t \cdot \operatorname{div}(f)) \text{ is strongly F-pure} \iff f^{r_e} \notin \mathfrak{m}^{[p^e]} \text{ for all } e \in \mathbb{N},$$

where $r_e = \lfloor tp^e \rfloor$. Since $f_n^{r_e} \notin \mathfrak{m}^{[p^e]}$ implies $f^{r_e} \notin \mathfrak{m}^{[p^e]}$, we may assume that f is a homogeneous polynomial of degree n in $x_1, \ldots, x_d$.

Assume $p \geq \mu := (m_1 + \cdots + m_d)/(d - nt)$ and let j_e be the integer such that $f^{j_e+1} \in \mathfrak{m}^{[p^e]}$ but $f^{j_e} \notin \mathfrak{m}^{[p^e]}$. Then it is sufficient to prove that $r_e \leq j_e$ for every $e \geq 0$. Assume to the contrary that there exists an e such that $r_e > j_e$, and choose the smallest one among all such e. Then $e > 0$, since $r_0 = 0$ by the assumption $t < 1$. Also, by the minimality of e, we have $f^{r_{e-1}} \notin \mathfrak{m}^{[p^{e-1}]}$, and this implies that $f^{pr_{e-1}} \notin \mathfrak{m}^{[p^e]}$. Indeed, if $f^{pr_{e-1}} \in \mathfrak{m}^{[p^e]}$, then $f^{r_{e-1}} = (f^{pr_{e-1}})^{1/p} \in \mathfrak{m}^{[p^{e-1}]}A^{1/p} \cap A = \mathfrak{m}^{[p^{e-1}]}$, because $A \to A^{1/p}$ is pure. Hence $pr_{e-1} \leq j_e < r_e$, and $j_e + 1$ is not divisible by p since $r_e - pr_{e-1} \leq p - 1$. Therefore, by differentiating $f^{j_e+1} \in \mathfrak{m}^{[p^e]}$ by x_i, we have $f^{j_e}\partial f/\partial x_i \in \mathfrak{m}^{[p^e]}$ for $i = 1, \ldots, d$, so that

$$f^{j_e} \in \mathfrak{m}^{[p^e]} : J \subseteq \mathfrak{m}^{[p^e]} : (x_1^{m_1}, \ldots, x_d^{m_d}) \subseteq \mathfrak{m}^{[p^e]} + (x_1^{p^e-m_1} \cdots x_d^{p^e-m_d})A.$$

Then f^{j_e} must have a nonzero term in $(x_1^{p^e-m_1} \cdots x_d^{p^e-m_d})A$, since $f^{j_e} \notin \mathfrak{m}^{[p^e]}$. Hence $p^e d - \sum_{i=1}^d m_i = \deg f^{j_e} = nj_e < nr_e \leq ntp^e$. This inequality, together with the assumption $t < d/n$, implies $p^e < \mu$, which contradicts to $p \geq \mu$. $\square$

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3. Main Theorem

First, we recall the definition of log terminal and log canonical singularities.

Let $f\colon X \to Y$ be a proper birational morphism of normal varieties over a field k and let $E = \bigcup_{i=1}^{s} E_i$ be the exceptional divisor of f with irreducible components $E_1, \ldots, E_s$. For a $\mathbb{Q}$-Weil divisor D on Y (resp. on X), we denote by $f_*^{-1}(D)$ (resp. $f_* D$) the strict transform of D in X (resp. in Y). A $\mathbb{Q}$-Weil divisor D on Y is said to be $\mathbb{Q}$-*Cartier* if rD is a Cartier divisor for some integer $r > 0$. Then the pull-back $f^*(rD)$ of rD is also a Cartier divisor on X, and we can define the pull-back f^*D of D as a $\mathbb{Q}$-Cartier divisor by $f^*D = \frac{1}{r} f^*(rD)$.

We denote the dualizing sheaves of X and Y by ω_X and ω_Y, respectively, and fix ω_X and ω_Y as divisorial subsheaves of the rational function field $k(X) = k(Y)$ so that they coincide with each other outside the exceptional locus of f. Then the canonical divisor K_X of X (resp. K_Y of Y) is also fixed by $\omega_X = \mathcal{O}_X(K_X)$ (resp. $\omega_Y = \mathcal{O}_Y(K_Y)$), and the strict transform of K_X in Y is $f_* K_X = K_Y$.

Now let Δ be an effective $\mathbb{Q}$-Weil divisor on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and denote by $\tilde{\Delta}$ the strict transform $f_*^{-1}(\Delta)$ of Δ in X. Let $r > 0$ be an integer such that $r(K_Y + \Delta)$ is Cartier. Then $r(K_X + \tilde{\Delta})$ and $r \cdot f^*(K_Y + \Delta) = f^*(r(K_Y + \Delta))$ have integer coefficients, and coincide with each other outside the exceptional locus of f. Hence $r(K_X + \tilde{\Delta}) = r \cdot f^*(K_Y + \Delta) + \sum_{j=1}^{s} b_j E_j$ for some $b_1, \ldots, b_s \in \mathbb{Z}$, and we have

$$(3.1.1) \qquad K_X + \tilde{\Delta} = f^*(K_Y + \Delta) + \sum_{j=1}^{s} a_j E_j, \quad \text{where } a_j = \frac{b_j}{r} \ (j = 1, \ldots, s).$$

We call $a_j \in \mathbb{Q}$ the *discrepancy* of E_j with respect to (Y, Δ). In the following definition, we keep this notation for any desingularization $f\colon X \to Y$ under consideration.

Definition 3.1. Let Y be a normal variety of characteristic zero, Δ be an effective $\mathbb{Q}$-Weil divisor, and assume that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier.

(1) (Y, Δ) is said to be *Kawamata log terminal* (or *KLT* for short), if $\lfloor \Delta \rfloor = 0$ and if for every resolution of singularities $f\colon X \to Y$ and every f-exceptional divisor E_j, the discrepancy a_j defined in (3.1.1) satisfies $a_j > -1$.

(2) (Y, Δ) is *purely log terminal* (or *PLT* for short), if for every resolution of singularities $f\colon X \to Y$ and every f-exceptional divisor E_j, one has $a_j > -1$.

(3) (Y, Δ) is *log canoninal* (or *LC* for short), if for every resolution of singularities $f\colon X \to Y$ and every f-exceptional divisor E_j, one has $a_j \geq -1$.

When the pair $(Y, 0)$ is KLT, or equivalently, PLT in this case (resp. LC), we say that Y has log terminal (resp. log canonical) singularities.

Remark 3.2. (1) Clearly, we have the implications "KLT $\Rightarrow$ PLT $\Rightarrow$ LC," and if (Y, Δ) is LC, then $\lceil \Delta \rceil$ is reduced, i.e., every coefficients of Δ is less than or equal to 1. Also, conditions (1) and (3) of Definition 3.1 is checked by referring *some* log resolution of (Y, Δ), that is, a resolution of singularities $f\colon X \to Y$ such that the union of the exceptional set and $\mathrm{Supp}(f^{-1}\Delta)$ is a simple normal crossing divisor. Namely, (Y, Δ) is KLT (resp. LC) if and only if $\lfloor \Delta \rfloor = 0$ (resp. $\lceil \Delta \rceil$ is reduced)

and there exists a log resolution $f\colon X \to Y$ such that $a_j > -1$ (resp. $a_j \geq -1$) for every f-exceptional divisor E_j. Also, (Y, Δ) is PLT if and only if there exists a log resolution $f\colon X \to Y$ such that $f_*^{-1}\lfloor \Delta \rfloor$ is smooth and that $a_j > -1$ for every f-exceptional divisor E_j.

