On α -Critical Edges in König-Egerváry Graphs

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Abstract

The stability number of a graph G, denoted by $\alpha(G)$, is the cardinality of a stable set of maximum size in G. If $\alpha(G - e) > \alpha(G)$, then e is an α -critical edge, and if $\mu(G - e) < \mu(G)$, then e is a μ -critical edge, where $\mu(G)$ is the cardinality of a maximum matching in G. G is a König-Egerváry graph if its order equals $\alpha(G) + \mu(G)$. Beineke, Harary and Plummer have shown that the set of α -critical edges of a bipartite graph is a matching. In this paper we generalize this statement to König-Egerváry graphs. We also prove that in a König-Egerváry graph α -critical edges are also μ -critical, and that they coincide in bipartite graphs. Eventually, we deduce that $\alpha(T) = \xi(T) + \eta(T)$ holds for any tree T, and characterize the König-Egerváry graphs enjoying this property, where $\xi(G)$ is the number of α -critical vertices of G, and $\eta(G)$ is the number of α -critical edges of G.

1 Introduction

Throughout this paper G = (V, E) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set V = V(G), edge set E = E(G), and order n(G) = |V(G)|. If $X \subset V$, then G[X] is the subgraph of G spanned by X. By G - W we mean the subgraph G[V - W], if $W \subset V(G)$. For $F \subset E(G)$, by G - F we denote the partial subgraph of G obtained by deleting the edges of F, and we use G - e, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \cup \{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subset V$.

A set S of vertices is stable if no two vertices from S are adjacent. A stable set of maximum size will be referred to as a maximum stable set of G. The stability number of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set of G. Let $\Omega(G)$ denotes the set $\{S : S \text{ is a maximum stable set of } G\}$, $\sigma(G) = |\cap\{V - S : S \in \Omega(G)\}|$ and $\xi(G) = |core(G)|$, where $core(G) = \cap\{S : S \in \Omega(G)\}$, [12]. In other words, $\xi(G)$ equals the number of α -critical vertices of G, (a vertex $v \in V(G)$ is α -critical provided $\alpha(G - v) < \alpha(G)$).

By P_n, C_n, K_n we mean the chordless path on $n \ge 3$, the chordless cycle on $n \ge 4$ vertices, and respectively the complete graph on $n \ge 1$ vertices.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a maximum matching, and a perfect matching is one covering all vertices of G. An edge $e \in E(G)$ is μ -critical provided $\mu(G - e) < \mu(G)$. By their definition, μ -critical edges of G belong to all maximum matchings of G.

If $\alpha(G) + \mu(G) = n(G)$, then G is called a König-Egerváry graph, [4], [17]. Properties of these graphs were presented in several papers, like of Sterboul [17], Deming [4], Lovász and Plummer [14], Korach [8], Bourjolly and Pulleyblank [2], Paschos and Demange [16], Levit and Mandrescu [11], [13]. It is worth observing that a disconnected graph is of König-Egerváry type if and only if all its connected components are König-Egerváry graphs. In this paper, by "graph" we mean a connected graph having at least one edge.

An edge $e \in E(G)$ is α -critical whenever $\alpha(G - e) > \alpha(G)$. Let denote by $\eta(G)$ the number of α -critical edges of G. Notice that there are graphs in which: (a) any edge is α -critical (so-called α -critical graphs); e.g., all C_{2n+1} for $n \geq 3$; (b) no edge is α -critical; e.g., all C_{2n} for $n \geq 2$. More generally, Haynes et al., [7], have proved that a graph G has no α -critical edge if and only if $|N(x) \cap S| \geq 2$ holds for any $S \in \Omega(G)$ and every $x \in V(G) - S$.

Beineke, Harary and Plummer, [1], have shown that any two incident α -critical edges of a graph lie on an odd cycle, and hence, they deduce that no two α -critical edges of a bipartite graph can have a common endpoint. Independently, Zito, [21], has proved the same result for trees using a different technique. Some variations and strengthenings of these results are discussed in [18], [20], and [19].

In this paper we generalize the above assertion to König-Egerváry graphs. We also show that α -critical edges are μ -critical in a König-Egerváry graph, and that they coincide in bipartite graphs. As a corollary, we obtain one result of Zito, [21], stating that a vertex v is in some but not in all maximum stable sets of a tree T if and only if v is an endpoint of an α -critical edge of T. In the sequel, we analyze other relationships between α -critical edges and μ -critical edges in a König-Egerváry graph, and its corresponding implications to equalities and inequalities linking $\alpha(G)$, $\xi(G)$, $\eta(G)$, $\sigma(G)$, and $\mu(G)$. Eventually, we infer that $\alpha(T) = \xi(T) + \eta(T), \sigma(T) + \eta(T) =$ $\mu(T)$ and $\xi(T) + 2\eta(T) + \sigma(T) = n(T)$ holds for any tree T, and characterize the König-Egerváry graphs having these properties.

