## THE MANIFOLD OF FINITE RANK PROJECTIONS IN THE SPACE  $\mathcal{L}(H)$ .

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December, 1999

ABSTRACT. Given a complex Hilbert space H and the von Neumann algebra  $\mathcal{L}(H)$  of all bounded linear operators in H, we study the Grassmann manifold M of all projections in  $\mathcal{L}(H)$  that have a fixed finite rank r. To do it we take the Jordan-Banach triple (or JB∗-triple) approach which allows us to define a natural Levi-Civita connection on  $M$  by using algebraic tools. We identify the geodesics and the Riemann distance and establish some properties of M.

## 0 INTRODUCTION

In this paper we are concerned with the differential geometry of the infinite-dimensional Grassmann manifold M of all projections in  $Z = \mathcal{L}(H)$ , the space of bounded linear operators  $z: H \to H$  in a complex Hilbert space H. Grassmann manifolds are a classical object in Differential Geometry and in recent years several authors have considered them in the Banach space setting. Besides the Grassmann structure, a Riemann and a Kähler structure has sometimes been defined even in the infinite-dimensional setting. Let us recall some aspects of the history of the topic that are relevant for our purpose.

The study of the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra was made by U. Hirzebruch in [4], who proved that such a manifold is a compact symmetric Riemann space of rank 1, and that every such a space arises in this way. Later on, Nomura in [13, 14] established similar results for the manifold of fixed finite rank projections in a topologically simple real Jordan-Hilbert algebra. On the other hand, the Grassmann manifold M of all projections in the space  $Z:=\mathcal{L}(H)$  of bounded linear operators has been discussed by Kaup in [7] and [10]. It is therefore reasonable to ask whether a Riemann structure can always be defined in  $M$  and how does it behave when it exists. It is known that  $M$  has several connected components  $M_r \subset M$  each of which consists of the projections in  $\mathcal{L}(H)$  that have a fixed rank  $r, 1 \leq r \leq \infty$ . We prove that  $M_r$  admits a Riemann structure if and only if  $r < \infty$  establishing a distinction between the finite and the infinite dimensional cases. We then assume  $r < \infty$  and proceed to discuss the behaviour of the Riemann manifold  $M_r$ , which looks very much like in the finite-dimensional case. One of the novelties is that we take JB<sup>∗</sup> -triple approach instead of the Jordan-algebra approach of [4] and [13]. As noted in [1] and [5], within this context the algebraic structure of JB<sup>\*</sup>-triple acts as a substitute for the Jordan algebra structure and provides a *local* scalar product known as the Levi form [10]. Although  $\mathcal{L}(H)$  is not a Hilbert space, the JB<sup>\*</sup>-triple approach and the use of the Levi form allows us to define a torsion-free affine connection  $\nabla$  on

<sup>1991</sup> Mathematics Subject Classification. 48 G 20, 72 H 51.

Key words and phrases. Grassmann manifolds, Riemann manifolds, JB\*-triples.

<sup>†</sup>Supported by Comisión Hispano-Húngara de Cooperación Científica y Tecnológica

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M<sub>r</sub> that is invariant under the group  $Aut^{\circ}(Z)$  of all surjective linear isometries of  $\mathcal{L}(H)$ . We integrate the equation of the geodesics and define an  $\text{Aut}^{\circ}(Z)$ -invariant Riemann metric on  $M_r$ with respect to which  $\nabla$  is a Levi-Civita connection. We prove that any two distinct points in  $M_r$  can be joined by a geodesic which (except for the case of a pair of antipodal points) is uniquely determined and is a minimizing curve for the Riemann distance, that is also computed. We prove that  $M_r$  is a symmetric manifold on which  $\text{Aut}^\circ(Z)$  acts transitively as a group of isometries.

# 1 JB<sup>\*</sup>-TRIPLES AND TRIPOTENTS.

For a complex Banach space Z, denote by  $\mathcal{L}(Z)$  the Banach algebra of all bounded linear operators on Z. A complex Banach space Z with a continuous mapping  $(a, b, c) \mapsto \{abc\}$  from  $Z \times Z \times Z$  to Z is called a *JB<sup>\*</sup>-triple* if the following conditions are satisfied for all  $a, b, c, d \in Z$ , where the operator  $a\Box b \in \mathcal{L}(Z)$  is defined by  $z \mapsto \{abz\}$  and  $[ , ]$  is the commutator product:

1  ${abc}$  is symmetric complex linear in a, c and conjugate linear in b.

2  $[a \Box b, c \Box d] = \{abc\} \Box d - c \Box \{dab\}.$ 

3 a $\Box a$  is hermitian and has spectrum  $\geq 0$ .

$$
4 \| \{aaa\} \| = \|a\|^3.
$$

If a complex vector space  $Z$  admits a JB<sup>\*</sup>-triple structure, then the norm and the triple product determine each other. A *derivation* of a JB<sup>\*</sup>-triple Z is an element  $\delta \in \mathcal{L}(Z)$  such that  $\delta\{zzz\} = \{(\delta z)zz\} + \{z(\delta z)z\} + \{zz(\delta z)\}\$ and an *automorphism* is a bijection  $\phi \in \mathcal{L}(Z)$  such that  $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}\$ for  $z \in \mathbb{Z}$ . The latter occurs if and only if  $\phi$  is a surjective linear isometry of Z. The group  $Aut(Z)$  of automorphisms of Z is a real Banach-Lie group whose Banach-Lie algebra is the set of derivations of Z . The connected component of the identity in Aut(Z) is denoted by Aut°(Z). Two elements  $x, y \in Z$  are orthogonal if  $x \Box y = 0$ . An element  $e \in Z$  is called a *tripotent* if  $\{eee\} = e$ . The set  $\text{Tri}(Z)$  of tripotents is endowed with the induced topology of Z. If  $e \in Tr(Z)$ , then  $e \Box e \in \mathcal{L}(Z)$  has the eigenvalues  $0, \frac{1}{2}$  $\frac{1}{2}$ , 1 and we have the topological direct sum decomposition

