# A generalization of Cayley submanifolds

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#### Abstract

Given a Kähler manifold of complex dimension 4, we consider submanifolds of (real) dimension 4, whose Kähler angles coincide. We call these submanifolds *Cayley*. We investigate some of their basic properties, and prove that (a) if the ambient manifold is a Calabi-Yau, the minimal Cayley submanifolds are just the Cayley submanifolds as defined by Harvey and Lawson  $[HL1]$ ; (b) if the ambient is a Kähler-Einstein manifold of non-zero scalar curvature, then minimal Cayley submanifolds have to be either complex or Lagrangian.

## 1 Introduction

Cayley submanifolds were defined by Harvey-Lawson[[HL1](#page-14-0)] and by McLean [[McL](#page-14-0)], as calibrated submanifolds of Spin(7)-manifolds. Each such manifold M admits a parallel calibration  $\Phi \in \Lambda^4(M)$  whose stabilizer is Spin(7). It is called the Cayley calibration, due to the link with the octonians, and the corresponding minimal varieties are called Cayley submanifolds.

If the ambient manifold is a Calabi-Yau manifold, the Cayley calibration is not unique. Indeed, given any parallel normalized<sup>1</sup> complex volume form  $\Omega \in \wedge^{4,0}(M)$ , the form

$$
\Phi_\Omega:=\mathrm{Re}\,\Omega+\frac{\omega^2}{2}
$$

is a calibration whose stabilizer is isomorphic to Spin(7). Therefore there is an S<sup>1</sup> -family of Cayley calibrations. A submanifold calibrated by any of

$$
\frac{\omega^4}{4!} = \text{vol}_g = \left(\frac{\sqrt{-1}}{2}\right)^4 \Omega \wedge \overline{\Omega}.
$$

This amounts to requiring that  $||\Omega||^* = 1$ , where  $|| \cdot ||^*$  denotes the comass norm.

<sup>1</sup> I.e. such that

<span id="page-1-0"></span>these forms, has the property that its Kähler angles coincide. It is therefore natural to consider the submanifolds defined by the latter condition, without any assumption relating to the calibrating forms. This makes sense whether or not the ambient manifold is Ricci-flat, and gives rise to an interesting family of not necessarily minimal submanifolds, including both the Lagrangian and the complex submanifolds as extreme cases.

In this paper we start collecting some facts from linear algebra, making precise the relation between the Kähler angles on one side, and the Cayley calibrations on the other. Then we define the submanifolds with equalKähler angles, which we call  $Cayley$ , and prove a formula ([26\)](#page-11-0) relating the angle, the mean curvature and the Ricci form of the ambient manifold. Finally we apply this formula to the case where the ambient manifold is Kähler-Einstein, obtaining the following two results:

#### Theorem 1

Let  $(M, J, g)$  be a Calabi-Yau manifold. Then a Cayley submanifold of M, is minimal iff it is calibrated by some parallel Cayley calibration.

#### Theorem 2

Let  $(M, J, g)$  be a Kähler-Einstein manifold of non-zero scalar curvature. Then any (connected) minimal Cayley submanifold of M is either complex or (minimal) Lagrangian.

The first result shows the relation with the theory of Harvey and Lawson. The last result has been obtained, independently and very recently, also by Isabel Salavessa and Giorgio Valli[[SV](#page-14-0)], by quite different methods.

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# 2 Linear algebra of real 4-planes in  $\mathbb{C}^4$

Let  $(V, J, g)$  be a Hermitian vector space of real dimension 8. Denote by  $\omega(X, Y) = q(JX, Y)$  the associated Kähler form. Given a subspace  $W \subset V$ we denote by  $\pi_W$  the orthogonal projection onto W, and we put

$$
B_W := \pi_W \circ J_{|W}.
$$

 $B_W$  is a skew-hermitian operator on W with respect to g. We let  $G(p, V)$ denote the Grassmannian of *oriented p*-planes in *V*.