2 α -Critical and μ -Critical Edges

According to a well-known result of König, [9], and Egerváry, [5], any bipartite graph is a König-Egerváry graph. It is easy to see that this class includes also some non-bipartite graphs (see, for instance, the graph $K_3 + e$ in Figure 1).

If $G_i = (V_i, E_i), i = 1, 2$, are two disjoint graphs, then $G = G_1 * G_2$ is defined as the graph with $V(G) = V(G_1) \cup V(G_2)$, and

$$E(G) = E(G_1) \cup E(G_2) \cup \{xy : for some \ x \in V(G_1) \ and \ y \in V(G_2)\}.$$

Clearly, if H_1, H_2 are subgraphs of a graph G such that $V(G) = V(H_1) \cup V(H_2)$



Figure 1: Graph $K_3 + e$.

and $V(H_1) \cap V(H_2) = \emptyset$, then $G = H_1 * H_2$, i.e., any graph of order at least two admits such decompositions. However, some particular cases are of special interest. For instance, if: $E(H_i) = \emptyset$, i = 1, 2, then $G = H_1 * H_2$ is bipartite; $E(H_1) = \emptyset$ and H_2 is complete, then $G = H_1 * H_2$ is a *split graph* [6].

The following result shows that the König-Egerváry graphs are, in this sense, between these two "extreme" situations. The equivalence of the first and the third parts of this proposition was proposed by Klee and included in [10] without proof (private communication).

Proposition 2.1 [13] The following assertions are equivalent:

(i) G is a König-Egerváry graph;

(*ii*) $G = H_1 * H_2$, where $V(H_1) = S \in \Omega(G)$ and $n(H_1) \ge \mu(G) = n(H_2)$;

(iii) $G = H_1 * H_2$, where $V(H_1) = S$ is a stable set in G, $|S| \ge n(H_2)$ and $(S, V(H_2))$ contains a matching M with $|M| = n(H_2)$.

In the sequel, we shall often represent a König-Egerváry graph G as G = S * H, where $S \in \Omega(G)$ and H = G[V - S] has $n(H) = \mu(G)$.

Lemma 2.2 [13] If G = (V, E) is a König-Egerváry graph, then any maximum matching of G is contained in (S, V - S), where $S \in \Omega(G)$.

Clearly, Lemma 2.2 is not valid for any graph. For instance, K_4 is a counterexample. Moreover, K_4 has α -critical edges that are incident. Nevertheless, there are graphs having only non-incident α -critical edges.

Theorem 2.3 If G is a König-Egerváry graph, then the following assertions hold:

- (i) for any α -critical edge e of G, the graph G e is still a König-Egerváry graph;
- (ii) any α -critical edge of G is also μ -critical;
- (iii) the α -critical edges of G form a matching.

Proof. (i) If e = xy is an α -critical edge G, then there is some $S \in \Omega(G)$ such that either $N(x) \cap S = \{y\}$ or $N(y) \cap S = \{x\}$. Suppose that $y \in S$. Since $S \in \Omega(G)$, we get, by Proposition 2.1, that G = S * H, where H = G[V - S] has $\mu(G) = n(H) = |M|$ and M is a maximum matching of G, included, by Lemma 2.2, in (S, V(G) - S). Hence, it follows that G - e = S' * V(H'), where $S' = S \cup \{x\} \in \Omega(G - e)$ and $n(H') = |M - \{e\}|$. According to Proposition 2.1(*iii*), we infer that G - e is also a König-Egerváry graph.

(*ii*) If $e \in E(G)$ is an α -critical edge of G, then according to (i) we obtain:

$$n(G) = \alpha(G) + \mu(G) \le \alpha(G - e) + \mu(G - e) = \alpha(G) + 1 + \mu(G - e) = n(G - e)$$

and this implies $\mu(G) = 1 + \mu(G - e)$, i.e., e is also μ -critical.

(*iii*) Let e_1, e_2 be two α -critical edges of G. We have to show that they are not incident. According to second part (*ii*), both edges are also μ -critical. Hence, it follows that $e_1, e_2 \in \cap \{M : M \text{ is a maximum matching of } G\}$ and this ensures that e_1, e_2 have no common endpoint. Consequently, the set of all α -critical edges of G yields a matching.

Notice that:

(a) Theorem 2.3(i) is not true for any μ -critical edge of a König-Egerváry graph; e.g., the edge e of $G = K_3 + e$ is μ -critical, but G - e is not a König-Egerváry graph; (b) Theorem 2.2(ii) is not true for any graph, and a solution of K and a gritical

(b) Theorem 2.3(*ii*) is not true for any graph; e.g., all the edges of K_3 are α -critical, but none is also μ -critical;

(c) the converse of Theorem 2.3(*ii*) is not valid for any König-Egerváry graph; e.g., the edge e of graph $K_3 + e$ is μ -critical, but is not also α -critical. However, as we shall see later, (namely Proposition 2.6), the μ -critical edges are also α -critical in the case of bipartite graphs.