$$
Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e)
$$

called the *Peirce decomposition* of Z. Here  $Z_k(e)$  is the k- eigenspace and the *Peirce projections* are

$$
P_1(e) = Q^2(e)
$$
,  $P_{1/2}(e) = 2(e\Box e - Q^2(e))$ ,  $P_0(e) = \text{Id} - 2e\Box e + Q^2(e)$ ,

where  $Q(e)z = \{eze\}$  for  $z \in Z$ . We will use the Peirce rules  $\{Z_i(e)Z_j(e)Z_k(e)\} \subset Z_{i-j+k}(e)$ where  $Z_l(e) = \{0\}$  for  $l \neq 0, 1/2, 1$ . We note that  $Z_1(e)$  is a complex unital JB<sup>\*</sup>-algebra in the product  $a \circ b$ : = { $aeb$ } and involution  $a^{\#}$ : = { $eae$ }. Let

$$
A(e) := \{ z \in Z_1(e) : z^\# = z \}.
$$

Then we have  $Z_1(e) = A(e) \oplus iA(e)$ . The Peirce spaces of Z with respect to a an orthogonal family of tripotents  $\mathcal{E} = (e_i)_{i \in I}$  are defined by

$$
Z_{ii} = Z_1(e_i)
$$
  
\n
$$
Z_{ij} = Z_{1/2}(e_i) \cap Z_{1/2}(e_j), \quad i \neq j
$$
  
\n
$$
Z_{i0} = Z_{0i} = Z_{1/2}(e_i) \bigcap_{j \neq i} Z_0(e_j)
$$
  
\n
$$
Z_{00} = \bigcap_{i \in I} Z_0(e_i)
$$

The Peirce sum  $P(\mathcal{E}) = \bigoplus_{i,j \in I} Z_{ij}$  relative to the family  $\mathcal{E}$  is direct and we have  $Z = P(\mathcal{E})$ whenever  $\mathcal E$  is a finite set. Every  $\mathcal E$ -Peirce space is a JB<sup>\*</sup>-subtriple of Z and the Peirce rules

$$
\{Z_{ij}Z_{jk}Z_{kl}\}\subset Z_{il}
$$

hold for all  $i, j, k, l \in I$ .

A tripotent e in a JB<sup>\*</sup>-triple Z is said to be minimal if  $P_1(e)Z = \mathbb{C}e$ , and we let  $Min(Z)$ be the set of them. Clearly  $e = 0$  lies in  $\text{Min}(Z)$  and is an isolated point there. If  $e \in \text{Min}(Z)$ and  $e \neq 0$  then  $||e|| = 1$  and by the Peirce multiplication rules we have  $\{ew\} \in Z_1(e) =$ Ce for all  $u, v \in Z_{1/2}(e)$ . Therefore we can define a sesquilinear form, called the Levi form,  $\langle \cdot, \cdot \rangle_e: Z_{1/2}(e) \times Z_{1/2}(e) \to \mathbb{C}$  by

$$
{\text{euv}} = \langle v, u \rangle_e e, \qquad u, v \in Z_{1/2}(e).
$$

It is known [10] that  $\langle \cdot, \cdot \rangle_e$  is positive definite hence it defines a scalar product in  $Z_{1/2}(e)$  whose norm, called the Levi norm and denoted by  $|\cdot|_e$ , satisfies

$$
|u|_e^2 \le ||u||^2, \qquad u \in Z_{1/2}(e)
$$

that is, we have the continuous inclusion  $(Z_{1/2}(e), \|\cdot\|) \hookrightarrow (Z_{1/2}(e), \|\cdot\|_e)$ . To simplify the notation, we shall omit the subindex e in both the Levi form and the Levi norm if no confusion is likely to occur.

 $JB*$ -triples include  $C*$ -algebras and  $JB*$ -algebras. A  $C*$ -algebra is a  $JB*$ -triple with respect to the triple product  $2{abc}$ : =  $(ab^*c + cb^*a)$ . Every JB<sup>\*</sup>-algebra with Jordan product  $(a, b) \mapsto a \circ b$ and involution  $a \mapsto a^*$  is a JB<sup>\*</sup>-triple with triple product  $\{abc\} = (a \circ b^*) \circ c - (c \circ a) \circ b^* + (b^* \circ c) \circ a$ .

We refer to [8,9,10,12] for the background of JB<sup>∗</sup> -triples theory.

#### $2$  The manifold  $M$  of minimal projections

Let  $Z:={\cal L}(H)$ , where H is a complex Hilbert space, and let  $M\subset Tr(Z)$  denote the set of all projections in Z endowed with its topology as subspace of Z. Fix any non zero projection  $e_0 \in M$  and denote by M the connected component of  $e_0$  in M. Then all elements in M have the same rank as  $e_0$  and  $\text{Aut}^\circ(Z)$  acts transitively on M which is an  $\text{Aut}^\circ(Z)$ - invariant real analytic manifold whose tangent space at a point  $e \in M$  is

$$
T_e M = Z_{1/2}(e)_s,
$$

the selfadjoint part of the  $\frac{1}{2}$ -eigenspace of e. If we set  $k_u = 2(u\Box e - e\Box u)$ , then by [1, th. 3.3] a local chart of M in a suitable neighbourhood U of 0 in  $Z_{1/2}(e)$  is given by

$$
u \mapsto f(u) := \exp k_u(e).
$$

Let  $\mathfrak{D}(M)$  be the Lie algebra of all real analytic vector fields on M, and as in [1], define an affine connection  $\nabla$  on M by

$$
(\nabla_X Y)_e = P_{1/2}(e) Y'_e X_e, \qquad e \in M, \qquad X, Y \in \mathfrak{D}(M)
$$
 (1).