(2) Log terminal (resp. log canonical) singularities behave similarly as strongly F-regular (resp. F-pure) rings under finite covers. Namely, if $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a finite local homomorphism of normal local rings which is étale in codimension 1, then A is log terminal (resp. log canonical) if and only if so is B.

(3) If A has log terminal singularities, then A has rational singularities. Conversely, if A has Gorenstein rational singularities, then A has log terminal singularities.

Theorem 3.3. *Let $(A, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Let $f\colon X \to Y = \operatorname{Spec} A$ be a proper birational morphism with X normal. Let $E = \bigcup_{j=1}^{s} E_j$ be the exceptional divisor of f and let*

$$K_X + \tilde{\Delta} = f^*(K_Y + \Delta) + \sum_{j=1}^{s} a_j E_j$$

as in (3.1.1). If the pair (A, Δ) is F-pure (resp. divisorially F-regular), then $a_j \geq -1$ (resp. $a_j > -1$) for every $j = 1, \ldots, s$.

We will give two different proofs to this theorem, in which we relax the properness assumption of $f\colon X \to Y = \operatorname{Spec} A$ as follows: f factorizes into $f\colon X \hookrightarrow \bar{X} \xrightarrow{\bar{f}} Y$, where $\bar{f}\colon \bar{X} \to Y$ is a proper birational morphism with $\bar{X}$ normal, and X is an open subset of $\bar{X}$ with $\operatorname{codim}(\bar{X} \setminus X, \bar{X}) \geq 2$. This assumption is preserved if we replace X by the nonsingular locus X_{reg} of X, and we do not lose any information about the discrepancies a_j by this replacement, since X_{reg} intersects every E_j. Hence we may assume without loss of generality that X *is Gorenstein* in what follows.

First proof. We will divide the proof into six steps. First (since $\bar{f}$ is proper and birational), f is an isomorphism on an open set U of Y with $\operatorname{codim}(Y \setminus U, Y) \geq 2$. Since $K_A^{(i)}$ is reflexive, we have

$$H^0(X, \omega_X^{\otimes i}) \subseteq H^0(U, \omega_U^{\otimes i}) = K_A^{(i)}$$

for every $i \in \mathbb{Z}$. Also, we can choose a nonzero element $b \in A$ such that

(3.3.1) $\qquad b \cdot K_A^{(-i)} \subseteq H^0(X, \omega_X^{\otimes (-i)}) \quad$ for each $i = 0, 1, \ldots, r - 1$.

(1) Since X is Gorenstein, we have

(3.3.2)
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/q}, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/q}, \omega_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}$$
$$\cong (\omega_X)^{1/q} \otimes \omega_X^{-1} \cong \mathcal{O}_X((1-q)K_X)^{1/q}.$$

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by the adjunction formula for the finite map $\mathcal{O}_X \hookrightarrow \mathcal{O}_X^{1/q}$. By a similar argument on U, we also have an isomorphism

$$\alpha\colon \operatorname{Hom}_A(A^{1/q}, A) \xrightarrow{\sim} (K_A^{(1-q)})^{1/q},$$

which induces $\operatorname{Hom}_A(A((q-1)\Delta)^{1/q}, A) \cong A((1-q)K_A - \lfloor (q-1)\Delta \rfloor)^{1/q}$ and is compatible with (3.3.2).

(2) Let us fix an embedding $\mathcal{O}_X(iK_X) \hookrightarrow L$, where $L = k(X)$ is the rational function field of X, and put $A(r(K_A + \Delta)) = wA$. Then $A((mr+i)(K_A + \Delta)) = w^m A(i(K_A + \Delta))$, and $\omega_X^{\otimes r} = \mathcal{O}_X(rK_X) = \mathcal{O}_X(r(f^*(K_Y + \Delta) + \sum a_j E_j - \tilde{\Delta})) = f^* w \cdot \mathcal{O}_X(r \sum a_j E_j - \tilde{\Delta})$.

(3) Now, assume that (A, Δ) is F-pure and let $\phi\colon A((q-1)\Delta)^{1/q} \to A$ be a splitting of $A \hookrightarrow A((q-1)\Delta)^{1/q}$. Then ϕ induces a splitting $\tilde{\phi}\colon L^{1/q} \to L$ of $L \hookrightarrow L^{1/q}$. Let $X' = X \setminus Z$, where $Z = \operatorname{Supp}((\tilde{\phi}(\mathcal{O}_X^{1/q}) + \mathcal{O}_X)/\mathcal{O}_X)$. Then X' is an open subset of X since $\mathcal{O}_X^{1/q}$ is a coherent $\mathcal{O}_X$-module (we always assume that A is F-finite). Also $\tilde{\phi}$ induces an $\mathcal{O}_{X'}$-linear map $\mathcal{O}_{X'}^{1/q} \to \mathcal{O}_{X'}$, which we denote by the same letter ϕ. Then $\phi\colon \mathcal{O}_{X'}^{1/q} \to \mathcal{O}_{X'}$ gives an F-splitting of X' since the composition map $\mathcal{O}_{X'} \hookrightarrow \mathcal{O}_{X'}^{1/q} \xrightarrow{\phi} \mathcal{O}_{X'}$ is the identity (cf. Lemma 2 of [MS]).

(4) Let ϕ be as in (3) and fix an irreducible component E_j of E. We want to examine if ϕ is defined at the generic point of E_j or not. Let ξ be the generic point of E_j and η be a regular parameter of $\mathcal{O}_{X,\xi}$. We write $q-1 = mr+i$ with $0 \le i < r$. Then by (3.3.1) we have

$$\alpha(\phi) \in w^{-m} A(-iK_A - \lfloor i\Delta \rfloor) \subseteq b^{-1} w^{-m} H^0(X, \omega_X^{\otimes(-i)}).$$

(5) Now, we will show that $a_j \ge -1$ if (A, Δ) is F-pure. Assume, on the contrary, that $a_j < -1$. Then, since $ra_j \in \mathbb{Z}$, we have $a_j \le -1 - 1/r$, and for a fixed integer s, we can take $q \gg 0$ so that $-mra_j \ge q + s$. This means that if κ is a local generator of the $\mathcal{O}_X^{1/q}$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/q}, \mathcal{O}_X)$ at ξ, then the stalk ϕ_ξ of ϕ at ξ lies in $[\eta^q \mathcal{O}_{X,\xi}]^{1/q} \kappa$. To see this let $s = v_{E_j}(b)$, the value of $b \in A$ at ξ, and let $-mra_j \ge q+s$. Then $\mathcal{O}_{X,\xi}^{1/q} \cdot \alpha_\xi(\kappa) = [w^{-m} \eta^{mra_j} \omega_{X,\xi}^{\otimes(-i)}]^{1/q} \supseteq [w^{-m} \eta^{-q-s} \omega_{X,\xi}^{\otimes(-i)}]^{1/q}$ by (2), and $\alpha(\phi)_\xi \in [\eta^{-s} w^{-m} \omega_{X,\xi}^{\otimes(-i)}]^{1/q}$ by (4). It follows that $\phi_\xi \in [\eta^q \mathcal{O}_{X,\xi}]^{1/q} \kappa$, whence $\phi_\xi(\mathcal{O}_{X,\xi}^{1/q}) \subseteq \eta \mathcal{O}_{X,\xi}$. This implies that $\xi \in X'$ but ϕ is not an F-splitting at ξ. This contradiction concludes that $a_i \ge -1$.