Corollary 2.4 A König-Egerváry graph is α -critical if and only if it is isomorphic to K_2 .

Since any bipartite graph is also a König-Egerváry graph, we obtain the following statement, due to Beineke, Harary and Plummer.

Theorem 2.5 [1] No two α -critical edges of a bipartite graph are incident.

Proposition 2.6 If G is a bipartite graph, then its α -critical edges coincide with its μ -critical edges.

Proof. By Theorem 2.3(*ii*), it suffices to show that any μ -critical edge e of G is also α -critical. Since G - e is still bipartite, and hence, also a König-Egerváry graph, it follows that $\alpha(G - e) + \mu(G - e) = n(G) = \alpha(G) + \mu(G) = \alpha(G) + 1 + \mu(G - e)$, and this implies $\alpha(G - e) > \alpha(G)$, i.e., e is an α -critical edge of G.

In Theorem 4.2 we will meet another type of König-Egerváry graphs with this property. Notice that there are also non-bipartite König-Egerváry graphs in which their μ -critical edges are α -critical (see the graph in Figure 2).



Figure 2: A Koenig-Egervary graph whose all μ -critical edges are α -critical.

It is well-known that if a tree has a perfect matching, then it is unique. Consequently, we obtain:

Corollary 2.7 A tree has a perfect matching if and only if the set of its α -critical edges forms a maximal matching of the tree.

Using the definition of König-Egerváry graphs and the fact that $\mu(G) \leq n(G)/2$ is true for any graph G, we get:

Lemma 2.8 If G admits a perfect matching, then G is a König-Egerváry graph if and only if $\alpha(G) = \mu(G)$. If G is a König-Egerváry graph, then $\mu(G) \leq \alpha(G)$.

Combining Corollary 2.7 and Lemma 2.8, we get the following result from [21].

Corollary 2.9 [21] If a tree T has a perfect matching M, then all the edges of M are α -critical and $2\alpha(T) = n(T)$.

Proposition 2.10 If G = (V, E) is a König-Egerváry graph, then the following assertions are true:

(i) any $S \in \Omega(G)$ meets each μ -critical edge in exactly one vertex;

(ii) any $S \in \Omega(G)$ meets each α -critical edge in exactly one vertex;

(iii) if G has a maximal matching consisting of only α -critical edges, then it is the unique perfect matching of G.

Proof. (i) and (ii) By Theorem 2.3(ii), any α -critical edge of G is also μ -critical. Consequently, we infer that

 $\{e \in E : e \text{ is } \alpha - critical\} \subseteq \cap \{M : M \text{ is a maximum matching of } G\} \subseteq (S, V - S)$

holds for any $S \in \Omega(G)$, according to Lemma 2.2. It follows that if e = xy is an α -critical or a μ -critical edge of G, then any $S \in \Omega(G)$ contains one of x and y, (since clearly, no stable set may contain both x and y).

(*iii*) Let M be a maximal matching of G consisting of only α -critical edges. By Theorem 2.3, all the edges of M are also μ -critical. Therefore, we infer that M is included in any maximum matching of G, and because M is a maximal matching, it results that M is the unique maximum matching of G. Suppose, on the contrary, that M is not perfect, and let $S \in \Omega(G)$. According to Proposition 2.1, G can be written as G = S * H, with $n(H) = |M| = \mu(G)$, and by Lemma 2.2 we have that $M \subseteq (S, V - S)$. Since G is a König-Egerváry graph without perfect matchings, Lemma 2.8 implies $|S| = \alpha(G) > \mu(G) = |M|$. Hence, it follows that there are at least two vertices $v_1, v_2 \in S$ having a common neighbor $w \in V(H)$ and such that one of them, say v_1 , is unmatched by M and $v_2w \in M$. Thus, $M \cup \{v_1w\} - \{v_2w\}$ is another maximum matching of G, in contradiction with the uniqueness of M. Consequently, M must be also perfect.

For trees, Proposition 2.10(ii) was proved by Zito in [21].

Notice that the matching in Proposition 2.10(iii) is not necessarily formed by pendant edges; e.g., P_6 has such a matching. Concerning the uniqueness of this matching, it is worth mentioning that: (a) if G is not a König-Egerváry graph, then it may have several different maximum matchings consisting of only α -critical edges (e.g., C_5); (b) if a König-Egerváry graph has a unique perfect matching, then it may contain non- α -critical edges (e.g., the edge e of $K_3 + e$ is not α -critical, but it belongs to the unique perfect matching of $K_3 + e$).