Then  $\nabla$  is a torsion-free Aut°(Z)-invariant affine connection on M. For each  $e \in M$  and  $u \in$  $Z_{1/2}(e)$  we let  $\gamma_{e,u}:\mathbb{R}\to M$  denote the curve  $\gamma_{e,u}(t):=\exp tk_u(e)$ . Clearly we have  $\gamma_{e,u}(0)=e$ and  $\dot{\gamma}_{e,u}(0) = u \in T_eM$ . By [1, th. 2.7],  $\gamma_{e,u}$  is a  $\nabla$ -geodesic of M. Let us introduce a binary product in Z by  $x \circ y = \{x \in y\}$ . Then  $(Z, \circ)$  is a complex Jordan algebra where, as usual,  $x^{(n)}$ 

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denotes the n-th power of x in  $(Z, \circ)$  for  $n \in \mathbb{N}$ . For  $u \in Z_{1/2}(e)$ , the real Jordan subalgebra of  $(Z, \circ)$  generated by the pair  $(e, u)$  is denoted by  $J[e, u]$  and we have  $\gamma_{e, u}(\mathbb{R}) \subset J[e, u]$ .

To make a more detailed study of the manifold  $M$ , we shall assume that  $e_0$  is *minimal*. In such a case  $J[e, u]$  coincides with the closed real linear span of the set  $\{e, u, u^{(2)}\}$ , in particular dim  $J[e, u] \leq 3$  and

$$
\gamma_{e,u}(t) = (\cos^2 t \theta) e + \left(\frac{1}{2\theta} \sin 2t \theta\right) u + \left(\frac{1}{\theta^2} \sin^2 t \theta\right) u^{(2)}, \qquad t \in \mathbb{R}
$$
 (2)

for some angle  $0 \le \theta < \frac{\pi}{2}$ . If a, b are two distinct minimal projections and they are not orthogonal (that is, if the Peirce projection  $P_1(a)b$  is invertible in the JB<sup>\*</sup>-algebra  $Z_1(a)$ ) then there is an unique geodesic  $\gamma_{a,u}(t)$  joining a with b in M. Moreover, due to the minimality of e the tangent space  $Z_{1/2}(e) \approx \{e\}^{\perp}$  appears naturally endowed with the Levi form  $\langle \cdot, \rangle_e$  and it turns out that the Levi norm  $|\cdot|_e$  and the operator norm  $\|\cdot\|$  are equivalent in  $Z_{1/2}(e)$  (see [6, th.5.1]). Thus  $(Z_{1/2}(e), \|\cdot\|_e)$  is a Hilbert space and an Aut<sup>°</sup>-invariant Riemann structure can be defined in M by

$$
g_e(X, Y) := \langle X_e, Y_e \rangle_e, \qquad X, Y \in \mathfrak{D}(M)
$$
\n(3)

where  $V_e \in Z_{1/2}(e)$  denotes the value taken by the vector field V at the point  $e \in M$ . By [1] g satisfies

$$
Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \qquad X, Y, Z \in \mathfrak{D}(M)
$$
 (4)

Therefore  $\nabla$  is the only Levi-Civita affine connection on M, and the geodesics are minimizing curves for the Riemann distance in  $M$ , which is given by the formula

$$
d(a, b) = \cos^{-1} (||P_1(a)b||^{\frac{1}{2}}) = \theta.
$$

M is symmetric Riemann manifold on which  $\text{Aut}^{\circ}(Z)$  acts transitively as a group of isometries and there is a real analytic diffeomorphism of M onto the projective space  $\mathbb{P}(H)$  over H, endowed with the Fubini-Study metric. We refer to [1,5,6,13] for proofs and background about these facts.

## 3 THE MANIFOLD OF FINITE RANK PROJECTIONS IN  $\mathcal{L}(H)$ .

In what follows we let M and  $M_r$  be the set of all projections in Z and the set of all projections that have a fixed finite rank r, respectively. If  $a \in M_r$  then a *frame* for a is any family  $(a_1, \dots, a_r)$ of pairwise orthognal minimal projections in Z such that  $a = \sum a_k$ . Note that then the  $a_k$  have the form  $a_k = (\cdot, \alpha_k) \alpha_k$  where  $(\alpha_k)$  is an orthonormal family of vectors in the range  $a(H)$ .

## **3.1 Proposition.** For every projection  $a \in M$  the following conditions are equivalent:

- (1) *The rank of* a *is finite.*
- (2) *The Banach space*  $Z_{1/2}(a)$  *is linearly homeomorphic to a Hilbert space.*

*Proof.* Let us choose an orthonormal basis  $(\alpha_i)_{i \in I}$  in the range  $a(H) \subset H$  of a. Then  $a_i =$  $(\cdot, \alpha_i)\alpha_i, i \in I$ , is a family of pairwise orthogonal minimal projections that satisfy

$$
a = \sum_{i \in I} a_i \qquad \text{strong operator convergence in } Z \tag{5}
$$

The space  $Z_{1/2}(a)_s$  consists of the operators  $u \in Z$  such that  $2\{aau\} = u$  and using (5) it is easy to check that u can be represented in the form