(6) If (A, Δ) is divisorially F-regular, then for any $c \ne 0 \in A$ which is not in any minimal prime of $A(-\lfloor \Delta \rfloor)$, there exists an A-homomorphism $\psi\colon A((q-1)\Delta)^{1/q} \to A$ sending $c^{1/q}$ to 1. Then $\phi = c^{1/q} \psi$ gives a splitting of $A \hookrightarrow A((q-1)\Delta)^{1/q}$, and we may think $\phi \in c^{1/q} \operatorname{Hom}_A(A((q-1)\Delta)^{1/q}, A) \cong [c \cdot A((1-q)K_A - \lfloor (q-1)\Delta \rfloor)]^{1/q}$. Let us choose c so that the value $t = v_{E_j}(c)$ of c at ξ satisfies $t \ge r+s = r + v_{E_j}(b)$. Then, arguing as in the F-pure case, we see that if $a_j = -1$, then $\phi_\xi \in [\eta^q \mathcal{O}_{X,\xi}]^{1/q} \kappa$ and again ϕ cannot be an F-splitting at ξ.

This completes the proof of the theorem. $\square$

Second proof. Next we shall give an alternative proof of Theorem 3.3 with different flavor. As we have seen in the first proof, the adjunction formula gives

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{1/q}, \mathcal{O}_X) \cong (\omega_X^{\otimes(1-q)})^{1/q} \cong \mathcal{O}_X((1-q)K_X)^{1/q},$$

$$\mathrm{Hom}_A(A((q-1)\Delta)^{1/q}, A) \cong A((1-q)K_A - \lfloor(q-1)\Delta\rfloor)^{1/q}.$$

(1) Now assume that (A, Δ) is F-pure and let $\phi\colon A((q-1)\Delta)^{1/q} \to A$ be a splitting of $A \hookrightarrow A((q-1)\Delta)^{1/q}$. We regard $\phi \in \mathrm{Hom}_A(A((q-1)\Delta)^{1/q}, A) \cong A((1-q)K_A - \lfloor(q-1)\Delta\rfloor)^{1/q}$ as a rational section of the sheaf $\omega_X^{\otimes(1-q)}$ and consider the corresponding divisor on X,

$$D = D_\phi = (\phi)_0 - (\phi)_\infty,$$

where $(\phi)_0$ (resp. $(\phi)_\infty$) is the divisor of zeros (resp. poles) of ϕ as a rational section of $\omega_X^{\otimes(1-q)}$. Clearly, D is linearly equivalent to $(1-q)K_X$ and $(\phi)_\infty$ is an f-exceptional divisor. Hence f_*D is linearly equivalent to $(1-q)K_Y$ and $f_*D \geq \lfloor(q-1)\Delta\rfloor$. We denote $X' = X \setminus \mathrm{Supp}\,(\phi)_\infty$. Then ϕ lies in

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}^{1/q}, \mathcal{O}_{X'}) \cong H^0(X', \omega_{X'}^{\otimes(1-q)}),$$

and gives an F-splitting of X'.

(2) We show that the coefficient of D in each irreducible component is $\leq q - 1$. Assume to the contrary that there exists an irreducible component of D, say D_0, whose coefficient is $\geq q$. Then ϕ lies in

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}(qD_0)^{1/q}, \mathcal{O}_{X'}) \cong H^0(X', \omega_{X'}^{\otimes(1-q)}(-qD_0)),$$

and gives a splitting of the map $\mathcal{O}_{X'} \hookrightarrow \mathcal{O}_{X'}(qD_0)^{1/q}$. But this map factors through $\mathcal{O}_X(D_0)$, and $\mathcal{O}_{X'} \hookrightarrow \mathcal{O}_{X'}(D_0)$ never splits as an $\mathcal{O}_{X'}$-module homomorphism, since D_0 intersects X'. Consequently, every coefficient of D must be $\leq q - 1$.

(3) Let

$$B = \frac{1}{q-1}D - \tilde{\Delta}.$$

Then B is $\mathbb{Q}$-linearly equivalent to $-(K_X + \tilde{\Delta})$, so that f_*B is $\mathbb{Q}$-linearly equivalent to $-f_*(K_X + \tilde{\Delta}) = -(K_Y + \Delta)$. Hence f_*B is $\mathbb{Q}$-Cartier, and we can define the pull-back f^*f_*B. Since $B + \sum_{i=1}^s a_iE_i$ is $\mathbb{Q}$-linearly equivalent to $-f^*(K_Y + \Delta)$, $(B - f^*f_*B) + \sum_{i=1}^s a_iE_i$ is an f-exceptional divisor which is $\mathbb{Q}$-linearly trivial relative to f. Hence

$$(3.3.3) \qquad\qquad (B - f^*f_*B) + \sum_{i=1}^s a_iE_i = 0.$$

(4) Now $f_*D - (q-1)\Delta \geq \lfloor(q-1)\Delta\rfloor - (q-1)\Delta \geq -\Delta'$ for some (effective $\mathbb{Q}$-) Cartier divisor Δ' on Y which is independent of q. This implies $f_*B \geq -\dfrac{1}{q-1}\Delta'$,

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whence $f^* f_* B \geq -\dfrac{1}{q-1} f^* \Delta'$. Therefore, if we take $q = p^e$ sufficiently large, then the coefficient of $f^* f_* B$ in E_i is greater than $-1/r$. On the other hand, we have seen in (2) that the coefficient of B in E_i is at most 1. Since $ra_i \in \mathbb{Z}$, it follows from (3.3.3) that $a_i \geq -1$.

(5) Assume now that (A, Δ) is divisorially F-regular. For a $\mathbb{Q}$-Cartier divisor Δ' as in (4), we choose $c \neq 0 \in A$ which is not in any minimal prime of $A(-\lfloor \Delta \rfloor)$ so that the value $v_{E_i}(c)$ is greater than the coefficient of $f^* \Delta'$ in E_i. Then there exists an A-linear map $\psi \colon A((q-1)\Delta)^{1/q} \to A$ sending $c^{1/q}$ to 1, and $\phi = c^{1/q}\psi$ gives a splitting of $A \hookrightarrow A((q-1)\Delta)^{1/q}$. Let D_ϕ and D_ψ be the divisors defined by ϕ and ψ as rational sections of $\omega_X^{\otimes(1-q)}$, respectively. Then the coefficient of D_ϕ in E_i is $\leq q-1$ by (2), so that the coefficient of $D_\psi + f^* \Delta'$ in E_i is $< q-1$, since $D_\phi = D_\psi + \mathrm{div}_X(c)$. Let $B = (1/(q-1))D_\psi - \tilde{\Delta}$ and argue as in the F-pure case. Then we have $a_i > -1$, as required. $\square$

Remark 3.4. The both proofs show that if the map $A \to A((q-1)\Delta)^{1/q}$ sending 1 to $c^{1/q}$ has a splitting for a *single* element c in a sufficiently high power of $H^0(X, \mathcal{O}_X(-E))$, then we have $a_j > -1$ for every j. In the case $\Delta = 0$, if the map $A \xrightarrow{c^{1/q}} A((q-1)\Delta)^{1/q}$ splits for an element c with $v_{E_i}(c) > 0$, then $a_i > -1$.

Remark 3.5. It follows that if (A, Δ) is strongly F-regular and if $K_Y + \Delta$ is $\mathbb{Q}$-Cartier, then $\lfloor \Delta \rfloor = 0$ and in (3.1.1), one has $a_j > -1$ for every j. But the above proof says something about strong F-regularity even when $K_Y + \Delta$ is not $\mathbb{Q}$-Cartier.