3 Equalities and Inequalities between Parameters

If $v \in N(core(G))$, then clearly follows that $v \in V(G) - S$, for any $S \in \Omega(G)$, that is $N(core(G)) \subseteq \cap \{V - S : S \in \Omega(G)\}$ holds for any graph G.

Lemma 3.1 [13] If G = (V, E) is a König-Egerváry graph, then $N(core(G)) = \cap \{V - S : S \in \Omega(G)\}.$

Notice that there are graphs that do not enjoy the above equality, for example, the graph G in Figure 3(a) has $N(core(G)) = \emptyset$ and $\cap \{V - S : S \in \Omega(G)\} = \{v\}$. There exist non-König-Egerváry graphs for which $N(core(G)) = \cap \{V - S : S \in \Omega(G)\}$, (see, for instance, the graph G from Figure 3(b)).



Figure 3: (a) G is non-König-Egerváry with $N(core(G)) \neq \cap \{V - S : S \in \Omega(G)\}$; (b) G is a non-König-Egerváry graph with $N(core(G)) = \cap \{V - S : S \in \Omega(G)\}$.

Proposition 3.2 If G = (V, E) is a König-Egerváry graph, $G_0 = G - N[core(G)]$ and $S \in \Omega(G)$, then the following assertions are true:

(i) $|core(G)| \ge |N(core(G))|;$

(ii) |S - core(G)| = |V - S - N(core(G))|;

(iii) G_0 has a perfect matching and it is also a König-Egerváry graph.

Proof. According to Proposition 2.1, G can be written as G = S * H, where H = G[V-S] has $n(H) = \mu(G)$. Let denote A = S-core(G) and B = V(H) - N(core(G)). In [12] it has been proved that $|A| \leq |B|$ holds for any graph G. Since $\cap \{V - S : S \in \Omega(G)\} \subseteq V(H)$, and $N(core(G)) = \cap \{V - S : S \in \Omega(G)\}$ (see Lemma 3.1), we obtain $B = V(H) - \cap \{V - S : S \in \Omega(G)\}$.

(i) Since $|A| + |core(G)| = \alpha(G) \ge \mu(G) = n(H) = |B| + |N(core(G))|$ and, on the other hand $|A| \le |B|$, it follows that $|core(G)| \ge |N(core(G))|$.

(*ii*) Let M be a maximum matching in G. Since G is a König-Egerváry graph, Lemma 2.2 ensures that M is included in (S, V(H)), and $|M| = \mu(G) = n(H)$. The matching M matches B into A, because there are no edges connecting B and core(G). Hence, $|B| \leq |A|$. Together with $|A| \leq |B| |A| \leq |B|$, it implies |A| = |B|, i.e., |S - core(G)| = |V - S - N(core(G))|, and that $M \cap (A, B)$ is a perfect matching of $G[A \cup B]$.

(*iii*) Since, in fact, $G_0 = G[A \cup B]$, it follows necessarily that G_0 has a perfect matching. In addition, because A is stable, we get $\alpha(G_0) \leq \mu(G_0) = |A| \leq \alpha(G_0)$, i.e., $\alpha(G_0) = \mu(G_0)$, and according to Lemma 2.8, G_0 must be also a König-Egerváry graph.

Corollary 3.3 If G is a König-Egerváry graph, then $\alpha(G) + \sigma(G) = \mu(G) + \xi(G)$.

Proof. By Lemma 3.1, $N(core(G)) = \cap \{V - S : S \in \Omega(G)\}$ and according to Proposition 3.2(*ii*), |S - core(G)| = |V - S - N(core(G))|. Hence, we obtain that $\alpha(G) - \xi(G) = |S - core(G)| = |V - S - N(core(G))| = \mu(G) - \sigma(G)$.

Let us observe that there exist non-König-Egerváry graphs satisfying the equality $\alpha(G) + \sigma(G) = \mu(G) + \xi(G)$ (see graph W_1 in Figure 6). It is also interesting to notice that there exists a non-König-Egerváry graph enjoying the property that its subgraph $G_0 = G - N[core(G)]$ has a perfect matching (see Figure 8). Figure 4 shows a non-König-Egerváry graph G whose G_0 has no perfect matching.



Figure 4: G is a non-König-Egervary graph with $core(G) = \{a, b\}$ and $G_0 = C_5$.

Lemma 3.4 Let G = (V, E) and $G_0 = G - N[core(G)]$. Then the following assertions are valid:

(i) no α -critical edge in G has an endpoint in N[core(G)];

(*ii*) $\alpha(G) = \alpha(G_0) + \xi(G), \Omega(G_0) = \{S \cap V(G_0) : S \in \Omega(G)\}, core(G_0) = \emptyset;$

(iii) e = xy is an α -critical edge of G if and only if e is an α -critical edge of G_0 .