<span id="page-2-0"></span>Let us recall an important lemma proved by Harvey and Lawson[[HL2](#page-14-0)], applied to our situation.

### Lemma 1 (Canonical form of a 4-plane over  $U(4)$ )

Let  $(V, J, g)$  be a Hermitian vector space of real dimension 8. Then, given  $\xi \in G(4, V)$ , there is a unitary basis  $u_1, u_2, u_3, u_4$  of V and angles  $\theta_1, \theta_2$ , with

$$
0 \le \theta_1 \le \frac{\pi}{2}
$$
  
\n
$$
\theta_1 \le \theta_2 \le \pi
$$
\n(1)

such that

$$
\xi = u_1 \wedge (\cos \theta_1 J u_1 + \sin \theta_1 u_2) \wedge \wedge u_3 \wedge (\cos \theta_2 J u_3 + \sin \theta_2 u_4).
$$
\n(2)

Therefore

$$
B_{\xi} = \begin{pmatrix} 0 & -\cos \theta_1 & 0 \\ \cos \theta_1 & 0 & 0 \\ 0 & 0 & -\cos \theta_2 \\ 0 & \cos \theta_2 & 0 \end{pmatrix}
$$
 (3)

and

$$
\omega_{|\xi} = \cos \theta_1 e^{12} + \cos \theta_2 e^{34}.\tag{4}
$$

The numbers  $\theta_1$  and  $\theta_2$  are called the Kähler angles of the 4-plane  $\xi$ .

#### Definition 1

 $\xi \in \mathrm{G}(4, V)$  is called a Cayley 4-plane if

$$
\omega_{|\xi} = *_{\xi} = \omega_{|\xi}.\tag{5}
$$

Here  $*_\xi$  is the Hodge operator of the metric  $g_{|\xi}$ .

#### Lemma 2

An oriented 4-plane  $\xi \in G(4, V)$  is a Cayley subspace if and only if its Kähler angles coincide. In this case, putting  $\cos \theta_1 = \cos \theta_2 = \lambda \in [0, 1]$ , we have

$$
B_{\xi}^{2} = -\lambda^{2} \operatorname{Id} \tag{6}
$$

$$
(\omega^2)_{|\xi} = 2\lambda^2 \operatorname{vol}_{g_{|\xi}} \tag{7}
$$

and there is a a positive orthonormal basis  $e_1, e_2, e_3, e_4$  of  $\xi$  such that

$$
\omega_{|\xi} = \lambda (e^{12} + e^{34})\tag{8}
$$

$$
B_{\xi} = \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix}
$$
 (9)

P R O O F. Justapply  $*_\xi$  to ([4](#page-2-0)). Q.D.E.

A positive orthonormal basis  $\{e_1, ..., e_4\}$  in which (8) hold is called a Cayley basis. If we put

$$
\mathfrak{X}:=\{\xi\in\mathrm{G}(4,V):\xi\text{ is Cayley}\},
$$

then we have a well defined function

$$
\lambda: \mathfrak{X} \longrightarrow [0,1].
$$

#### Lemma 3

- (a)  $\mathfrak X$  is a closed subset of G(4, V) and  $\lambda$  is a continuous function.
- (b)  $\lambda^{-1}(1)$  is the Grassmannian of complex planes in  $(V, J)$ , while

$$
\mathfrak{X}_r:=\{\xi\in\mathfrak{X}:\lambda(\xi)<1\}
$$

consist of totally real subspaces.

(c)  $\lambda^{-1}(0)$  is the (oriented) Lagrangian grassmannian, while every  $\xi \in \mathfrak{X}$ with  $\lambda(\xi) > 0$  is a symplectic subspace of  $(V, \omega)$ .

P R O O F.