Let $f \colon X \to Y = \mathrm{Spec}\, A$ be a resolution of singularities admitting an f-ample Cartier divisor H supported on the exceptional locus of f, and assume that (A, Δ) is strongly F-regular. Then we can prove that there exists a $\mathbb{Q}$-divisor G on X satisfying the following conditions:

(i) $\lceil G \rceil$ is an effective f-exceptional divisor;
(ii) $-f_* G \geq \Delta$;
(iii) $G - K_X$ is f-ample.

To see this, for an effective $\mathbb{Q}$-divisor Δ' as in (4) above, choose $c \neq 0 \in A$ such that $\mathrm{div}_X(c) \geq f^* \Delta'$ and that this is a strict inequality for the coefficients of E_j's. Then there is an A-linear map $\theta \colon A((q-1)\Delta)^{1/q} \to A$ sending $c^{2/q}$ to 1, by strong F-regularity. Let $\psi = c^{1/q}\theta$ and define the divisor D_ψ as in the step (5) above. Now we put $G = -\dfrac{1}{q-1}D_\psi + \varepsilon H$ for $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon \ll 1$. Then G satisfies conditions (ii) and (iii). Also, the coefficient of G in each irreducible component G_j is ≥ -1 and is > -1 if G_j is f-exceptional (resp. ≤ 0 if G_j is not f-exceptional). Since $\lfloor \Delta \rfloor = 0$ and ampleness is an open condition, we can perturb coefficients of G slightly so that G satisfies condition (i) as well as (ii) and (iii).

Example 3.6. Let us take a look at two typical examples in the case $\Delta = 0$.

(1) Let A be the localization of $k[X, Y, Z, W]/(X^4 + Y^4 + Z^4 + W^4)$ at the unique graded maximal ideal, where k is a field of characteristic p. By a criterion by R. Fedder [F], A is F-pure if (and only if) $p \equiv 1 \pmod 4$. On the other hand, if $f \colon X \to \mathrm{Spec}\, A$ is the blowing-up of the maximal ideal of A, then X is regular with

$\omega_X \cong \mathcal{O}_X(-E_0)$, where E_0 is the exceptional divisor of f. Then the discrepancy of E_0 is $a_0 = -1$.

(2) Let A be the localization of the rth Veronese subring $k[X_1, \dots, X_n]^{(r)}$ of $k[X_1, \dots, X_n]$ at the unique graded maximal ideal, where k is a field of characteristic p. Then A is $\mathbb{Q}$-Gorenstein with index $r/(r, n)$ and is surely strongly F-regular being a pure subring of a regular ring. Again, let X be the blowing-up of the maximal ideal of A. Then X is regular and if E_0 is the exceptional divisor, we have $a_0 = -1 + n/r$.

Now, let us discuss rings essentially of finite type over a field of characteristic zero. Our goal is the following

Theorem 3.7. *Let A be a normal local ring essentially of finite type over a field k of characteristic zero and let Δ be an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Carier. If (A, Δ) is of strongly F-regular (resp. divisorially F-regular, F-pure) type, that means the reduction modulo p of (A, Δ) is strongly F-regular (resp. divisorially F-regular, F-pure) for infinitely many prime p, then (Y, Δ) is KLT (resp. PLT, LC).*

Proof. Let Z be a normal affine variety over k such that $A \cong \mathcal{O}_{Z,z}$ for some $z \in Z$. We may assume that Δ comes from a $\mathbb{Q}$-divisor Δ_Z such that $K_Z + \Delta_Z$ is $\mathbb{Q}$-Cartier. Let $g \colon \tilde{X} \to Z$ be a log resolution of (Z, Δ_Z) (such that $g_*^{-1} \lfloor \Delta_Z \rfloor$ is smooth, in order to prove PLT). Then since $Z, \Delta_Z, g, \tilde{X}$ are defined by finite elements of k, we can choose a finitely generated $\mathbb{Z}$-subalgebra R of k, schemes $Z_R, \tilde{X}_R$ of finite type over R, a $\mathbb{Q}$-divisor Δ_R and an R-morphism $g_R \colon \tilde{X}_R \to Z_R$, which give $Z, \tilde{X}, \Delta_Z$ and g after tensoring k over R. Then, taking suitable open subset of $\operatorname{Spec} R$, we may assume $Z_R, \tilde{X}_R$ and each irreducible component of Δ_R are flat over R, $\tilde{X}_R$ is smooth over R with each irreducible component of the exceptional divisor and Z_R is normal over R (cf. [EGA, IV, 12.1.7]). Also, we can choose R to be regular with trivial dualizing module. Then we may assume that the numbers a_i in the formula $K_X + \tilde{\Delta} = f^*(K_Y + \Delta) + \sum_{i=1}^r a_i E_i$ are preserved in the fibers of $g_R \colon \tilde{X}_R \to Z_R$ over $\operatorname{Spec} R$ in an open neighborhood of the generic point of $\operatorname{Spec} R$. By our assumption, this open neighborhood contains a maximal ideal $\mathfrak{p}$ of R, such that the base change of (Z_R, Δ_R) to $k(\mathfrak{p})$ over R is strongly F-regular (resp. divisorially F-regular, F-pure) at $\bar{z}$, a specialization of z (to be F-regular or F-pure is preserved under base field extension or restriction and so depends only on characteristic). This implies $a_i > -1$ (resp. $a_i \geq -1$) by Theorem 3.3 and our assertion is proved. $\square$

4. APPLICATIONS

4.1. The graded case. (cf. [W1, W2]) Let $R = \bigoplus_{n \geq 0} R_n$ be a normal graded ring over a perfect field $R_0 = k$ of characteristic $p > 0$. Given a homogeneous element T of degree 1 in the quotient field of R, there is an ample $\mathbb{Q}$-Cartier divisor D on $X = \operatorname{Proj} R$ such that

$$R = R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) T^n.$$

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If $D = \sum_{i=1}^{s}(e_i/d_i)D_i$ for distict prime divisors $D_1, \ldots, D_s$ and coprime integers d_i and e_i with $d_i > 0$ ($1 \leq i \leq s$), we denote $D' = \sum_{i=1}^{s}((d_i - 1)/d_i)D_i$ and call it the "fractional part" of D ([W1]). Then $K_R^{(r)}$ is free if and only if $r(K_X + D')$ is linearly equivalent to bD for some integer b.

If $Z = \mathrm{Spec}_X\left(\bigoplus_{n \geq 0} \mathcal{O}_X(nD)T^n\right)$ is the "graded blowing-up" of $\mathrm{Spec}\, R$ and if $E_0 \cong X$ is the exceptional divisor of this blowing-up, its discrepancy is $a_0 = -1 - b/r$. Consequently, if R is strongly F-regular (resp. F-pure) then $b < 0$ (resp. $b \leq 0$). (In this case, although Z is not Gorenstein in general, the number a_0 is preserved after we make more blowing-ups and get a Gorenstein (or regular) scheme.)

Now let $\Delta = \sum_{j=1}^{r} t_j \Delta_j$ be an effective $\mathbb{Q}$-Weil divisor on $\mathrm{Spec}\, R$ which is stable under the k^*-action, i.e., each irreducible component Δ_j of Δ is defined by a *homogeneous* prime ideal of R of height 1. Then there exist effective $\mathbb{Q}$-Weil divisors $\Gamma_1, \ldots, \Gamma_r$ on $X = \mathrm{Spec}\, R$ such that for every $i \in \mathbb{Z}$ and $j = 1, \ldots, r$,

$$R(i\Delta_j) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(i\Gamma_j + nD))T^n.$$

Let $\Gamma = \sum_{j=1}^{r} t_j \Gamma_j$ and D' be the "fractional part" of D as above. Then we can rephrase Proposition 2.4 using this language, cf. [W2].