Proof. (i) Let e = xy be an α -critical edge in G, and let $\overline{S} \in \Omega(G - e)$. Since $|\overline{S}| = \alpha(G - e) > \alpha(G)$, it follows that $x, y \in \overline{S}$ and $\overline{S} - \{x\}, \overline{S} - \{y\} \in \Omega(G)$. Now, the inclusion $N(core(G)) \subseteq \cap \{V - S : S \in \Omega(G)\}$ completes the proof that no α -critical edge in G has an endpoint in N(core(G)), and respectively, in core(G).

(*ii*) By definition of G_0 , if $S \in \Omega(G)$, then $S - core(G) = S \cap V(G_0)$, and therefore

$$\alpha(G) - \xi(G) = |S - core(G)| \le \alpha(G_0).$$

For any $S_{G_0} \in \Omega(G_0)$ we have that $S_{G_0} \cup core(G)$ is stable, and hence

$$|S_{G_0} \cup core(G)| = \alpha(G_0) + \xi(G) \le \alpha(G).$$

Consequently, we get $\alpha(G) = \alpha(G_0) + \xi(G)$. Now it is easy to check that $\Omega(G_0) = \{S \cap V(G_0) : S \in \Omega(G)\}$ and $core(G_0) = \emptyset$.

(*iii*) Let e = xy be an α -critical edge of G. By (*i*), we infer that $e \in E(G_0)$, and as we saw above, there is some stable set S_{xy} such that $S_{xy} \cup \{x\}, S_{xy} \cup \{y\} \in \Omega(G)$ and $S_{xy} \cup \{x, y\} \in \Omega(G - e)$. Hence, (*ii*) implies that

$$V(G_0) \cap (S_{xy} \cup \{x\}), V(G_0) \cap (S_{xy} \cup \{y\}) \in \Omega(G_0) \text{ and } V(G_0) \cap (S_{xy} \cup \{x,y\}) \in \Omega(G_0-e),$$

because $V(G_0) \cap (S_{xy} \cup \{x, y\})$ is stable in $G_0 - e$ and larger than $V(G_0) \cap (S_{xy} \cup \{x\})$. Therefore, e is α -critical in G_0 , as well. Similarly, we can show that any α -critical edge of G_0 is α -critical in G too. Proposition 3.5 If G is a König-Egerváry graph, then

(i) $\xi(G) + \eta(G) \le \alpha(G);$ (ii) $\sigma(G) + \eta(G) \le \mu(G);$ (iii) $\xi(G) + 2\eta(G) + \sigma(G) \le n(G).$

Proof. For any $S \in \Omega(G)$, we have that $core(G) \subseteq S$, and by Lemma 3.4(*i*), no α -critical edge has an endpoint in core(G). In addition, according to Proposition 2.10(*ii*), S meets each α -critical edge in exactly one vertex. Hence, it follows that $\xi(G) + \eta(G) \leq \alpha(G)$, and using Corollary 3.3 we obtain (*ii*). Clearly, (*iii*) follows from (*i*) and (*ii*).

Notice that $\xi(K_3 + e) + \eta(K_3 + e) = \alpha(K_3 + e)$ and also $\eta(K_3 + e) + \sigma(K_3 + e) = \mu(K_3 + e)$, but there are König-Egerváry graphs satisfying $\xi(G) + \eta(G) < \alpha(G)$ and $\eta(G) + \sigma(G) < \mu(G)$. For instance, $G = C_6$, and also the graph W in Figure 5 is a König-Egerváry non-bipartite graph that has $\eta(W) = |\{e\}| = 1, \xi(W) = |\{a\}| = 1 = \sigma(W), \alpha(W) = \mu(W) = 4$.



Figure 5: W is a non-bipartite Koenig-Egervary graph and $\xi(W) + \eta(W) < \alpha(W)$.

Observe that Proposition 3.5 is not true for general graphs; e.g., the graph W_1 in Figure 6 has $\alpha(W_1) = 3$, $\mu(W_1) = 2$, $\eta(W_1) = 3$, $\xi(W_1) = 2$, $\sigma(W_1) = 1$. However, there are non-König-Egerváry graphs satisfying $\xi(G) + \eta(G) < \alpha(G)$ and $\eta(G) + \sigma(G) < \mu(G)$, for example, the graph W_2 in Figure 6 has $\alpha(W_2) = 3$, $\eta(W_2) = |\{ab, cd\}|, \xi(W_2) = \sigma(W_2) = 0$. There also exist non-König-Egerváry graphs satisfying $\xi(G) + \eta(G) = \alpha(G)$ and $\eta(G) + \sigma(G) = \mu(G)$, e.g., the graph W_3 in Figure 6. Nevertheless, $\xi(K_5 - e) + \eta(K_5 - e) = \alpha(K_5 - e)$, but $\eta(K_5 - e) + \sigma(K_5 - e) > \mu(K_5 - e)$.



