On α -Critical Edges in König-Egerváry Graphs

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Abstract

The *stability number* of a graph G, denoted by $\alpha(G)$, is the cardinality of a stable set of maximum size in G. If $\alpha(G - e) > \alpha(G)$, then e is an α -critical edge, and if $\mu(G - e) < \mu(G)$, then e is a μ -critical edge, where $\mu(G)$ is the cardinality of a maximum matching in G . G is a König-Egerváry graph if its order equals $\alpha(G) + \mu(G)$. Beineke, Harary and Plummer have shown that the set of α -critical edges of a bipartite graph is a matching. In this paper we generalize this statement to König-Egerváry graphs. We also prove that in a König-Egerváry graph α -critical edges are also μ -critical, and that they coincide in bipartite graphs. Eventually, we deduce that $\alpha(T) = \xi(T) + \eta(T)$ holds for any tree T , and characterize the König-Egerváry graphs enjoying this property, where $\xi(G)$ is the number of α -critical vertices of G, and $\eta(G)$ is the number of α -critical edges of G.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$, edge set $E = E(G)$, and order $n(G) = |V(G)|$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F, and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set ${e = ab : a \in A, b \in B, e \in E}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\},\$ and $N(A) = \bigcup \{N(v) : v \in A\},\ N[A] = A \cup N(A)$ for $A \subset V$.

A set S of vertices is *stable* if no two vertices from S are adjacent. A stable set of maximum size will be referred to as a *maximum stable set* of G. The *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set of G. Let $\Omega(G)$ denotes the set $\{S : S \text{ is a maximum stable set of } G\}, \sigma(G) = |\bigcap \{V - S : S \in \Omega(G)\}\$ and $\xi(G) = |core(G)|$, where $core(G) = \bigcap \{S : S \in \Omega(G)\}\$, [[12\]](#page-13-0). In other words, $\xi(G)$ equals the number of α -critical vertices of G, (a vertex $v \in V(G)$ is α -critical provided $\alpha(G - v) < \alpha(G)$).

By P_n, C_n, K_n we mean the chordless path on $n \geq 3$, the chordless cycle on $n \geq 3$ 4 vertices, and respectively the complete graph on $n \geq 1$ vertices.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G. An edge $e \in E(G)$ is μ -*critical* provided $\mu(G - e) < \mu(G)$. By their definition, μ -critical edges of G belong to all maximum matchings of G .

If $\alpha(G) + \mu(G) = n(G)$, then G is called a *König-Egerváry graph*, [[4\]](#page-12-0), [[17\]](#page-13-0). Properties of these graphs were presented in several papers, like of Sterboul[[17\]](#page-13-0), Deming [[4\]](#page-12-0), Lovász and Plummer [\[14](#page-13-0)],Korach [[8\]](#page-13-0), Bourjolly and Pulleyblank [[2\]](#page-12-0), Paschos and Demange[[16\]](#page-13-0), Levit and Mandrescu[[11\]](#page-13-0), [\[13\]](#page-13-0). It is worth observing that a disconnected graph is of König-Egerváry type if and only if all its connected components are König-Egerváry graphs. In this paper, by "graph" we mean a connected graph having at least one edge.

An edge $e \in E(G)$ is α -*critical* whenever $\alpha(G - e) > \alpha(G)$. Let denote by $\eta(G)$ the number of α-critical edges of G. Notice that there are graphs in which: (*a*) any edge is α -critical (so-called α -critical graphs); e.g., all C_{2n+1} for $n \geq 3$; (b) no edge is α -critical; e.g., all C_{2n} for $n \geq 2$. More generally, Haynes et al., [\[7](#page-13-0)], have proved that a graph G has no α -critical edge if and only if $|N(x) \cap S| \ge 2$ holds for any $S \in \Omega(G)$ and every $x \in V(G) - S$.