Proposition 4.2. *Let the notation be as in 4.1 and let* $\dim R = d + 1 \geq 2$.

    (1) (R, Δ) *is F-pure if and only if the map*

$$F^e: H^d(X, \mathcal{O}_X(K_X)) \to H^d(X, \mathcal{O}_X(q(K_X + D') + (q-1)\Gamma))$$

    is injective for every $q = p^e$.

    (2) (R, Δ) *is divisorially (resp. strongly) F-regular if and only if for every* $n \geq 0$ *and every* $c \neq 0 \in H^0(X, \mathcal{O}_X(nD)) \setminus \bigcup_{t_j \geq 1} H^0(X, \mathcal{O}_X(-\Gamma_j + nD))$ *(resp.* $c \neq 0 \in H^0(X, \mathcal{O}_X(nD)))$, *there exists* $q = p^e$ *such that the map*
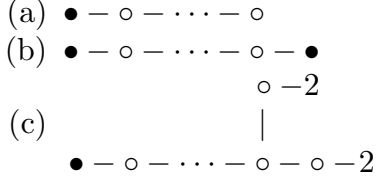
$$cF^e: H^d(X, \mathcal{O}_X(K_X)) \to H^d(X, \mathcal{O}_X(q(K_X + D') + (q-1)\Gamma + nD))$$

    is injective.

Example 4.3. Let $R = R(\mathbb{P}^1, D)$ with $D = \frac{1}{2}((0) + (\infty))$ and let Δ be the divisor defined by $R(-\Delta) = \bigoplus_{n \geq 0} H^0(\mathbb{P}^1, \mathcal{O}(-(1) + nD))T^n$. Then (R, Δ) is F-pure if and only if $p \neq 2$. This is a special case of the type (c) in 4.4 below.

4.4. Two-dimensional F-regular and F-pure pairs with integer coefficient boundary. Combining Theorem 3.3 and the technique used in [Ha2], we can classify F-regular and F-pure pairs (A, Δ) such that A is a two-dimensional normal local ring of characteristic $p > 0$ and that Δ is an integer coefficient effective Weil divisor on $\mathrm{Spec}\, A$. The case $\Delta = 0$ is treated in [Ha2], so we treat here only the case $\Delta \neq 0$.

Two-dimensional LC pairs with nonzero reduced boundaries are classified in terms of the dual graph of the union of the exceptional divisor E and the strict transform $\tilde{\Delta}$ of Δ for the minimal log resolution $f: X \to Y = \mathrm{Spec}\, A$, and such a pair is one of the following three types ([A], [Ka]):

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(a) $\bullet - \circ - \cdots - \circ$

(b) $\bullet - \circ - \cdots - \circ - \bullet$

$\qquad\qquad\qquad \circ -2$

(c) $\qquad\qquad\qquad |$

$\quad \bullet - \circ - \cdots - \circ - \circ -2$

Here, a blank circle $\circ$ (resp. a solid circle $\bullet$) denotes an irreducible component $E_i \cong \mathbb{P}^1$ of E (resp. an irreducible component of $\tilde{\Delta} = f_*^{-1}(\Delta)$), and the numbers "$-2$" outside the circles in (c) mean that the corresponding components have self-intersection number equal to -2. Among the above three types, type (a) is PLT but types (b) and (c) are not PLT.

Theorem 3.3 tells us that a two-dimensional F-pure (resp. divisorially F-regular) pair with reduced boundary is of type (a), (b) or (c) (resp. type (a)) as above. We can show that the converse is true but one exception.

Theorem 4.5. *Let $(A, \mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field $k = A/\mathfrak{m}$ of characteristic $p > 0$ and let Δ be a nonzero reduced Weil divisor on $Y = \operatorname{Spec} A$. Let $f\colon X \to Y$ be the minimal resolution with exceptional divisor E and let $\tilde{\Delta} = f_*^{-1}(\Delta)$.*

(1) *(A, Δ) is divisorially F-regular if and only if the dual graph of $E \cup \tilde{\Delta}$ is of type (a) in 4.4.*

(2) *(A, Δ) is F-pure but not divisorially F-regular if and only if the dual graph of $E \cup \tilde{\Delta}$ is of type (b) in 4.4, or of type (c) in 4.4 and $p \neq 2$.*

Proof. (1) Let the dual graph of $E \cup \tilde{\Delta}$ be of type (a) as in 4.4, and fix any nonzero element $c \in A$ which is not in $A(-\Delta) = H^0(X, \mathcal{O}_X(-\tilde{\Delta}))$. By Proposition 2.4, our goal is to show that there exists $q = p^e$ such that the map

(4.5.1) $\qquad cF^e\colon H^2_{\mathfrak{m}}(K_A) \to H^2_{\mathfrak{m}}(A(qK_A + (q-1)\Delta))$

is injective. Blowing up at $E \cap \tilde{\Delta}$ finitely many times if necessary, we may assume that $\tilde{\Delta}$ does not intersect the strict transform of $\operatorname{div}_Y(c)$ on X and that the graph of $E \cup \tilde{\Delta}$ is still of type (a) as follows.

$$\tilde{\Delta} - E_0 - E_1 - \cdots - E_l$$

Note that $f\colon X \to Y$ may not be the minimal resolution any longer, but E_i is a (-1)-curve only if $i = 0$. By Lemma 3.9 of [Ha2], one has an effective f-exceptional $\mathbb{Q}$-divisor

$$D = E_0 + \frac{d-1}{d} E_1 + (\text{terms of } E_2, \ldots, E_l)$$

with $d \in \mathbb{N}$ such that $(K_X + D)E_0 = -1 - 1/d$ and $(K_X + D)E_j = 0$ for $j = 1, \ldots, l$. Let $D' = D - E_0$.

Let n be the integer with $c \in H^0(X, \mathcal{O}_X(-nE_0)) \setminus H^0(X, \mathcal{O}_X(-(n+1)E_0))$. Since $H^2_{\mathfrak{m}}(K_A) \cong H^2_E(\omega_X)$ by $H^i(X, \omega_X) = 0$ $(i = 1, 2)$, the map (4.5.1), followed by $H^2_{\mathfrak{m}}(A(qK_A + (q-1)\Delta)) \to H^2_E(\mathcal{O}_X(q(K_X + D') + (q-1)(\tilde{\Delta} + E_0) - nE_0))$, is

(4.5.2) $\quad cF^e\colon H^2_E(\omega_X) \to H^2_E(\mathcal{O}_X(q(K_X + D') + (q-1)(\tilde{\Delta} + E_0) - nE_0)),$

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and it is sufficient to show that this map is injective for some $q = p^e$.

Let $P, Q \in E_0$ be the points of intersection of E_0 with $E_1, \tilde{\Delta}$, respectively, and let $\mathfrak{d}' = D'|_{E_0} = \frac{d-1}{d}P$ and $\mathfrak{a} = -E_0|_{E_0}$ as ($\mathbb{Q}$-)divisors on $E_0 \cong \mathbb{P}^1$. Then $c \in H^0(X, \mathcal{O}_X(-nE_0))$ restricts to a nonzero element $\bar{c} \in H^0(E_0, \mathcal{O}_{E_0}(n\mathfrak{a}))$, and $\bar{c} \notin H^0(E_0, \mathcal{O}_{E_0}(n\mathfrak{a} - Q))$ since $\tilde{\Delta} \cap f_*^{-1} \operatorname{div}_Y(c) = \emptyset$.