Let us consider the following subset of the Stiefel manifold of orthonormal quadruples of vectors in  $V$ :

$$
\mathcal{Y} = \{ (e_1, e_2, e_3, e_4) : \omega(e_1, e_2) = \omega(e_3, e_4) \omega(e_1, e_3) = \omega(e_1, e_4) = \omega(e_2, e_3) = \omega(e_2, e_4) = 0 \} (10)
$$

 $\mathcal Y$  is a closed subset, and the projection  $\pi : \mathcal Y \to \mathfrak X$  is onto, therefore it is an identification, i.e.  $\mathfrak X$  has the quotient topology. As  $\lambda \circ \pi(e_1, e_2, e_3, e_4) =$  <span id="page-4-0"></span> $\omega(e_1, e_2)$ ,  $\lambda \circ \pi$  is a continuos function on  $\mathcal{Y}$ , hence the same is true of  $\lambda$ . The remaining statements are trivial. Q.D.E.

### Lemma 4

Let  $\xi$  be a non-complex, hence totally real Cayley 4-plane. Given any Cayley basis  $\{e_i\}$  of  $\xi$ , we put

$$
u_1 = e_1 \qquad \qquad u_2 = \frac{1}{\sqrt{1 - \lambda^2}} (e_2 - \lambda J e_1)
$$
  

$$
u_3 = e_3 \qquad \qquad u_4 = \frac{1}{\sqrt{1 - \lambda^2}} (e_4 - \lambda J e_3).
$$
 (11)

Then  $\{u_j\}$  is a unitary basis of V and

$$
\xi = u_1 \wedge \left( \lambda(\xi) J u_1 + \sqrt{1 - \lambda^2(\xi)} u_2 \right) \wedge \wedge u_3 \wedge \left( \lambda(\xi) J u_3 + \sqrt{1 - \lambda^2(\xi)} u_4 \right).
$$
\n(12)

PROOF.

A straightforward computation shows that

$$
g(u_i, u_j) = \delta_{ij} \qquad \omega(u_i, u_j) = 0.
$$

(12) follows immediately from (11). Q.D.E.

### Lemma 5

(a) If  $\xi \in G(4, V)$  is totally real (i.e. if  $\cos \theta_1 \neq 0 \neq \cos \theta_2$ ) there exists a unique normalized (4,0)-form  $\Omega_\xi$  such that

$$
\Omega_{\xi}(\xi) > 0.
$$

If we write  $\xi$  in the form ([2](#page-2-0)), then  $\vec{u} = u_1 \wedge u_2 \wedge u_3 \wedge u_4$  satisfies  $\Omega_{\xi}(\vec{u}) = 1$ . In particular, two basis  $\{u_i\}$  such that ([2](#page-2-0)) hold differ by an element of SU(4).

(b) If ξ is Cayley and totally real, it is calibrated by the Cayley calibration associated to  $\Omega_{\xi}$ :

$$
\Phi_{\xi} = \text{Re}\,\Omega_{\xi} + \frac{\omega}{2} \qquad \Phi_{\xi}(\xi) = 1.
$$

(c) If  $\xi$  is calibrated by some Cayley calibration,  $\Phi_{\Omega}(\xi) = 1$ , then it is a Cayley subspace, and  $\Omega_{\xi} = \Omega$ .

PROOF. From the constraints [\(1](#page-2-0)) descends that

$$
\sin \theta_1 \sin \theta_2 \ge 0
$$
  

$$
\cos \theta_1 \ge 0.
$$

If  $\xi$  is totally real, then  $\sin \theta_i \neq 0$ , and  $\sin \theta_1 \sin \theta_2 > 0$ . If we let  $\Omega_{\xi}$  be the unique (4,0)-form such that  $\Omega_{\xi}(\vec{u}) = 1$ , then  $\Omega_{\xi}(\xi) = \sin \theta_1 \sin \theta_2 > 0$ . This shows  $\Omega_{\xi}$  only depends on  $\xi$  and proves (a). Using the representation [\(12\)](#page-4-0) we see that

$$
\Omega_{\xi}(\xi) = 1 - \lambda^2(\xi) \qquad \omega^2(\xi) = 2\lambda^2(\xi),
$$

thus proving (b). On the other hand, using the representation([2\)](#page-2-0) we see that

$$
\Omega_{\xi}(\xi) = \sin \theta_1 \sin \theta_2 \qquad \omega^2(\xi) = 2 \cos \theta_1 \cos \theta_2.
$$