 $u = \sum_{i \in I} (\cdot, \xi_i) \alpha_i + (\cdot, \alpha_i) \xi_i$  strong operator convergence in Z

where  $\xi_i = u(\alpha_i)$  are vectors in H that satisfy  $\xi_i \in a(H)^{\perp}$ . By (4) each  $u \in Z_{1/2}(a)_s$  is determined by the family  $(\xi_i)_{i \in I}$ . To simplify the notation, set  $K := a(H)^{\perp}$  and  $L := \ell_{\infty}(I, K)$  for the Banach space of the families  $(\xi_i)_{i\in I} \subset K$  with the norm of the supremun  $\|(\xi_i)\| := \sup_{i\in I} \|\xi_i\|$ . Then the mapping

$$
L \to Z_{1/2}(a)_s, \qquad (\xi_i) \mapsto u_{\xi} := \Sigma_{i \in I} [(\cdot, \alpha_i) \xi_i + (\cdot, \xi_i) \alpha_i]
$$

is a continuous *real linear* vector space isomorphism, hence a homeomorphism . Thus if the operator norm in  $Z_{1/2}(a)$  is equivalent to a Hilbert space norm the same must occur with  $\ell_{\infty}(I, K)$ , hence I must be a finite set which means that  $a = \Sigma a_i$  has finite rank. The converse is easy.  $\square$ 

**3.2 Lemma.** Let  $a, b \in M_r$  with  $a = \sum a_k$  where the  $(a_k)$  is a frame for a, and let  $Q(a_k)b =$  $\lambda_k a_k$ ,  $(k = 1, \dots, r)$ *. If*  $P_1(a)b$  *is invertible in the JB<sup>\*</sup>-algebtra*  $Z_1(a)$ *, then*  $\lambda_k \neq 0$  *for all* k*. The set of all elements*  $b \in M_r$  *for which*  $P_1(a)b$  *is invertible in*  $Z_1(a)$  *is dense in*  $M_r$ *.* 

*Proof.* Suppose that  $a_k = (\cdot, \alpha_k) \alpha_k$  and  $b_j = (\cdot, \beta_j) \beta_j$  are frames for a and b respectively. Then for each fixed  $k$  we have

$$
Q(a_k)b = \{a_kba_k\} = (\Sigma_j | (\alpha_k, \beta_j)|^2)a_k = \lambda_k a_k
$$

where  $\lambda_k \geq 0$ . Moreover  $\lambda_k = 0$  if and only if  $\alpha_k \in {\beta_1, \dots, \beta_r}^{\perp}$  which is equivalent to  $a_k \perp b$ . But in such a case range $(a_k) \subset \text{ker}\{a_kba_k\} = \text{ker}P_1(a)b$  which contradicts the invertibility of  $P_1(a)b$ . To simplify the notation set  $K: = a(H) \subset H$  and note that  $\dim K = \operatorname{rank} a = r < \infty$ The operators in  $Z_1(a) = aZa$  can be viewed as operators in  $\mathcal{L}(K)$ , therefore the *determinant* function is defined in  $Z_1(a)$  and an element  $z \in Z_1(a)$  is invertible if and only if  $\det(z) \neq 0$ . Thus the set of the operators  $b \in Z$  for which  $P_1(a)b$  is invertible in  $Z_1(a)$  is an open dense subset of  $M_r$ .  $\Box$ 

**3.3 Lemma.** If a, p and q are projections in  $M_r$  and  $P_{1/2}(a)p = P_{1/2}(a)q$ , then  $p = q$ .

*Proof.* Take frames for a, p, q, compute  $P_{1/2}(a)p = 2(D(a\Box a) - Q(a)^2)p$  and proceed similarly with q. An elementary exercise of linear algebra yields range (p)=range (q), hence  $p = q$ .  $\Box$ 

Let  $a \in M_r$  and choose any frame  $(a_1, a_2, \dots, a_r)$  for a. As above  $Z_{1/2}(a)_s$  consists of the operators  $u = \Sigma(\cdot, \xi_k) \alpha_k + (\cdot, \alpha_k) \xi_k$  where  $\xi_k = u(\alpha_k)$  are vectors in H that satisfy  $\xi_k \in a(H)^{\perp}$ . Write  $u_k:=(\cdot,\xi_k)\alpha_k+(\cdot,\alpha_k)\xi_k$ . Then we have  $u=\Sigma u_k$  where the  $u_k$  are selfadjoint operators in  $Z = \mathcal{L}(H)$  (in fact  $u_k \in Z_{1/2}(a_k)_s$ ) that satisfy

$$
u_j \Box a_k = a_k \Box u_j = 0, \t j \neq k, \t (j, k = 1, 2, \cdots, r)
$$
 (6).

The above properties of the  $a_k$ ,  $u_k$  hold whatever is the frame  $(a_1, a_2, \dots, a_r)$ . There are many families in those conditions and we are going to prove that, by making an appropriate choice of the  $a_k$  (a choice in which the tangent vector  $u \in Z_{1/2}(a)$  is also involved) we can additionally have

$$
u_k \Box u_j = u_j \Box u_k = 0, \qquad j \neq k, \qquad (j, k = 1, 2, \cdots, r)
$$
\n
$$
(7)
$$

This will simplify considerably the calculations in the sequel. We need some material.

**3.4 Lemma.** With the above notation the set of minimal tripotents in  $Z_{1/2}(a)$  is

$$
\{ (\cdot, \alpha)\xi + (\cdot, \xi)\alpha : \alpha \in a(H), \ \xi \in a(H)^{\perp}, \ ||\alpha|| = 1 = ||\xi|| \}
$$

*Proof.* Let  $x \in Z$  be of the form  $x = (\cdot, \alpha)\xi + (\cdot, \xi)\alpha$  where  $\alpha, \xi \in H$  satisfy the above conditions. It is a matter of routine calculation to see that then  $2{aax} = x$  hence  $x \in Z_{1/2}(a)_s$ . Moreover  $\{xxx\} = x$  so that x is a tripotent and we can easily see that  $\{xZ_{1/2}(a)x\} \subset \mathbb{C}x$  which proves the minimality of x in  $Z_{1/2}(a)$ . The converse is similar.  $\square$ 

The following result should be compared to [14, prop. 3.4]