Beineke,Harary and Plummer, [[1\]](#page-12-0), have shown that any two incident α -critical edges of a graph lie on an odd cycle, and hence, they deduce that no two α -critical edges of a bipartite graph can have a common endpoint. Independently, Zito,[[21\]](#page-14-0), has proved the same result for trees using a different technique. Some variations and strengthenings of these results are discussed in [\[18](#page-13-0)],[[20\]](#page-14-0), and[[19\]](#page-13-0).

In this paper we generalize the above assertion to König-Egerváry graphs. We also show that α -critical edges are μ -critical in a König-Egerváry graph, and that they coincide in bipartite graphs. As a corollary, we obtain one result of Zito,[[21\]](#page-14-0), stating that a vertex v is in some but not in all maximum stable sets of a tree T if and only if v is an endpoint of an α -critical edge of T. In the sequel, we analyze other relationships between α -critical edges and μ -critical edges in a König-Egerváry graph, and its corresponding implications to equalities and inequalities linking $\alpha(G)$, $\xi(G)$, $\eta(G)$, $\sigma(G)$, and $\mu(G)$. Eventually, we infer that $\alpha(T) = \xi(T) + \eta(T)$, $\sigma(T) + \eta(T) =$ $\mu(T)$ and $\xi(T) + 2\eta(T) + \sigma(T) = n(T)$ holds for any tree T, and characterize the König-Egerváry graphs having these properties.

2 α -Critical and μ -Critical Edges

Accordingto a well-known result of König, $[9]$ $[9]$, and Egerváry, $[5]$ $[5]$, any bipartite graph is a König-Egerváry graph. It is easy to see that this class includes also some nonbipartite graphs (see, for instance, the graph $K_3 + e$ in Figure [1](#page-2-0)).

If $G_i = (V_i, E_i), i = 1, 2$, are two disjoint graphs, then $G = G_1 * G_2$ is defined as the graph with $V(G) = V(G_1) \cup V(G_2)$, and

$$
E(G) = E(G_1) \cup E(G_2) \cup \{xy : for \ some \ x \in V(G_1) \ and \ y \in V(G_2) \}.
$$

Clearly, if H_1, H_2 are subgraphs of a graph G such that $V(G) = V(H_1) \cup V(H_2)$

Figure 1: Graph $K_3 + e$.

and $V(H_1) \cap V(H_2) = \emptyset$, then $G = H_1 * H_2$, i.e., any graph of order at least two admits such decompositions. However, some particular cases are of special interest. For instance, if: $E(H_i) = \emptyset$, $i = 1, 2$, then $G = H_1 * H_2$ is bipartite; $E(H_1) = \emptyset$ and H_2 is complete, then $G = H_1 * H_2$ is a *split graph* [\[6](#page-13-0)].

The following result shows that the König-Egerváry graphs are, in this sense, between these two "extreme" situations. The equivalence of the first and the third parts of this proposition was proposed by Klee and included in[[10\]](#page-13-0) without proof (private communication).

Proposition 2.1 *[\[13](#page-13-0)] The following assertions are equivalent:*

 (i) *G is a König-Egerváry graph;*

(ii) $G = H_1 * H_2$ *, where* $V(H_1) = S \in \Omega(G)$ *and* $n(H_1) \ge \mu(G) = n(H_2)$ *;*

 (iii) $G = H_1 * H_2$, where $V(H_1) = S$ *is a stable set in* $G, |S| \ge n(H_2)$ *and* $(S, V(H_2))$ *contains a matching* M *with* $|M| = n(H_2)$ *.*

In the sequel, we shall often represent a König-Egerváry graph G as $G = S * H$. where $S \in \Omega(G)$ and $H = G[V - S]$ has $n(H) = \mu(G)$.

Lemma 2.2 *[\[13](#page-13-0)]* If $G = (V, E)$ *is a König-Egerváry graph, then any maximum matching of* G *is contained in* $(S, V - S)$ *, where* $S \in \Omega(G)$ *.*

Clearly, Lemma 2.2 is not valid for any graph. For instance, K_4 is a counterexample. Moreover, K_4 has α -critical edges that are incident. Nevertheless, there are graphs having only non-incident α -critical edges.