Figure 6: Non-Koenig-Egervary graphs.

Proposition 3.6 If G is a König-Egerváry graph, then the following assertions are equivalent:

(i) $\xi(G) + \eta(G) = \alpha(G);$ (ii) $\sigma(G) + \eta(G) = \mu(G);$ (iii) $\xi(G) + 2\eta(G) + \sigma(G) = n(G).$ **Proof.** Suppose that $\xi(G) + \eta(G) = \alpha(G)$. According to Corollary 3.3, we get that $\mu(G) = \alpha(G) + \sigma(G) - \xi(G) = \xi(G) + \eta(G) + \sigma(G) - \xi(G) = \eta(G) + \sigma(G)$. The converse is proven in the same way.

Suppose $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$. Proposition 3.5 claims that $\xi(G) + \eta(G) \leq \alpha(G)$ and $\sigma(G) + \eta(G) \leq \mu(G)$. Together with $\alpha(G) + \mu(G) = n(G)$, which is true for König-Egerváry graphs, it gives us the two equalities needed. Conversely, if, for instance, $\xi(G) + \eta(G) = \alpha(G)$ then, as we already proved, $\sigma(G) + \eta(G) = \mu(G)$. Summing these two equalities we obtain $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$.

4 König-Egerváry Graphs for which $\xi + \eta = \alpha$

Lemma 4.1 Let G be a König-Egerváry graph and $G_0 = G - N[core(G)]$. If G_0 has a unique perfect matching then its α -critical edges coincide with its μ -critical edges.

Proof. By Theorem 2.3, it is enough to show that all the edges of M (the unique perfect matching of G_0) are also α -critical.

According to Proposition 2.1, we may write G as G = S * H, where $S \in \Omega(G)$ and H = G[V - S] has $n(H) = \mu(G)$. By virtue of Lemma 3.4(*ii*), G_0 has $\alpha(G_0) = |S - core(G)| = q$ and $core(G_0) = \emptyset$. Let $M = \{a_i b_i : 1 \le i \le q\}$ and suppose that $\{a_i : 1 \le i \le q\} = A \subseteq S$. We shall show that any $a_i b_i \in M$ is α -critical, by exhibiting a maximum stable set S_0 in G_0 that satisfies: $b_i \in S_0$ and $S_0 \cap N(a_i) = \{b_i\}$. For the sake of simplicity, let us take i = 1. In the sequel, if $D \subseteq V(G_0)$, then by M(D) we mean the set of vertices, which D is matched onto.

Claim 1. There exists some $S_0 \in \Omega(G_0)$ with $b_1 \in S_0$.

Otherwise, any $W \in \Omega(G_0)$ contains a_1 , because |M| = |W| and $|W \cap \{a_j, b_j\}| = 1$ holds for every $j \in \{1, 2, ..., q\}$. Hence, it follows that $a_1 \in core(G_0)$, in contradiction with $core(G_0) = \emptyset$.

Claim 2. The following procedure gives rise to some $S_0 \in \Omega(G_0)$ that contains b_1 .

Input: $G_0, A = \{a_1, a_2, ..., a_q\}, b_1 \in B = \{b_1, b_2, ..., b_q\} = M(A);$ Output: $b_1 \in S_0 \in \Omega(G_0);$ $S_0 := \{b_1\};$ $D := \{b_1\};$ while $(N(D) \cap A) - M(S_0) \neq \emptyset$ do begin Step 1. $S_1 := S_0;$ Step 2. $S_0 := S_0 \cup M((N(D) \cap A) - M(S_0));$ Step 3. $D := S_0 - S_1;$ end Step 4. $S_0 := S_0 \cup M(B - S_0).$

Clearly, $|S_0| = q$ and no edge of G_0 joins some $a_l \in S_0$ to any $b_j \in S_0$, according to building procedure of S_0 . Any maximum stable set $W \in \Omega(G_0)$ that contains b_1 must contain also all $b_j \in S_0$, because |W| = |M| and $|W \cap \{a_j, b_j\}| = 1$ holds for every $j \in \{1, 2, ..., q\}$. Hence, the set $\{b_j : b_j \in S_0\}$ is stable, and consequently, we obtain that $S_0 \in \Omega(G_0)$. An example of $S_0 \in \Omega(G_0)$ obtained by this procedure is illustrated in Figure 7.



Figure 7: The graph G_0 has a unique perfect matching and $\xi(G_0) = 0$.