Therefore, if  $\Omega = e^{\sqrt{-1} \alpha} \Omega_{\xi}$ ,

$$
\Phi_{\Omega}(\xi) = \text{Re}\left(e^{\sqrt{-1}\alpha}\Omega_{\xi}(\xi)\right) + \cos\theta_1\cos\theta_2 =
$$
  
=  $\cos\alpha\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2$ 

and this can be 1, only if  $\cos \alpha = 1$  and  $\theta_1 = \theta_2$ . Q.D.E.

## 3 Cayley submanifolds of Kähler manifolds

Let  $(M, J, g)$  be a Kähler manifold of complex dimension 4. We consider an oriented submanifold  $N \subset M$  of real dimension 4.

We let  $*_N$  denote the Hodge operator of the metric  $g_{|N}$ .

### Definition 2

We call  $N$  a Cayley submanifold if the equation

$$
\omega_{|N} = *_{N} \omega_{|N} \tag{13}
$$

is satisfied on N.

<span id="page-6-0"></span>This just means that for any point  $x$  of  $N$ , the oriented tangent space  $T_xN$  is a Cayley subspace of  $T_xM$ .

We stress that this definition does NOT agree with the one given by Harvey and Lawson, which makes senses on any Spin(7)-manifolds and implies that the submanifold is volume-minimizing. The above definition on the contrary makes sense on any Kähler manifold, and does not imply minimality. Just consider that any Lagrangian submanifold has equal (and zero) Kähler angles, and is therefore Cayley, according to the above definition.

The relation between this definition and the one of Harvey and Lawson, in the case where the ambient manifold is Calabi-Yau, is the subject of theorem [1.](#page-1-0)

As the tangent spaces to N are Cayley subspaces, if we denote by  $B_x$ the endomorphism  $\pi \circ J_{x|T_xN}$ , then  $B_x^2$  is a multiple of the identity at each point x of N. We can define a function  $\lambda = \lambda(x) \geq 0$ , such that

$$
B_x^2 = -\lambda^2(x) \operatorname{Id}.
$$

As  $\omega^2|_N = 2\lambda^2$  vol, we deduce that  $\lambda^2$  is a smooth function on N, with values in  $[0, 1]$ .

Given any 4-dimensional submanifold of  $M$ , not necessarily Cayley, we denote by  $N_r$  the totally real part of N, and by  $N_c$  the set of complex points. If N is Cayley, then  $N_r = \{x \in N : \lambda(x) < 1\}$  and  $N_c = \lambda^{-1}(1)$ . In particular  $N = N_r \sqcup N_c$ . Taking the square root of  $\lambda^2$  we deduce that  $\lambda$  is a continuos function on  $N$ , smooth on  $N_r$ , i.e. away from complex points.

On  $N_r$  is defined a section  $\Omega_N$  of  $\wedge^{4,0}(M)_{\vert N} = K_{M \vert N}$ , determined by the condition that  $\Omega_N$  be normalized and satisfy  $\Omega_N(T_xN) > 0$  at each point. This is seen applying lemma [5](#page-4-0).

#### Lemma 6

Given a Cayley submanifold N, near each non-complex point  $x$  of N, one can find a smooth Cayley frame  $\{e_1, e_2, e_3, e_4\}$ , and a smooth unitary frame  ${u_1, u_2, u_3, u_4}$  in  $TM_{|N}$  such that

$$
T_x N = u_1 \wedge \left(\lambda(x) J u_1 + \sqrt{1 - \lambda^2(x)} u_2\right) \wedge
$$
  
 
$$
\wedge u_3 \wedge \left(\lambda(x) J u_3 + \sqrt{1 - \lambda^2(x)} u_4\right).
$$
 (14)

In particular

 $\Omega_N(u_1, u_2, u_3, u_4) = 1$ ,

and therefore  $\Omega_N : N \longrightarrow K_{M|_N}$  is a smooth section.