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**3.5 Lemma.** *Two minimal tripotents*  $x = (\cdot, \alpha)\xi + (\cdot, \xi)\alpha$  *and*  $y = (\cdot, \beta)\eta + (\cdot, \eta)\beta$  *in*  $Z_{1/2}(a)_s$ *are orthogonal if and only if*  $\alpha \perp \beta$  *and*  $\xi \perp \eta$ *. In particular*  $Z_{1/2}(a)$  *has rank* r *for all*  $a \in M$ 

*Proof.* By [2, p. 18] x and y are orthogonal if and only if the conditions  $xy^* = 0 = y^*x$  hold. Now it is elementary to complete the proof of the first statement. For the second part, let  $(u_i)_{i\in I}$ be a family of pairwise minimal orthogonal tripotents in  $Z_{1/2}(a)$ . Then  $u_i = (\cdot, \alpha_i)\xi_i + (\cdot, \xi_i)\alpha_i$ where  $(\alpha_i) \subset a(H)$  and  $\xi_i \subset a(H)^{\perp}$  are orthonormal families of vectors in H. In particular  $a_i = (\cdot, \alpha_i)\alpha_i$  is a family of pairwise orthogonal projections with  $\Sigma a_i \leq a$ . Since rank(a)=r, we have cardinal  $(I) \leq r$ . The converse is easy.  $\Box$ 

Let  $a \in M$  be a fixed projection and take any tangent vector  $u \in Z_{1/2}(a)_s$  to M at a. By lemma 3.2  $Z_{1/2}(a)$  has finite rank, hence [9, cor 4.5] u has a spectral decomposition in the JB<sup>\*</sup>-triple  $Z_{1/2}(a)$  of the form

 $u = \rho_1 u_1 + \dots + \rho_s u_s,$   $0 \le \rho_1 \le \dots \le \rho_s = ||u||,$   $1 \le s \le r$  (8)

where the  $u_k$  are pairwise *orthogonal minimal tripotents* in  $Z_{1/2}(a)$ . Therefore

$$
u_k = (\cdot, \alpha_k)\xi_k + (\cdot, \xi_k)\alpha_k, \ \alpha_k \in a(H), \ \xi_k \in a(H)^{\perp},
$$
  

$$
\|\alpha_k\| = 1 = \|\xi_k\|, \ \alpha_j \perp \alpha_k, \ \xi_j \perp \xi_k, \ j \neq k
$$

Then  $a_k:=(\cdot,\alpha_k)\alpha_k$  are pairwise orthogonal minimal projections in Z and  $\Sigma a_k \leq a$ . In case  $s < r$ , which occurs if some of the  $\rho_k = 0$ , we pick additional minimal orthogonal projections  $a_{s+1}, \dots, a_r$  so as to have  $a = \sum a_k$ . For the family  $(a_1, \dots, a_r)$  so constructed, called a *frame* associated to the pair  $(a, u)$ , both properties  $(6)$  and  $(7)$  hold. Remark that this frame needs not be unique, it depends on a and on u as well, and it is invariant under the group  $Aut^{\circ}(Z)$ . In fact some more properties are valid now.

In accordance with section §1, each pair  $(a_k, u_k)$  gives rise to a real Jordan algebra  $J_k$ :  $J[a_k, u_k]$  with the product  $x \circ_k y = \{xa_k y\}$ . We have  $\dim(J_k) = 3$  and  $\{a_k, u_k, u_k^{(2)}\}$  $\binom{2}{k}$  is a basis of  $J_k$ . Moreover,  $J_k$  is invariant under the operator  $g_k := 2(a_k \Box u_k - u_k \Box a_k)$  where triple products are computed in  $Z = \mathcal{L}(H)$ . In case  $s < \text{rank}(a)$  we set  $J_n := \mathbb{R}a_n$  as real Jordan algebras.

**3.6 Lemma.** *The Jordan algebras*  $J_k$  *and*  $J_l$  *with*  $k \neq l$ ,  $(k, l = 1, \dots, r)$  *are orthogonal in the*  $JB^*$ -triple sense in Z, that is  $\{J_kJ_lZ\} = 0$ .

*Proof.* For  $n \in \{k, l\} \subset \{1, \dots, s\}$  with  $k \neq l$ , let  $z_n$  be any element in the basis  $\{a_n, u_n, u_n^{(2)}\}$ of  $J_n$ . Clearly it suffices to show that  $z_k z_l = 0 = z_l z_k$ . As an example, we shall prove that  $u_k^{(2)}$  $\binom{2}{k}u_l^{(2)}=0.$  It is a routine to check that  $u_ku_l=0.$  Then

$$
u_k^{(2)}u_l^{(2)} = \{u_k a_k u_k\} \{u_l a_l u_l\} = (u_k a_k u_k) (u_l a_l u_l) = u_k a_k (u_k u_l) a_l u_l = 0
$$

as we wanted to see.  $\Box$ 

Consider now the vector space direct sum  $J := \bigoplus_{i=1}^{r} J_k$ , and define a product  $z \circ w := \{zaw\}$ in  $J$  by

$$
z \circ w := \{zaw\} = \frac{1}{2}(zaw + waz) = \frac{1}{2}\sum_{1}^{r}(z_ka_kw_k + w_ka_kz_k) = \sum_{1}^{r}z_k \circ_k w_k
$$

where  $z_k$ ,  $w_k$  are respectively the  $J_k$ -component of z and w. It is now clear that J is a real Jordan algebra, that the product in J induces in each  $J_k$  its own product  $z \circ_k w = \{za_kw\}$  and that the  $J_k$  are orthogonal as Jordan subalgebras of J. It is also clear that J coincides with the closed real linear span of the set  $\bigcup_{1}^{r} \{a_k, u_k, u_k^{(2)}\}$  $\{e^{(2)}\}\$ , in particular dim  $J \leq 3r < \infty$ . Finally  $J[a, u] \subseteq J$  and we conjecture that the equality holds (see [14, prop. 3.5 & th. 3.6].