Claim 3. $S_0 \cup \{a_1\} \in \Omega(G_0 - a_1b_1)$, and hence, the edge a_1b_1 is α -critical in G_0 . Firstly, no $a_i \in S_0$ is adjacent to a_1 , because $a_i, a_1 \in A$. Secondly, no $b_j \in S_0 - \{b_1\}$ is adjacent to a_1 , otherwise there exists an even cycle C, with half of its edges belonging to M, which means that $(M - E(C)) \cup (E(C) - M)$ is another perfect matching in G_0 , in contradiction with the premises on G_0 . Therefore, $S_0 \cup \{a_1\} \in \Omega(G_0 - a_1b_1)$ and this implies that the edge a_1b_1 is α -critical in G_0 . Since a_1b_1 is an arbitrary edge of M, we may conclude that all the edges of M are α -critical in G_0 .

It is interesting to notice that if G_0 were bipartite for every König-Egerváry graph G, then it would be possible to prove Lemma 4.1 using only Proposition 2.6. Figures 2, 7 show that Proposition 2.6 is not enough for our purposes, because there exist non-bipartite König-Egerváry graphs G with nonempty cores and whose $G_0 = G - N[core(G)]$ have a unique perfect matching.

Theorem 4.2 Let G be a König-Egerváry graph and $G_0 = G - N[core(G)]$. Then the following assertions are equivalent:

(i) G_0 has a unique perfect matching;

(ii) α -critical edges of G_0 form a maximal matching in G_0 ;

- (*iii*) $\xi(G) + \eta(G) = \alpha(G);$ (*iv*) $\sigma(G) + \eta(G) = \mu(G);$

Proof. According to Proposition 3.2, G_0 is also a König-Egerváry graph and has a perfect matching, say M_0 .

 $(i) \Leftrightarrow (ii)$ If M_0 is the unique perfect matching of G_0 , all its edges are μ -critical and, by Lemma 4.1, α -critical, as well. In other words, the α -critical edges of G_0 form a maximal matching. The converse is true according to Proposition 2.10(*iii*).

 $(i) \Rightarrow (iii)$ Assume that M_0 is the unique perfect matching of G_0 . By Lemma 3.4(*ii*), it follows that $\alpha(G_0) = \alpha(G) - \xi(G)$. Lemma 3.4(*iii*) and the uniqueness of M imply that $\alpha(G_0) = \eta(G_0) = \eta(G)$. Hence, it results in $\xi(G) + \eta(G) = \alpha(G)$.

 $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ It is the claim of Proposition 3.6.

 $(v) \Rightarrow (ii)$ By Proposition 3.1,

 $|N(core(G))| = |\cap \{V - S : S \in \Omega(G)\}| = \sigma(G).$

Hence, $n(G_0) = n(G) - \xi(G) - \sigma(G)$. Now, our premise claims that $2\eta(G) = n(G_0)$. By Lemma 3.4(*iii*) we obtain $2\eta(G_0) = n(G_0)$. According to Theorem 2.3(*iii*) the set of α -critical edges of G form a matching, say M. Applying again Lemma 3.4(*iii*), we see that $M_0 = M$ and it consists of α -critical edges of G_0 .

Notice that Theorem 4.2 fails for non-König-Egerváry graphs. In Figure 8 is presented a non-König-Egerváry graph G having $\xi(G) = |\{v\}| = 1, \eta(G) = 10$, (all the edges of the two C_5 are α -critical), $\alpha(G) = 5 < \mu(G) = 6$, but $G_0 = G - N[core(G)]$ owns a unique perfect matching.



Figure 8: A non-Koenig-Egervary graph satisfying $\xi(G) + \eta(G) < \alpha(G)$.

Now using Theorem 4.2 we are giving a new characterization of the bipartite graphs that have a unique perfect matching (see some previous discussions of this topic in [3] and [15]). This result generalizes Corollary 2.9.

Corollary 4.3 Let G be a bipartite graph. Then the following assertions are equivalent:

(i) G has a unique perfect matching; (ii) α -critical edges of G form a maximal matching; (iii) $\eta(G) = \alpha(G)$; (iv) $\eta(G) = \mu(G)$; (v) $2\eta(G) = n(G)$.

Proof. $(i) \Leftrightarrow (ii)$ If M is the unique perfect matching of G, all its edges are μ -critical and, by Proposition 2.6, α -critical, as well. In other words, the α -critical edges of G form a maximal matching. The converse is true according to Proposition 2.10(*iii*).

The other equivalences follow from Theorem 4.2, and the observation that if a bipartite graph has a perfect matching, then the two stable sets of its standard partition are maximum, and, consequently, $\xi(G) = 0$.

It is interesting to notice that the equality $2\alpha(G) = n(G)$ mentioned in Corollary 2.9 follows from Corollary 4.3, but it can not join the above series of equivalences (see, for example, C_4).