Now if $q = p^e$ is sufficiently large, then $-q(K_X + D) - (q-1)\tilde{\Delta} + nE_0$ is f-nef, so that $R^1 f_* \omega_X(\lceil -q(K_X + D) - (q-1)\tilde{\Delta} + nE_0 \rceil) = 0$ (cf. [Ha2, Lemma 3.3]), or dually, $H^1_E(\mathcal{O}_X(q(K_X + D) + (q-1)\tilde{\Delta} - nE_0)) = 0$. Hence we have the following commutative diagram with the vertical arrows being injective for $q = p^e \gg 0$.

$$
\begin{array}{ccc}
H^2_E(\omega_X) & \xrightarrow{\;cF^e\;} & H^2_E(\mathcal{O}_X(q(K_X + D') + (q-1)(\tilde{\Delta} + E_0) - nE_0)) \\[2mm]
\uparrow & & \uparrow \\[2mm]
H^1(E_0, \omega_{E_0}) & \xrightarrow{\;\bar{c}F^e\;} & H^1(E_0, \mathcal{O}_{E_0}(q(K_{E_0} + \mathfrak{d}') + (q-1)Q + n\mathfrak{a}))
\end{array}
$$

Note also that the map $H^1(E_0, \omega_{E_0}) \to H^2_E(\omega_X)$ on the left is identified with $k = A/\mathfrak{m} \hookrightarrow E_A(A/\mathfrak{m})$, whence an essential extension. By computing Čech cohomologies on $E_0 \cong \mathbb{P}^1$, we can verify that the map $\bar{c}F^e$ at the bottom is injective for $q = p^e \gg 0$, from which follows the injectivity of the map (4.5.2) at the top.

(2) If the graph is of type (b) or (c), we use the anti-discrepancy of f in place of D in the proof of (1) above, i.e., $D = f^*(K_Y + \Delta) - (K_X + \tilde{\Delta})$. Let $D_0 = \lfloor D \rfloor$, $D' = D - D_0$ and $\mathfrak{d}' = D'|_{D_0}$, and argue as in (1) for $c = 1$. Then it follows that (A, Δ) is F-pure if and only if the map

$$
F^e \colon H^1(D_0, \omega_{D_0}) \to H^1(D_0, \mathcal{O}_{D_0}(q(K_{D_0} + \mathfrak{d}') + (q-1)\tilde{\Delta}|_{D_0}))
$$

is injective for all $q = p^e$. We can show as in [Ha2, Claim 4.8.1] that this map is injective if and only if the graph of $E \cup \tilde{\Delta}$ is of type (b), or of type (c) and $p \neq 2$. See also [Ha2] for details. $\square$

4.6. Strongly F-regular rings vs. admissible singularities. The correspondence of $\mathbb{Q}$-Gorenstein (strongly) F-regular rings and log terminal singularities (in the case $\Delta = 0$) is now well established. Namely, we have seen that a ring of characteristic zero has log terminal singularities if it is of (strongly) F-regular type and $\mathbb{Q}$-Gorenstein. The converse of this implication is also proved to be true [Ha3]. Thus we are tempted to consider what non-$\mathbb{Q}$-Gorenstein strongly F-regular rings are. We have no established answer to this question, but there is a candidate which is expected to correspond strongly F-regular rings even in non-$\mathbb{Q}$-Gorenstein case.

In [N], Nakayama introduced the notion of admissible singularities as an analog of log terminal singularities in the absence of $\mathbb{Q}$-Gorensteinness. Let Y be a normal variety of characteristic zero and let Δ be an effective $\mathbb{Q}$-Weil divisor on Y. Then the pair (Y, Δ) is said to be *strictly admissible* if there exist a birational morphism $f \colon X \to Y$ from a nonsingular X and a $\mathbb{Q}$-divisor G on X satisfying the following conditions:

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(i) $\lceil G \rceil$ is an effective f-exceptional divisor;

(ii) $-f_* G \geq \Delta$;

(iii) $G - K_X$ is f-ample;

(iv) $\mathrm{Supp}(G - \lfloor G \rfloor)$ is a normal crossing divisor.

On the other hand, we observed in Remark 3.5 that if (Y, Δ) is a strongly F-regular pair in characteristic $p > 0$ and $f \colon X \to Y$ is a resolution, then we can construct a $\mathbb{Q}$-divisor G satisfying conditions (i), (ii) and (iii). But we do not have condition (iv), and even worse, this construction depends on characteristic p. However, the following example may be a positive evidence to the correspondence of strong F-regularity and admissible singularity.

Example 4.7. Let X be a smooth Fano variety of characteristic zero (i.e., a smooth projective variety with ample anti-canonical divisor $-K_X$), and D be any ample Cartier divisor on X. Let $R = R(X, D)$ and let $f \colon Z \to \mathrm{Spec}\, R$ be the "graded blowing-up" as in (4.1). The exceptional set of f is a smooth divisor $E \cong X$, so if we put $G = (\varepsilon - 1)E$ with $0 < \varepsilon \leq 1$, conditions (i), (ii), (iv) in (4.6) are satisfied. Also, if we choose $\varepsilon \ll 1$, then $(G - K_Z)|_E = (-(K_Z + E) + \varepsilon E)|_E = -K_E - \varepsilon D$ is ample since $E \cong X$ is Fano, so that condition (iii) is satisfied. Consequently, R has an admissible singularity whereas it is not $\mathbb{Q}$-Gorenstein in general.

On the other hand, $R(X, -K_X)$ has a Gorenstein log terminal singularity, so that it is of F-regular type [Ha3]. However, since the strong F-regularity of $R(X, D)$ depends only on the "fractional part" D' of D [W2] and the fractional parts of D and $-K_X$ are both 0 in this case, $R = R(X, D)$ also has strongly F-regular type.

We study more about similarity of F-regularity and F-purity of pairs and "singularities of pairs" in characteristic zero [Ko], [Sh]. The following theorem generalizes [W2, Theorem 2.7]. See Remark 1.2 (5).

Theorem 4.8. *Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a finite local homomorphism of F-finite normal local rings which is étale in codimension 1. Let Δ_A be an effective $\mathbb{Q}$-Weil divisor on $\mathrm{Spec}\, A$ and let $\Delta_B = \pi^* \Delta_A$ be the pull-back of Δ_A by the induced morphism $\pi \colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$. If (A, Δ_A) is F-pure (resp. divisorially or strongly F-regular), then so is (B, Δ_B), too.*

Proof. Let $d = \dim A = \dim B$. Since $A \to B$ is étale in codimension 1, the natural maps $A^{1/q} \otimes_A B \to B^{1/q}$ and $A(qK_A + (q-1)\Delta_A) \otimes_A B \to B(qK_B + (q-1)\Delta_B)$ are isomorphic in codimension 1. Hence, by [W2, Lemma 2.2], we have

$$H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta_A)^{1/q}) \otimes_A B$$

$$\cong H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta_A)^{1/q} \otimes_A B)$$

$$\cong H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta_A)^{1/q} \otimes_{A^{1/q}} A^{1/q} \otimes_A B)$$

$$\cong H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta_A)^{1/q} \otimes_{A^{1/q}} B^{1/q})$$

$$\cong H^d_{\mathfrak{m}}((A(qK_A + (q-1)\Delta_A) \otimes_A B)^{1/q})$$

$$\cong H^d_{\mathfrak{n}}(B(qK_B + (q-1)\Delta_B)^{1/q}).$$

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Therefore, tensoring the map

(4.8.1)
$$F^e \colon H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_A + (q-1)\Delta_A))$$

with B over A yields

(4.8.2)
$$F^e \colon H^d_{\mathfrak{n}}(K_B) \to H^d_{\mathfrak{n}}(B(qK_B + (q-1)\Delta_B)).$$

Now, if (A, Δ_A) is F-pure, then for every $q = p^e$, the map (4.8.1) is injective, and even a *splitting* injective map, since $H^d_{\mathfrak{m}}(K_A) \cong E_A(A/\mathfrak{m})$ is an injective A-module. Hence the map (4.8.2) is also injective for every $q = p^e$, so that (B, Δ_B) is F-pure.