<span id="page-7-0"></span>PROOF. For  $x \in N_r$ , let us consider the endomorphism

$$
j_x = \frac{B_x}{\lambda(x)}
$$

of  $T_xN$ . It is a  $g_{|N}$ -orthogonal almost complex structure, compatible with the orientation of  $N$ , smooth on all of  $N_r$ . Therefore we know that near any  $x \in N_r$  we can find a smooth j-unitary frame  $\{e_1, e_3\}$  in TN, i.e. a positive orthonormal frame in TN of the form  $\{e_1, je_1, e_3, je_3\}$ . Putting  $e_2 = je_1$ ,  $e_4 = je_3$  $e_4 = je_3$  $e_4 = je_3$  we obtain the Cayley frame. Using the formulas ([11\)](#page-4-0) to define  $\{u_i\}$ we find a smooth unitary frame of  $TM_{|N}$  with the desidered properties. Q.D.E.

We let  $\nabla$  denote the Levi-Civita connection of the Kähler metric g on  $M$ , and  $D$  the induced connection on the submanifold  $N$ . The metric  $g$ being Kähler,  $\nabla$  gives a connection on  $K_M$ , and this in turn can be pulled back to a connection on  $K_{M|N}$ . We denote both connections by  $\nabla$ , too. Let  $\nu_N$  denote the normal bundle to  $N \subset M$ ,  $h : TN \otimes TN \to \nu_N$  the II fundamental form, and  $\vec{H}$  the mean curvature vector.

#### Proposition 1

If  $N$  is a Cayley submanifold

$$
g(h(X,Y),JZ) - g(h(X,Z),JY) = (D_X \omega)(Z,Y) \tag{15}
$$

$$
\omega(X, \vec{H}) = \sum_{i=1}^{4} g(h(X, e_i), Je_i)
$$
 (16)

where  $X, Y, Z$  are arbitrary vectors tangent to N, and  $\{e_i\}$  is any orthonormal basis of  $TN$ .

P R O O F.

$$
g(h(X,Y),JZ) = g((\nabla_X Y)^{\perp},JZ) =
$$
  
= 
$$
g(\nabla_X Y, JZ) - g(D_X Y, JZ) =
$$
  
= 
$$
g(\nabla_X Y, JZ) - \omega(Z, D_X Y),
$$
 (17)

therefore

$$
g(h(X, Y), JZ) - g(h(X, Z)JY) =
$$
  
= 
$$
g(\nabla_X Y, JZ) - g(\nabla_X Z, JY) + \omega(Y, D_X Z) - \omega(Z, D_X Y)
$$
 (18)

now

$$
g(\nabla_X Y, JZ) - g(\nabla_X Z, JY) =
$$
  
=  $Xg(Y, JZ) - g(Y, J\nabla_X Z) - g(\nabla_X Z, JY) =$  (19)  
=  $X \omega(Z, Y)$ .

Therefore

$$
g(h(X,Y),JZ) - g(h(X,Z)JY) =
$$
  
=  $X\omega(Z,Y) - \omega(D_XZ,Y) - \omega(Z,D_XY) =$   
=  $(D_X\omega)(Z,Y).$  (20)

This proves([15](#page-7-0)).

The second formula follows by taking the trace,

$$
\sum_{i=1}^{4} \Big{ g(h(e_i, X)Je_i) - g(h(e_i, e_i), JX) \Big} = \sum_{i=1}^{4} (D_{e_i} \omega)(e_i, X),
$$

and

$$
\sum_{i=1}^{4} (D_{e_i} \omega)(e_i, X) = -d^* (\omega_{|N})(X).
$$

N being Cayley, the restriction of  $\omega$  to N is selfdual (and closed), hence coclosed (with respect to the metric  $g_{|N}$ ). Therefore  $d^*\omega_{|M} = 0$ , and

$$
\sum_{i=1}^{4} g(h(e_i, X)Je_i) = \sum_{i=1}^{4} g(h(e_i, e_i), JX) = \omega(X, \vec{H}).
$$
 (21)

Q.D.E.