Let us also observe that for the bipartite graph G in Figure 9, the subgraph $G_0 = G - N[core(G)]$ has more than one perfect matching.



Proposition 4.4 If G is a König-Egerváry graph and there is some $S \in \Omega(G)$ such

$$\xi(G) + \eta(G) = \alpha(G), \sigma(G) + \eta(G) = \mu(G), \text{ and } \xi(G) + 2\eta(G) + \sigma(G) = n(G).$$

that the set W = (S, V(G) - S) generates a forest, then

Proof. If $G_0 = G - N[core(G)]$, A = S - core(G), B = V(G) - S - N(core(G)), then Proposition 3.2(*iii*) implies that G_0 is also a König-Egerváry graph and has a perfect matching, say M. Let G_1 be the partial graph of G_0 having $W \cap E(G_0)$ as edge set. Then, M is a perfect matching in G_1 , as well. Since G_1 is a forest, M is unique. By Lemma 2.2, any maximum matching of G_0 is contained in (A, B), and since the edges from (A, B) yield a unique perfect matching, namely M, it follows that M is the unique perfect matching of G_0 itself. Hence, according to Theorem 4.2, we obtain that $\xi(G) + \eta(G) = \alpha(G)$. By Proposition 3.5(*iii*), it implies $\sigma(G) + \eta(G) = \mu(G)$, and immediately $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$.

It is worth observing that if (S, V(G) - S) generates a forest for some $S \in \Omega(G)$, this is not necessarily true for all maximum stable sets of G. For example, the graph G presented in Figure 10(*i*) and Figure 10(*ii*) has the partition $\{S_1 = \{a, b, c, d\} \in \Omega(G), V(G) - S_1\}$ such that $(S_1, V(G) - S_1)$ does not generate a forest, (see Figure 10(*i*)), while for the partition $\{S_2 = \{a, b, y, z\} \in \Omega(G), V(G) - S_2\}$ the set $(S_2, V(G) - S_2)$ generates a forest (see Figure 10(*ii*)). Let us also remark that the converse of Proposition 4.4 is not generally true. For instance, the graph in Figure 10(*ii*) is a counterexample.



Figure 10: König-Egervary graphs satisfying $\xi(G) + \eta(G) = \alpha(G)$.

Corollary 4.5 If T is a tree, then

 $\xi(T) + \eta(T) = \alpha(T), \sigma(T) + \eta(T) = \mu(T), \text{ and } \xi(T) + 2\eta(T) + \sigma(T) = n(T).$

As a consequence of Corollary 4.5, we obtain:

Corollary 4.6 [21] If T is a tree, then a vertex $v \in V(T)$ is in some but not in all maximum stable sets of T if and only if v is an endpoint of an α -critical edge.

Proof. If $v \in V(T)$ is in some but not in all maximum stable sets of T, then there exists $S \in \Omega(T)$ such that $v \in S - core(T)$. By Theorem 2.3, α -critical edges of T form a matching. Proposition 2.6 ensures that they are also μ -critical, because T is bipartite. Consequently, these edges belong to any maximum matching, which, according to Lemma 2.2, is included in (S, V(T) - S). Since, by Lemma 3.4(*i*), no α -critical edge has an endpoint in N[core(T)], and Corollary 4.5 ensures that $\eta(T) = \alpha(T) - \xi(T) = |S - core(T)|$, we infer that v must be an endpoint of an α -critical edge.

Conversely, let e = vw be an α -critical edge in T and $\overline{S} \in \Omega(T - e)$. Since $|\overline{S}| = \alpha(T - e) > \alpha(T)$, it follows that $v, w \in \overline{S}$ and therefore, $\overline{S} - \{v\}, \overline{S} - \{w\} \in \Omega(T)$. Hence, v is in some, namely, in $\overline{S} - \{w\}$, but not in all maximum stable sets of T, namely, not in $\overline{S} - \{v\}$.

Notice that Corollary 4.5 and Corollary 4.6 are not valid for general bipartite graphs (see, for instance, the graph in Figure 9).

5 Conclusions

In this paper we state several properties of α -critical and μ -critical edges belonging to König-Egerváry graphs. These findings generalize some previously known results for trees and bipartite graphs. We have proved that for bipartite graphs and for some special König-Egerváry graphs, their sets of α -critical edges and μ -critical edges coincide. It seems to be interesting to characterize all the graphs having this property. From the other point of view, since the α -critical edges of a König-Egerváry graph span disjoint cliques of order two, one may be interested in describing the type of graphs where their α -critical edges span disjoint cliques of order larger than two. Another challenging problem is to describe classes of non-König-Egerváry graphs G satisfying $\xi(G) + \eta(G) = \alpha(G), \ \xi(G) + \eta(G) \leq \alpha(G), \ and/or \ \alpha(G) + \sigma(G) = \mu(G) + \xi(G).$

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