Theorem 2.3 If G is a König-Egerváry graph, then the following assertions hold:

- (i) for any α -critical edge e of G, the graph $G e$ is still a König-Egerváry graph;
- *(ii)* any α -critical edge of G is also μ -critical;
- (iii) the α -critical edges of G form a matching.

Proof. (*i*) If $e = xy$ is an α -critical edge G, then there is some $S \in \Omega(G)$ such that either $N(x) \cap S = \{y\}$ or $N(y) \cap S = \{x\}$. Suppose that $y \in S$. Since $S \in \Omega(G)$, we get, by Proposition 2.1, that $G = S * H$, where $H = G[V - S]$ has $\mu(G) = n(H) = |M|$ and M is a maximum matching of G, included, by Lemma 2.2, in $(S, V(G) - S)$. Hence, it follows that $G - e = S' * V(H')$, where $S' = S \cup \{x\} \in \Omega(G - e)$ and $n(H') = |M - \{e\}|$. According to Proposition 2.1(*iii*), we infer that $G - e$ is also a König-Egerváry graph.

(*ii*) If $e \in E(G)$ is an α -critical edge of G, then according to (*i*) we obtain:

$$
n(G) = \alpha(G) + \mu(G) \le \alpha(G - e) + \mu(G - e) = \alpha(G) + 1 + \mu(G - e) = n(G - e),
$$

and this implies $\mu(G) = 1 + \mu(G - e)$, i.e., e is also μ -critical.

(*iii*) Let e_1, e_2 be two α -critical edges of G. We have to show that they are not incident. According to second part (ii) , both edges are also μ -critical. Hence, it follows that $e_1, e_2 \in \bigcap \{M : M \text{ is a maximum matching of } G\}$ and this ensures that e_1, e_2 have no common endpoint. Consequently, the set of all α -critical edges of G yields a matching. \blacksquare

Notice that:

(*a*) Theorem [2.3](#page-2-0)(*i*) is not true for any μ -critical edge of a König-Egerváry graph; e.g., the edge e of $G = K_3 + e$ is μ -critical, but $G - e$ is not a König-Egerváry graph; (*b*) Theorem [2.3\(](#page-2-0)*ii*) is not true for any graph; e.g., all the edges of K_3 are α -critical,

but none is also μ -critical;

 (c) the converse of Theorem [2.3](#page-2-0)(*ii*) is not valid for any König-Egerváry graph; e.g., the edge e of graph $K_3 + e$ is μ -critical, but is not also α -critical. However, as we shall see later, (namely Proposition 2.6), the μ -critical edges are also α -critical in the case of bipartite graphs.

Corollary 2.4 *A K¨onig-Egerv´ary graph is* α*-critical if and only if it is isomorphic to* K2*.*

Since any bipartite graph is also a König-Egerváry graph, we obtain the following statement, due to Beineke, Harary and Plummer.

Theorem 2.5 *[\[1](#page-12-0)] No two* α -critical edges of a bipartite graph are incident.

Proposition 2.6 *If* G *is a bipartite graph, then its* α*-critical edges coincide with its* µ*-critical edges.*

Proof. By Theorem [2.3\(](#page-2-0)*ii*), it suffices to show that any μ -critical edge e of G is also α -critical. Since $G - e$ is still bipartite, and hence, also a König-Egerváry graph, it follows that $\alpha(G-e) + \mu(G-e) = n(G) = \alpha(G) + \mu(G) = \alpha(G) + 1 + \mu(G-e)$, and this implies $\alpha(G - e) > \alpha(G)$, i.e., e is an α -critical edge of G.

In Theorem [4.2](#page-9-0) we will meet another type of König-Egerváry graphs with this property. Notice that there are also non-bipartite König-Egerváry graphs in which their μ -critical edges are α -critical (see the graph in Figure 2).

Figure 2: A Koenig-Egervary graph whose all μ -critical edges are α -critical.