We will now use a Cayley frame and a unitary basis as in [6](#page-6-0), defined in some open subset of the totally real part of  $N$ , to prove some formulas relating  $\lambda$  and  $\vec{H}$ .

### Lemma 7

$$
\Omega_N(u_1, ..., \nabla_X u_k, ..., u_4) = \sqrt{-1} g(\nabla_X u_k, Ju_k).
$$
 (22)

P R O O F.

$$
\nabla_X u_k = \sum_{j=1}^4 \Big\{ g(\nabla_X u_k, u_j) u_j + g(\nabla_X u_k, J u_j) J u_j \Big\}
$$

therefore, using the fact that  $\Omega_N$  is a complex form, i.e. of type (4,0), we compute

$$
\Omega_N(u_1, ..., \nabla_X u_k, ..., u_4) =
$$
\n
$$
= \sum_j g(\nabla_X u_k, u_j) \Omega_N(u_1, ..., u_j, ..., u_4) +
$$
\n
$$
+ \sum_j g(\nabla_X u_k, Ju_j) \Omega_N(u_1, ..., Ju_j, ..., u_4) =
$$
\n
$$
= g(\nabla_X u_k, u_k) \Omega_N(u_1, ..., u_k, ..., u_4) +
$$
\n
$$
+ g(\nabla_X u_k, Ju_k) \sqrt{-1} \Omega_N(u_1, ..., u_k, ..., u_4) =
$$
\n
$$
= \sqrt{-1} g(\nabla_X u_k, Ju_k) \Omega_N(u_1, ..., u_k, ..., u_4)
$$

because

$$
g(\nabla_X u_k, u_k) = \frac{1}{2}X||u_k||^2 = 0.
$$

Q.D.E.

Lemma 8

$$
(1 - \lambda^2) \sum_{k=1}^{4} g(\nabla_X u_k, Ju_k) = \omega(X, \vec{H}).
$$
 (23)

P R O O F. Letus use the definition ([11\)](#page-4-0) of  $u_j$ :

$$
\nabla_X u_2 = \left(X \frac{1}{\sqrt{1 - \lambda^2}}\right)(e_2 - \lambda Je_1) + \frac{1}{\sqrt{1 - \lambda^2}}\left(\nabla_X e_2 - (X\lambda)Je_1 - \lambda J\nabla_X e_1\right)
$$

$$
g(\nabla_X u_2, Ju_2) =
$$
  
=  $\left(X \frac{1}{\sqrt{1 - \lambda^2}}\right) \frac{1}{\sqrt{1 - \lambda^2}} g\left(e_2 - \lambda Je_1, J(e_2 - \lambda Je_1)\right) +$   
+  $\frac{1}{1 - \lambda^2} g\left(\nabla_X e_2 - (X\lambda)Je_1 - \lambda J\nabla_X e_1, Je_2 + \lambda e_1\right) =$   
=  $0 + \frac{1}{1 - \lambda^2} g(\nabla_X e_2, Je_2) + \frac{\lambda}{1 - \lambda^2} g(\nabla_X e_2, e_1) +$   
-  $\frac{X\lambda}{1 - \lambda^2} g(Je_1, Je_2) - \frac{X\lambda\lambda}{1 - \lambda^2} g(Je_1, e_1) +$   
-  $\frac{\lambda}{1 - \lambda^2} g(\nabla_X e_1, e_2) + \frac{\lambda^2}{1 - \lambda^2} g(\nabla_X e_1, Je_1)$ 

therefore

$$
(1 - \lambda^2)g(\nabla_X u_2, Ju_2) = g(\nabla_X e_2, Je_2) + \lambda^2 g(\nabla_X e_1, Je_1) +- \lambda g(\nabla_X e_1, e_2) + \lambda g(\nabla_X e_2, e_1) == g(\nabla_X e_2, Je_2) + \lambda^2 g(\nabla_X e_1, Je_1) +- g(\nabla_X e_1, Be_1) - g(\nabla_X e_2, Be_2)
$$
  