Similarly, if (A, Δ_A) is strongly F-regular, then for every $c \neq 0 \in A$, there exists $q = p^e$ such that the map

$$cF^e \colon H^d_{\mathfrak{n}}(K_B) \to H^d_{\mathfrak{n}}(B(qK_B + (q-1)\Delta_B))$$

is injective. This is true for every $c \neq 0 \in B$, since $A \to B$ is a finite extension of normal domains, so that $cB \cap A \neq 0$. Hence (B, Δ_B) is strongly F-regular.

To prove the assertion for divisorial F-regularity, note that if $c \in B$ is not in any minimal prime ideal of $B(-\lfloor \Delta_B \rfloor)$, then there is an element of $cB \cap A$ which is not in any minimal prime ideal of $A(-\lfloor \Delta_A \rfloor)$. Then the argument for strong F-regularity works also for divisorial F-regularity. $\square$

Theorem 4.9. (cf. [Ko, Theorem 7.5]) *Let $(A, \mathfrak{m})$ be an F-finite normal local ring of characteristic $p > 0$ with $Y = \operatorname{Spec} A$ and let $x \in \mathfrak{m}$ be a nonzero element.*

(1) *If the pair $(A, \operatorname{div}_Y(x))$ is divisorially F-regular, then the ring A/xA is strongly F-regular.*

(2) *Assume that A is $\mathbb{Q}$-Gorenstein and that the order r of the canonical class in the divisor class group $\operatorname{Cl}(A)$ is not divisible by p. Then, if the ring A/xA is strongly F-regular, the pair $(A, \operatorname{div}_Y(x))$ is divisorially F-regular.*

Proof. Let $B = A/xA$ and $d = \dim A$. We will look at the Frobenius actions on $E_A = E_A(A/\mathfrak{m}) \cong H^d_{\mathfrak{m}}(K_A)$ and $E_B = E_B(B/\mathfrak{m}B) \cong H^{d-1}_{\mathfrak{m}}(K_B)$ in slightly different ways in proving (1) and (2), respectively.

(1) Assume that $(A, \operatorname{div}_Y(x))$ is divisorially F-regular. Then for every element $c \in A$ which is not in any minimal prime ideal of xA, there exists $q = p^e$ such that the map $cF^e \colon H^d_{\mathfrak{m}}(K_A) \to H^d_{\mathfrak{m}}(A(qK_Y + (q-1)\operatorname{div}_Y(x)))$ is injective, or equivalently, the map

$$\alpha \colon E_A = E_A \otimes_A A \to E_A \otimes_A A^{1/q}$$

sending $\xi \in E_A$ to $\xi \otimes (cx^{q-1})^{1/q} \in E_A \otimes_A A^{1/q}$ is injective (see the proof of Proposition 2.6). On the other hand, we have an inclusion $\imath \colon E_B \cong (0 : x)_{E_A} \hookrightarrow E_A$, and this map gives rise to a well-defined $A^{1/q}$-homomorphism

$$\jmath \colon E_B \otimes_B B^{1/q} \to E_A \otimes_A A^{1/q}$$

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sending $\xi \otimes \bar{a}^{1/q} \in E_B \otimes_B B^{1/q}$ to $\imath(\xi) \otimes (ax^{q-1})^{1/q} \in E_A \otimes_A A^{1/q}$, where $\bar{a}$ denotes the image of $a \in A$ in $B = A/xA$. Then we have the following commutative diagram.

$$
\begin{array}{ccc}
E_B & \xrightarrow{\;\imath\;} & E_A \\
{\scriptstyle 1\otimes\bar{c}^{1/q}}\downarrow & & \downarrow{\scriptstyle \alpha} \\
E_B \otimes_B B^{1/q} & \xrightarrow{\;\jmath\;} & E_A \otimes_A A^{1/q}
\end{array}
$$

Hence the injectivity of the map α implies that $1_{E_B} \otimes \bar{c}^{1/q}\colon E_B \to E_B \otimes_B B^{1/q}$ is injective. Consequently, we have that B is strongly F-regular by Remark 1.2 (1).

(2) Assume that B is strongly F-regular. Then B is a Cohen–Macaulay normal domain, so that A and K_A are also Cohen–Macaulay. For any $q = p^e$ and $c \in A\backslash xA$, we consider the following commutative diagram with exact rows,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_A & \xrightarrow{\;x\;} & K_A & \longrightarrow & K_A/xK_A & \longrightarrow & 0 \\
& & {\scriptstyle cx^{q-1}F^e}\downarrow & & {\scriptstyle cF^e}\downarrow & & {\scriptstyle cF^e}\downarrow & & \\
0 & \longrightarrow & K_A^{(q)} & \xrightarrow{\;x\;} & K_A^{(q)} & \longrightarrow & K_A^{(q)}/xK_A^{(q)} & \longrightarrow & 0,
\end{array}
$$

where $K_A/xK_A \cong K_B$. Since $H_{\mathfrak{m}}^{d-1}(K_A^{(q)}/xK_A^{(q)}) \cong H_{\mathfrak{m}}^{d-1}(K_B^{(q)})$ by the normality of B, this diagram induces the following commutative diagram of local cohomologies.

$$
\begin{array}{ccc}
H_{\mathfrak{m}}^{d-1}(K_B) & \longrightarrow & H_{\mathfrak{m}}^d(K_A) \\
{\scriptstyle cF^e}\downarrow & & \downarrow{\scriptstyle cx^{q-1}F^e} \\
H_{\mathfrak{m}}^{d-1}(K_B^{(q)}) & \longrightarrow & H_{\mathfrak{m}}^d(K_A^{(q)})
\end{array}
$$

Here, the map $H_{\mathfrak{m}}^{d-1}(K_B) \to H_{\mathfrak{m}}^d(K_A)$ upstairs is an essential extension, and if $q \equiv 1$ (mod r), then the map $H_{\mathfrak{m}}^{d-1}(K_B^{(q)}) \to H_{\mathfrak{m}}^d(K_A^{(q)})$ at the bottom is also injective since $K_A \cong K_A^{(q)}$ is Cohen–Macaulay.