It is well-known that if a tree has a perfect matching, then it is unique. Consequently, we obtain:

Corollary 2.7 *A tree has a perfect matching if and only if the set of its* α -critical *edges forms a maximal matching of the tree.*

Using the definition of König-Egerváry graphs and the fact that $\mu(G) \leq n(G)/2$ is true for any graph G , we get:

Lemma 2.8 If G admits a perfect matching, then G is a König-Egerváry graph if *and only if* $\alpha(G) = \mu(G)$ *. If* G *is a König-Egerváry graph, then* $\mu(G) \leq \alpha(G)$ *.*

Combining Corollary [2.7](#page-3-0) and Lemma 2.8, we get the following result from[[21\]](#page-14-0).

Corollary 2.9 *[\[21](#page-14-0)] If a tree* T *has a perfect matching* M*, then all the edges of* M *are* α -*critical and* $2\alpha(T) = n(T)$ *.*

Proposition 2.10 If $G = (V, E)$ is a König-Egerváry graph, then the following as*sertions are true:*

(i) any $S \in \Omega(G)$ meets each *µ*-critical edge in exactly one vertex;

(ii) any $S \in \Omega(G)$ meets each α -critical edge in exactly one vertex;

(iii) if G has a maximal matching consisting of only α -critical edges, then it is the *unique perfect matching of* G*.*

Proof. (*i*) *and* (*ii*) By Theorem [2.3\(](#page-2-0)*ii*), any α -critical edge of G is also μ -critical. Consequently, we infer that

 ${e \in E : e \text{ is } \alpha-critical} \subseteq \bigcap \{M : M \text{ is a maximum matching of } G\} \subseteq (S, V-S)$

holds for any $S \in \Omega(G)$, according to Lemma [2.2](#page-2-0). It follows that if $e = xy$ is an α -critical or a μ -critical edge of G, then any $S \in \Omega(G)$ contains one of x and y, (since clearly, no stable set may contain both x and y).

 (iii) Let M be a maximal matching of G consisting of only α -critical edges. By Theorem [2.3](#page-2-0), all the edges of M are also μ -critical. Therefore, we infer that M is included in any maximum matching of G , and because M is a maximal matching, it results that M is the unique maximum matching of G . Suppose, on the contrary, that M is not perfect, and let $S \in \Omega(G)$. According to Proposition [2.1](#page-2-0), G can be written as $G = S * H$, with $n(H) = |M| = \mu(G)$, and by Lemma [2.2](#page-2-0) we have that $M \subseteq (S, V - S)$. Since G is a König-Egerváry graph without perfect matchings, Lemma 2.8 implies $|S| = \alpha(G) > \mu(G) = |M|$. Hence, it follows that there are at least two vertices $v_1, v_2 \in S$ having a common neighbor $w \in V(H)$ and such that one of them, say v_1 , is unmatched by M and $v_2w \in M$. Thus, $M \cup \{v_1w\} - \{v_2w\}$ is another maximum matching of G , in contradiction with the uniqueness of M . Consequently, M must be also perfect. \blacksquare

For trees, Proposition 2.10(*ii*) was proved by Zito in[[21\]](#page-14-0).

Notice that the matching in Proposition 2.10(*iii*) is not necessarily formed by pendant edges; e.g., P_6 has such a matching. Concerning the uniqueness of this matching, it is worth mentioning that: (a) if G is not a König-Egerváry graph, then it may have several different maximum matchings consisting of only α-critical edges (e.g., C_5); (*b*) if a König-Egerváry graph has a unique perfect matching, then it may contain non- α -critical edges (e.g., the edge e of $K_3 + e$ is not α -critical, but it belongs to the unique perfect matching of $K_3 + e$.

3 Equalities and Inequalities between Parameters

If $v \in N(core(G))$, then clearly follows that $v \in V(G) - S$, for any $S \in \Omega(G)$, that is $N(core(G)) \subseteq \bigcap \{V - S : S \in \Omega(G)\}\$ holds for any graph G.