$$
(1 - \lambda^2) \Big[ g(\nabla_X u_1, Ju_1) + g(\nabla_X u_2, Ju_2) \Big] == g(\nabla_X e_1, Je_1) + g(\nabla_X e_2, Je_2) +- g(\nabla_X e_1, Be_1) - g(\nabla_X e_2, Be_2) == g(\nabla_X e_1, (Je_1)^{\perp}) + g(\nabla_X e_2, (Je_2)^{\perp}) == g(h(X, e_1), Je_1) + g(h(X, e_2)Je_2).
$$

The same computation works for the last two indices, 3 and 4. Summing the two terms and using[\(16\)](#page-7-0) one gets

$$
(1 - \lambda^2) \sum_{k=1}^4 g(\nabla_X u_k, Ju_k) = \sum_i g(h(X, e_i), Je_i) = \omega(X, \vec{H}).
$$

Q.D.E.

Proposition 2

$$
\sqrt{-1} \left( \nabla_X \Omega_N \right) (\vec{u}) = \frac{i_{\vec{H}} \omega}{\lambda^2 - 1} (X). \tag{24}
$$

<span id="page-11-0"></span>P R O O F. By construction  $\Omega_N(\vec{u}) \equiv 1$ .

$$
0 = X.\Omega_N(\vec{u}) = (\nabla_X \Omega)(\vec{u}) + \sum_{k=1}^4 \Omega_N(u_1, ..., \nabla_X u_k, ..., u_4).
$$

Using the last two lemmas

$$
\sqrt{-1}(\nabla_X \Omega)(\vec{u}) = \sum_{k=1}^4 g(\nabla_X u_k, Ju_k) = \frac{\omega(X, \vec{H})}{1 - \lambda^2}.
$$

Q.D.E.

Let  $\rho$  denote the Ricci form of  $\omega$  and let us define  $\gamma \in \wedge^1(N_r)$  by

$$
\gamma(X) := \sqrt{-1} \left( \nabla_X \Omega_N \right) (\vec{u}) =
$$
  
= 
$$
\frac{i_{\vec{H}}\omega}{\lambda^2 - 1} (X) = \sum_{k=1}^4 g(\nabla_X u_k, Ju_k).
$$
 (25)

Theorem 3

$$
d\gamma = \rho_{|N}.\tag{26}
$$

P R O O F.

$$
X((\nabla_Y\Omega_N)(\vec{u}) = (\nabla_X\nabla_Y\Omega_N)(\vec{u}) + (\nabla_Y\Omega_N)(\nabla_X\vec{u}).
$$

 $(M, g)$  being Kähler,  $\Omega_N \in \Lambda^{4,0}$  implies that  $\nabla_Y \Omega_N \in \Lambda^{4,0}$  too, therefore

the same computations as above apply:

$$
\nabla_Y \Omega_N (u_1, ..., \nabla_X u_k, ..., u_4) =
$$
  
\n
$$
= \sum_{j=1}^4 \left\{ g(\nabla_X u_k, u_j) (\nabla_Y \Omega_N) (u_1, ..., u_j, ..., u_4) + g(\nabla_X u_k, Ju_j) (\nabla_Y \Omega_N) (u_1, ..., Ju_j, ..., u_4) \right\} =
$$
  
\n
$$
= g(\nabla_X u_k, u_k) (\nabla_Y \Omega_N) (u_1, ..., u_k, ..., u_4) +
$$
  
\n
$$
+ \sqrt{-1} \sum_{j=1}^4 \left\{ g(\nabla_X u_k, Ju_j) (\nabla_Y \Omega_N) (u_1, ..., u_j, ..., u_4) \right\} =
$$
  
\n
$$
= \sqrt{-1} g(\nabla_X u_k, Ju_k) (\nabla_Y \Omega_N) (\vec{u}),
$$
  
\n
$$
(\nabla_Y \Omega_N) (\nabla_X \vec{u}) = \sqrt{-1} \left[ \sum_k g(\nabla_X u_k, Ju_k) \right] (\nabla_Y \Omega_N) (\vec{u}) = \gamma(X) \gamma(Y).
$$

Then applying the usual formula for the differential of a 1-form we find

$$
d\gamma(X,Y) = -\sqrt{-1} \big( R_{XY} \,\Omega_N \big)(\vec{u})
$$

but

$$
R_{XY}\,\Omega_N = \sqrt{-1}\,\rho(X,Y)\Omega_N.
$$

Q.D.E.