#### 4 Geodesics and the exponential mapping.

Consider  $M_r$  endowed with the affine connection  $\nabla$  given by (1). To discuss its geodesics, let us define an operator  $g \in Z = \mathcal{L}(H)$  by

$$
g\mathpunct{:}=g_{a,u}\mathpunct{:}=2(u\Box a-a\Box u)=2\Sigma\rho_k(u_k\Box a_k-a_k\Box u_k)=\Sigma\rho_k g_{a_k,u_k}
$$

where  $u = \sum \rho_k u_k$  is the spectral decomposition of  $u \in Z_{1/2}(a)$ , the  $a_k$  is any frame associated to the pair  $(a, u)$  and  $g_k := g_{a_k, u_k}$  is defined in a obvious manner. If the spectral decomposition of u (see (8)) has  $s < r$  non zero summands then we define  $g_n := 0$  for  $n = s + 1, \dots, r$ . Then  $g_k$  is a commutative family of operators in Z, more precisely we have  $g_k(J_l) = \{0\}$ ,  $g_k g_l = g_l g_k = 0$ for all  $k \neq l$ ,  $(k, l = 1, \dots, r)$  and g leaves invariant all the spaces J and  $J_k$ . Thus

$$
\gamma_{a,u}(t) := \exp t g(a) = \Sigma \exp t g_k(a_k), \qquad t \in \mathbb{R}
$$

By section §1 this curve is a geodesic in  $M_r$  and  $\gamma_{a,u}(\mathbb{R}) \subset J[a,u] \subset J$ . We can collect now the above discussion in the following statement (see [14, prop. 5.1  $\&$  5.4]

**4.1 Theorem.** Suppose that we are given a point  $a \in M_r$  and a tangent vector  $u \in Z_{1/2}(a)$  to M<sup>r</sup> *at* a*. Then the geodesic of* M<sup>r</sup> *that passes through* a *with velocity* u *is the curve*

$$
\gamma_{a,u}(t) = \Sigma \gamma_{a_k, u_k}(t), \qquad t \in \mathbb{R},
$$

*where*  $\gamma_k := \gamma_{a_k, u_k}$  *is given by* 

$$
\gamma_k(t) := \gamma_{a_k, u_k}(t) = (\cos^2 \theta_k t) \ a_k \ + (\frac{1}{2\theta_k} \sin 2\theta_k t) \ u_k + (\frac{1}{\theta_k^2} \sin^2 \theta_k t) \ u_k^{(2)} \tag{G}
$$

*Here*  $u = \sum \rho_k u_k$  *is the spectral decomposition of* u *in*  $Z_{1/2}(a)$ *, the*  $a_k$  *form* a *frame* associated *to the pair*  $(a, u)$  *and the numbers*  $\theta_k$  *are given by*  $\cos^2 \theta_k := \rho_k$  *with*  $0 \le \theta_k < \frac{\pi}{2}$  $\frac{\pi}{2}$ .

Now we are in a position to define the exponential mapping. Suppose the tangent vector  $u$ lies in the unit ball  $B_1(a) \subset Z_{1/2}(a)$ , i.e.  $||u|| < 1$ . For  $t = 1$  the expression (G) yields

$$
\gamma(1) = \Sigma(\cos^2 \theta_k) a_k + \Sigma(\frac{1}{2\theta_k} \sin 2\theta_k) u_k + \Sigma(\frac{1}{\theta_k^2} \sin^2 \theta_k) u_k^{(2)}
$$
(E)

and a real analytic mapping form the unit ball  $B_1(0) \subset Z_{1/2}(a)$  to the manifold M can be defined by

$$
Exp_a(u) := \gamma_{a,u}(1)
$$

An inspection of (E) yields that the Peirce decomposition of  $\gamma_{a,u}(1)$  relative to a is

$$
P_1(a)\gamma_{a,u}(1) = \Sigma(\cos^2\theta_k)a_k, \qquad P_{1/2}(a)\gamma_{a,u}(1) = \Sigma(\frac{1}{2\theta_k}\sin^2\theta_k) u_k
$$

$$
P_0(a)\gamma_{a,u}(1) = \Sigma(\frac{1}{\theta_k^2}\sin^2\theta_k) u_k^{(2)}
$$

Remark that  $0 < \cos^2 \theta_k \le 1$ , hence in particular  $P_1(a)\gamma_{a,u}(1)$  lies in the set of all  $\mathcal{N}_a$  of all invertible elements in the JB<sup>\*</sup>-algebra  $Z_1(a)$ . Clearly  $\mathcal{N}_a$  is an open neighbourhood of a in  $Z_1(a)$ . Remark also that  $0 \leq \frac{1}{2\theta}$  $\frac{1}{2\theta_k} \sin^2 \theta_k = \rho_k \le ||u|| < 1$ , hence  $\Sigma(\frac{1}{2\theta_k} \sin^2 \theta_k) u_k$  is the spectral decomposition of  $P_{1/2}(a)\gamma_{a,u}(1)$  in  $Z_{1/2}(a)$ . Thus  $Exp_aB_1(a) \subset \mathcal{N}_a \subset M$ . We refer to  $Exp_a$  as the *exponential* mapping.

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#### 5 Geodesics connecting two given points. The logaritm mapping.