Now, since B is strongly F-regular, the map $cF^e\colon H_{\mathfrak{m}}^{d-1}(K_B) \to H_{\mathfrak{m}}^{d-1}(K_B^{(q)})$ in the above diagram is injective for all $q = p^e \gg 0$, by (2) of Propositions 2.2 and 2.4. Since r is not divisible by p, we can choose $q = p^e$ so that $q \equiv 1$ (mod r) and that $cF^e\colon H_{\mathfrak{m}}^{d-1}(K_B) \to H_{\mathfrak{m}}^{d-1}(K_B^{(q)})$ is injective. Then the diagram implies that the map $cx^{q-1}F^e\colon H_{\mathfrak{m}}^d(K_A) \to H_{\mathfrak{m}}^d(K_A^{(q)})$ is injective for such a $q = p^e$, or equivalently,

$$
cF^e\colon H_{\mathfrak{m}}^d(K_A) \to H_{\mathfrak{m}}^d(A(qK_Y + (q-1)\operatorname{div}_Y(x)))
$$

is injective. Hence $(A, \operatorname{div}_Y(x))$ is divisorially F-regular. $\square$

Remark 4.10. (1) In the situation of Theorem 4.9 (2), assume in addition that $B = A/xA$ is normal and Cohen–Macaulay. Then we can also prove that $(A, \operatorname{div}_Y(x))$ is F-pure if and only if $B = A/xA$ is F-pure.

(2) The proof of Theorem 4.9 suggests a more general statement as follows: Let $(A, \mathfrak{m})$ and $x \in \mathfrak{m}$ be as in Theorem 4.9. Let $S = \operatorname{div}_Y(x) \cong \operatorname{Spec} A/xA$ and Δ

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be an effective $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$ such that $r(K_Y + \Delta)$ is Cartier for some positive integer r which is not divisible by p. Then $(A, S + \Delta)$ is divisorially F-regular if and only if $(A/xA, \Delta|_S)$ is strongly F-regular.

If we replace "divisorially F-regular" and "strongly F-regular" in this assertion by "PLT" and "KLT" respectively, we find the so-called "inversion of adjunction" in characteristic zero [Ko, Theorem 7.5], [Sh].

Corollary 4.11. (cf. [AKM]) *Let $(A, \mathfrak{m})$ be an F-finite local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a nonzero element. Assume that A is $\mathbb{Q}$-Gorenstein and that the order r of the canonical class in the divisor class group of $Y = \operatorname{Spec} A$ is not divisible by p. If A/xA is strongly F-regular, then A is also strongly F-regular.*

Proof. If A/xA is strongly F-regular, then A/xA is F-rational. This implies that A is F-rational, whence normal [FW], [HH3, 4.2]. Thus we can apply Theorem 4.9, which implies that $(A, \operatorname{div}_Y(x))$ is divisorially F-regular. It follows that $(A, 0)$ is divisorially F-regular, meaning that A is strongly F-regular. $\square$

In fact, we do not have to assume that the index r of A is not divisible by p in Corollary 4.11 (see Aberbach, Katzman and MacCrimmon [AKM]). On the other hand, an example by Singh [Si] shows that Corollary 4.11 fails in the absense of $\mathbb{Q}$-Gorensteinness.

5. Open Problems

So far, we have seen many positive evidences to the correspondence of F-purity (resp. divisorial, strong F-regularity) and LC (resp. PLT, KLT) property, which enable us to ask about the converse of Theorem 3.7.

5.1. Let A be a normal ring essentially of finite type over a field of characteristic zero and let Δ be a $\mathbb{Q}$-Weil divisor on $Y = \operatorname{Spec} A$.

Conjecture 5.1.1. If (Y, Δ) is KLT (resp. PLT), then it is of open strongly F-regular type (resp. open divisorially F-regular type), i.e., the modulo p reduction of (A, Δ) is strongly F-regular (resp. divisorially F-regular) for all $p \gg 0$.

Conjecture 5.1.1 is true when $\Delta = 0$ [Ha3]. Also, Theorem 4.8 and [Ko, Theorem 7.5] shows that a pair $(A, \operatorname{div}_Y(x))$ is PLT if and only if it is of divisorially F-regular type and A is $\mathbb{Q}$-Gorenstein, since A/xA is log terminal if and only if it is of F-regular type and $\mathbb{Q}$-Gorenstein.

Problem 5.1.2. If (Y, Δ) is LC, then is it always of (dense) F-pure type, i.e., the modulo p reduction of (A, Δ) is F-pure for infinitely many p?

5.2. Log canonical thresholds. ([Ko, §8], [Sh]) An affirmative answer to (5.1) suggests a Frobenius characterization of an important invariant called the log canonical threshold. Let Y be a variety in characteristic zero with only log terminal singularity at a point $y \in Y$ and Δ be an effective $\mathbb{Q}$-Cartier divisor on Y. The *log canonical threshold* of Δ at $y \in Y$ is defined by

$$LCTh_y(Y, \Delta) = \sup\{t \in \mathbb{R} \mid (Y, t\Delta) \text{ is LC at } y \in Y\}.$$

Theorem 3.3 tells us that this invariant is greater than or equal to the supremum of $t \in \mathbb{R}$ such that reduction modulo p of $(\mathcal{O}_{Y,y}, t\Delta)$ is F-pure for infinitely many p.

Conjecture 5.2.1. In the notation as above, the following are equal to each other:

(i) $LCTh_y(Y, \Delta)$;
(ii) $\sup\{t \in \mathbb{R} \mid$ reduction mod p of $(\mathcal{O}_{Y,y}, t\Delta)$ is F-pure for infinitely many $p\}$;
(iii) $\sup\{t \in \mathbb{R} \mid$ reduction mod p of $(\mathcal{O}_{Y,y}, t\Delta)$ is strongly F-regular for $p \gg 0\}$.

We expect that $LCTh_y(Y, \Delta)$ is computable in terms of characteristic p method. For example, by a Fedder-type criterion we can show that $(k[[x, y]], t \cdot \mathrm{div}(x^2 - y^3))$ is F-pure if and only if $t \leq 5/6$ (Example 2.9 (3)), and 5/6 is nothing but the log canonical threshold of the hypersurface $x^2 - y^3 = 0$ in $\mathbb{A}_k^2$ at the origin. Also, compare Proposition 2.10 with [Ko, Lemma 8.10].

5.3. Finally, we propose other open problems.

Problem 5.3.1. If (A, Δ) is of strongly F-regular type ($K_Y + \Delta$ is not necessarily $\mathbb{Q}$-Cartier), then does the pair (Y, Δ) have admissible singularities? How about the converse implication?

Problem 5.3.2. Let $(A, \mathfrak{m})$ be a d-dimensional normal local ring of characteristic $p > 0$ and let Δ be an effective $\mathbb{Q}$-Weil divisor on $Y = \mathrm{Spec}\, A$ such that $\lfloor \Delta \rfloor = 0$. Define the *tight closure of zero submodule in* $E = E_A(A/\mathfrak{m}) \cong H_{\mathfrak{m}}^d(K_A)$ *with respect to the pair* (A, Δ), $0_E^{*\Delta} \subset E = H_{\mathfrak{m}}^d(K_A)$, by

$$z \in 0_E^{*\Delta} \Longleftrightarrow$$

$$\exists c \neq 0 \in A \text{ such that } cz^q = 0 \text{ in } H_{\mathfrak{m}}^d(A(qK_A + (q-1)\Delta)) \text{ for } q = p^e \gg 0,$$

where z^q denotes the image of $z \in E$ by $F^e \colon H_{\mathfrak{m}}^d(K_A) \to H_{\mathfrak{m}}^d(A(qK_A + (q-1)\Delta))$. We ask for a geometric interpretation of the ideal $\tau(A, \Delta) = \mathrm{Ann}_A(0_E^{*\Delta})$. Especially, if $K_Y + \Delta$ is $\mathbb{Q}$-Cartier, is $\tau(A, \Delta)$ equal to the multiplier ideal of the pair (Y, Δ) in characteristic $p \gg 0$? The answer to this question is affirmative if $\Delta = 0$ ([Ha4], [S2]). See also [HH1,3] for tight closure, and [E] for multiplier ideals.

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