Finally we make the following remark.

### Proposition 3

Let  $\overline{M}$  be a Kähler manifold and let  $N, N'$  be two closed Cayley submanifolds in the same homology class,

$$
[N] = [N'] \in H_4(M, \mathbb{Z}).
$$

Then, if  $N$  is Lagrangian, the same is true of  $N'$ .

P R O O F. It is enough to observe that

$$
||\lambda_N||^2_{L^2(N)} = \int_N \lambda_N^2 \, \mathrm{dvol} = \frac{1}{2} \int_N \omega^2 = \frac{1}{2} < \omega^2, [N] >
$$

is a topological invariant. If N is Lagrangian,  $||\lambda_{N'}||_{L^2}^2 = ||\lambda_N||_{L^2}^2 = 0$ , therefore  $\lambda_{N'} \equiv 0$  and  $N'$  is Lagrangian. Q.D.E.

# 4 Minimal Cayley submanifolds in Kähler-Einstein manifolds

Wenow apply the formula ([26\)](#page-11-0) to the cases where  $\rho = s\omega$ , i.e. when the ambient manifold is Kähler-Einstein. This will yield proofs of theorems [1](#page-1-0) and [2](#page-1-0).

P R O O F OF THEOREM [1.](#page-1-0)

Let  $\Omega \in \Gamma(K_M)$  be a parallel normalized (4,0)-form. On N we can write

$$
\Omega_N = e^{\sqrt{-1}\,\theta}\Omega
$$

for some locally defined real valued function  $\theta = \theta(x)$ . Then

$$
\nabla_X \Omega_N = \nabla_X (e^{\sqrt{-1}\theta} \Omega) =
$$
  
=  $\sqrt{-1}(X\theta) e^{\sqrt{-1}\theta} \Omega =$   
=  $\sqrt{-1}(X\theta) \Omega_N$ .  

$$
\gamma(X) = \sqrt{-1} (\nabla_X \Omega_N) (\vec{u}) =
$$
  
=  $-(X\theta) \Omega_N(\vec{u}) =$   
=  $-X\theta$ 

i.e.  $\gamma = -d\theta$ . But  $\gamma = 0$  because  $\vec{H} = 0$ . Therefore  $\theta \equiv \theta_0$  is a constant, and  $N$  is calibrated by

$$
\Phi_0 = \text{Re}\big(e^{\sqrt{-1}\,\theta_0}\Omega\big) + \frac{\omega^2}{2}.
$$

On the other side, if  $N$  is calibrated by some parallel Cayley calibration, then it is obviously minimal, and thanks to lemma [5](#page-4-0) (c), it is Cayley also according to our definition. Q.D.E.

P R O O F OF THEOREM [2.](#page-1-0)

Let  $\rho = s\omega$ , with  $s \neq 0$ .

<span id="page-14-0"></span>If  $N \subset M$  is not complex, then  $N_r$  is not empty. But  $\vec{H} \equiv 0$  implies that  $\gamma$ , hence  $d\gamma$  vanish identically. Therefore

$$
\omega_{|N_r} = \frac{1}{s}\rho_{|N_r} = d\gamma = 0
$$

i.e.  $N_r$  is a Lagrangian submanifold, and  $\lambda = 0$  on it. This means that  $N_r = \lambda^{-1}(0)$  is a closed and open set. Then  $N = N_r$ , and N is Lagrangian. Q.D.E.

# References