Now we discuss the possibility of joining two given projections a and b such that  $P_1(a)b$  is invertible in the Jordan algebra  $Z_1(a)$ , by means of a geodesic in M. The remarks in the precedent section show how to proceed. First we compute the spectral decomposition of  $u = P_{1/2}(a)b$  in the JB<sup>\*</sup>-triple  $Z_{1/2}(a)$ . Assume it to be

$$
u = P_{1/2}(a)b = \sum \rho_k u_k, \qquad 0 \le \rho_1 \le \cdots \le \rho_r = ||u|| < 1, \qquad 1 \le k \le r
$$

where the  $u_k$  are pairwise orthogonal minimal tripotents in  $Z_{1/2}(a)$ . Hence By lemma 3.4 the  $u_k$ have the form  $u_k = (\cdot, \alpha_k)\xi_k+(\cdot, \xi_k)\alpha_k$  for some orthonormal families of vectors  $(\alpha_k) \subset a(H)$  and  $(\xi_k) \subset a(H)^\perp$ . By lemma 3.2  $Q(a_k)b = \{a_kba_k\} = \lambda_k$  where  $\lambda_k \neq 0$  since  $P_1(a)b$  is invertible in  $Z_1(a)$ . Also  $|\lambda_k| = ||a_kba_k|| \le 1$ . Thus  $0 < \lambda_k \le 1$  and a unique angle  $0 \le \theta_k < \frac{\pi}{2}$  $rac{\pi}{2}$  is determined by  $\cos^2 \theta_k = \lambda_k$ . In this way we have got all the elements appearing in (E). Let us define  $\tilde{\gamma}(t) := \Sigma \tilde{\gamma}_k(t)$  for  $t \in \mathbb{R}$  where

$$
\tilde{\gamma}_k(t) := (\cos^2 t \theta_k) a_k + \left(\frac{1}{2\theta_k} \sin 2t \theta_k\right) u_k + \left(\frac{1}{\theta_k^2} \sin^2 t \theta_k\right) u_k^{(2)}
$$

By section §1, each  $\tilde{\gamma}_k(t)$  is a geodesic in the manifold  $M_1$  of all rank 1 projections. By the previous discussion  $\tilde{\gamma}_i(t)$  and  $\tilde{\gamma}_k(t)$  are orthogonal whenever  $j \neq k$ ,  $t \in \mathbb{R}$ , hence  $\tilde{\gamma}(t) := \sum \tilde{\gamma}_k(t)$ ,  $t \in \mathbb{R}$ , is a curve in the manifold M of projections of rank r. Clearly  $\tilde{\gamma}(0) = \Sigma \tilde{\gamma}_k(0) = \Sigma a_k = a$  and we shall now show that  $\tilde{b} = \gamma(1)$  coincides with b. As above  $P_{1/2}(a) = \tilde{b} = \sum_{k=1}^{\infty} \frac{1}{2} \sin 2\theta_k u_k = \sum_{k=1}^{\infty} \rho_k u_k$ is the spectral decomposition of  $P_{1/2}(a)\tilde{b}$  in  $Z_{1/2}(a)$ , which by construction is the spectral decomposition of  $P_{1/2}(a)b$ . Hence by lemma 3.3,  $b = \tilde{\gamma}(1) = b$ . This gives a geodesic  $\gamma(t)$  that connects a with b in the manifold  $M_r$  and passes through the point a with the velocity  $u = P_{1/2}(a)b$ . It is uniquely determined by the data a, b and the property  $\gamma_{a,u}(1) = b$ .

Now we are in a position to define the logaritm mapping. Fix a point  $a \in M$  and let  $\mathcal{N}_a \subset M$ be the set of all projections  $b \in M$  such that  $P_1(a)b$  is invertible in the JB<sup>\*</sup>-algebra  $Z_1(a)$ . Define a mapping  $\text{Log}_a$  from  $\mathcal{N}_a \subset M$  to the unit ball  $B_1(a) \subset Z_{1/2}(a)$  by declaring  $\text{Log}_a(b)$  to be the velocity at  $t = 0$  of the unique geodesic  $\gamma_{a,u}(t)$  that joins a with b in M and  $\gamma_{a,u}(1) = b$ , in other words  $\text{Log}_a(b) := P_{1/2}(a)b$ . We refer to  $\text{Log}_a$  as the *logaritm* mapping. Clearly  $\text{Log}_a$  and  $\text{Exp}_a$ are real analytic inverse mappings. In particular, the family  $\{(\mathcal{N}_a, \text{Log}_a) : a \in M\}$  is an atlas of M. We remark the fact that  $\gamma_{a,u}[0,1] \subset \mathcal{N}_a$  for all  $u \in B_1(a)$  which shall be needed later on to apply the Gauss lemma [11, 1.9] and summarize the above discussion in the statement (see [14, th. 5.7 & prop. 5.8])

**5.1 Theorem.** Let a and b be two given projections in  $M_r$  and assume that  $P_1(a)$  is invertible *in the Jordan algebra*  $Z_1(a)$ *. Then there is exactly one geodesic*  $\gamma_{a,u}(t)$  *that joins* a *with* b *in* M *and*  $\gamma_{a,u}(1) = b$ *.* 

#### 6 The Riemann structure on M.

Let  $a \in M_r$  and choose any frame  $(a_k)$  for a. By section §1 we have vector space direct sum decomposition

$$
Z_{1/2}(a) = \bigoplus_{1}^{r} Z_{1/2}(a_k)
$$
\n(9)

which suggests to define a scalar product in  $Z_{1/2}(a)$  by

$$
\langle u, v \rangle := \frac{1}{\sqrt{r}} \Sigma \langle u_k, v_k \rangle_{a_k} \tag{10}
$$

where  $\langle \cdot, \cdot \rangle_{a_k}$  stands for the Levi form on  $Z_{1/2}(a_k)$ . First we prove

**6.1 Lemma.** With the above notation, (9) defines an Aut<sup>°</sup>-invariant scalar product on  $Z_{1/2}(a)$ *that does not depend of the frame*  $a = \Sigma_k$  *and converts*  $Z_{1/2}(a)$  *into a Hilbert space.* 

*Proof.* Let  $\Sigma a_k$  and  $\Sigma a'_k$  denote two frames for a where  $a_k = (\cdot, \alpha_k) \alpha_k$  and  $a'_k = (\cdot, \alpha'_k) \alpha'_k$  for some orthonormal families  $(\alpha_k)$ ,  $(\alpha'_k) \subset a(H)$ . Extend them to two orthonormal basis of H and let  $u \in \mathcal{L}(H)$  be the unitary operator that exchanges these bases. Then u induces an isometry  $U \in \text{Aut}^{\circ}(Z)$  by  $Uz = uzu^{-1}$  that satisfies  $Ua'_{k} = a_{k}$ . The invariance of the Levi form together with (10) yields part of the result. The remainder is trivial.  $\Box$ 

A Riemann structure can now be defined in  $M_r$  in the following way. Let  $X, Y \in \mathfrak{D}(M)$ vector fields on  $M_r$ , and for  $a \in M_r$  take any frame  $a = \Sigma a_k$ . Then (9) gives representation  $X = \Sigma X_k$ ,  $Y = \Sigma Y_k$  with  $X_k$ ,  $Y_k \in Z_{1/2}(a_k)$  and we set

$$
g_a(X, Y) := \langle X, Y \rangle = \frac{1}{\sqrt{r}} \Sigma \langle X_k, Y_k \rangle_{a_k} = \frac{1}{\sqrt{r}} \Sigma g_{a_k}(X_k, Y_k)
$$

This is a well defined Aut<sup>°</sup>-invariant Riemann structure on  $M_r$ . By section §1 each  $g_{a_k}$  has property  $(4)$  and a routine argument gives the same property for g. Thus g is the only Levi-Civita connection in  $M_r$  and we can apply the Gauss lemma [11, 1.9] to conclude that the ∇-geodesics are minimizing curves for the Riemann distance.

Recall that for a tripotent  $a \in Z$ , the mapping  $\sigma_a: x_1 + x_{1/2} + x_0 \mapsto x_1 - x_{1/2} + x_0$ , where  $x \in Z$  and  $x_1 + x_{1/2} + x_0$  is the Peirce decomposition of x with respect to a, called the Peirce symmetry of  $Z$  with center  $a$ , is an involutory automorphism of  $Z$  that induces an isometric symmetry of  $M_r$  (see [6, th. 5.1]). We let Isom $M_r$  and  $\mathfrak S$  denote the group of all isometries of the Riemann manifold  $M_r$  and the subgroup generated by the set  $S = {\sigma_a : a \in M_r}$ , respectively.

6.2 Proposition. *With the above notation,* M<sup>r</sup> *is a symmetric Riemann manifold in which the group* S *acts transitively.*

*Proof.* Let  $a, b \in M_r$  be such that  $b \in \mathcal{N}_a$ . Then a and b can be joined in  $M_r$  by a unique geodesic with  $\gamma(0) = a, \gamma(1) = b$ . If  $c = \gamma(\frac{1}{2})$  $\frac{1}{2}$ ), then  $\sigma_c$  is a symmetry of  $M_r$  such that  $\sigma_c(a) = b$ . Thus the set S is transitive in  $\mathcal{N}_a$  and S is locally transitive in  $M_r$ . Consider now the case  $b \notin \mathcal{N}_a$ . Since  $M_r$  is pathwise connected, we can join a with b by a curve  $\Gamma$  in  $M_r$  and by a standard compactness argument there exists a finite set  $\{b_0, \dots, b_s\} \subset \Gamma[0,1]$  such that  $b_0 = a, b_s = b$  and  $b_{k+1} \in \mathcal{N}_{b_k}$  for  $k = 1, \dots, s$ . An application of the above argument to each pair of consecutive points gives the result.  $\square$ 

We now compute the Riemann distance in  $M_r$ . Consider first the case of two points  $a, b \in M_r$ with  $b \in \mathcal{N}_a$ . Let  $\gamma_{a,u}(t)$  be the unique geodesic that joins a with b in  $M_r$  and satisfies  $b = \gamma_{a,u}(1)$ . Since  $\text{Aut}^{\circ}(Z)$  is transitive in  $\mathcal{N}_a$  and the Levi norm is  $\text{Aut}^{\circ}(Z)$ -invariant, we have

$$
|\dot{\gamma}_{au}(t)|_{\gamma_{au}(t)} = |\dot{\gamma}_{au}(0)|_{\gamma_{au}(0)} = |u|_a
$$

On the other hands, since the Levi norm in  $Z_{1/2}(a)$  is the direct hilbertian sum of the Levi norms in the  $Z_{1/2}(a_k)$ , we have by section §1

$$
|u|_a^2 = \frac{1}{r} \Sigma |u_k|_{a_k}^2 = \frac{1}{r} \Sigma \theta_k^2
$$
 (D)

where  $u = \sum \rho_k u_k$  is the spectral decomposition of u in  $Z_{1/2}(a)$ ,  $(a_k)$  is the frame associated to the pair  $(a, u)$  and  $\cos^2 \theta_k = \rho_k$ . Therefore

$$
d(a,b) = \int_0^1 |\dot{\gamma}_{au}(t)|_{\gamma_{au}(t)} dt = \int_0^1 |u|_a dt = |u|_a = \frac{1}{\sqrt{r}} (\Sigma \theta_k^2)^{1/2}
$$

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Consider now the case  $b \notin \mathcal{N}_a$ . By lemma 3.2 we can take a sequence  $(b_n)_{n\in\mathbb{N}}$  in  $\mathcal{N}_a$  such that  $b = \lim_{n \to \infty} b_n$  since (D) holds for all  $b_n$  and the Riemann distance is continuous, we get the validity (D) for all  $a, b \in M_r$ .  $\Box$ 

Note that expression (D) is a generalization of the classical formula for the Fubini-Study metric in the projective space  $\mathbb{P}(H)$ .

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