Lemma 3.1 [\[13](#page-13-0)] If $G = (V, E)$ is a König-Egerváry graph, then $N(core(G)) = \bigcap \{V - S : S \in \Omega(G)\}.$

Notice that there are graphs that do not enjoy the above equality, for example, the graph G in Figure 3(*a*) has $N(\text{core}(G)) = \emptyset$ and $\bigcap \{V - S : S \in \Omega(G)\} = \{v\}$. There exist non-König-Egerváry graphs for which $N(core(G)) = \bigcap \{V - S : S \in \Omega(G)\}\)$, (see, for instance, the graph G from Figure $3(b)$).

Figure 3: (a) G is non-König-Egerváry with $N(core(G)) \neq \bigcap \{V - S : S \in \Omega(G)\}\;$ (b) G is a non-König-Egerváry graph with $N(core(G)) = \bigcap \{V - S : S \in \Omega(G)\}.$

Proposition 3.2 *If* $G = (V, E)$ *is a König-Egerváry graph,* $G_0 = G - N[core(G)]$ and $S \in \Omega(G)$, then the following assertions are true:

 (i) $|core(G)| \geq |N(core(G))|$;

 (iii) $|S - core(G)| = |V - S - N(core(G))|;$

 $(iii) G_0$ has a perfect matching and it is also a König-Egerváry graph.

Proof. According to Proposition [2.1](#page-2-0), G can be written as $G = S * H$, where H $G[V-S]$ has $n(H) = \mu(G)$. Let denote $A = S - core(G)$ and $B = V(H) - N(core(G))$. In[[12\]](#page-13-0) it has been proved that $|A| \leq |B|$ holds for any graph G. Since $\bigcap \{V - S : S \in$ $\Omega(G) \subseteq V(H)$, and $N(core(G)) = \bigcap \{V - S : S \in \Omega(G)\}\$ (see Lemma 3.1), we obtain $B = V(H) - \bigcap \{V - S : S \in \Omega(G)\}.$

(*i*) Since $|A| + |core(G)| = \alpha(G) \ge \mu(G) = n(H) = |B| + |N(core(G)|$ and, on the other hand $|A| \leq |B|$, it follows that $|core(G)| \geq |N(core(G))|$.

 (ii) Let M be a maximum matching in G. Since G is a König-Egerváry graph, Lemma [2.2](#page-2-0) ensures that M is included in $(S, V(H))$, and $|M| = \mu(G) = n(H)$. The matching M matches B into A , because there are no edges connecting B and $core(G)$. Hence, $|B| \leq |A|$. Together with $|A| \leq |B| |A| \leq |B|$, it implies $|A| = |B|$, i.e., $|S - core(G)| = |V - S - N(core(G))|$, and that $M \cap (A, B)$ is a perfect matching of $G[A \cup B]$.

(*iii*) Since, in fact, $G_0 = G[A \cup B]$, it follows necessarily that G_0 has a perfect matching. In addition, because A is stable, we get $\alpha(G_0) \leq \mu(G_0) = |A| \leq \alpha(G_0)$, i.e., $\alpha(G_0) = \mu(G_0)$, and according to Lemma [2.8](#page-4-0), G_0 must be also a König-Egerváry graph.

Corollary 3.3 *If* G *is a König-Egerváry graph, then* $\alpha(G) + \sigma(G) = \mu(G) + \xi(G)$ *.*

Proof. By Lemma [3.1](#page-5-0), $N(core(G)) = \bigcap \{V - S : S \in \Omega(G)\}\$ and according to Proposition [3.2](#page-5-0)(*ii*), $|S - core(G)| = |V - S - N(core(G))|$. Hence, we obtain that $\alpha(G) - \xi(G) = |S - core(G)| = |V - S - N(core(G))| = \mu(G) - \sigma(G).$

Let us observe that there exist non-König-Egerváry graphs satisfying the equality $\alpha(G) + \sigma(G) = \mu(G) + \xi(G)$ (see graph W_1 in Figure [6\)](#page-7-0). It is also interesting to notice that there exists a non-König-Egerváry graph enjoying the property that its subgraph $G_0 = G - N[core(G)]$ has a perfect matching (see Figure [8\)](#page-10-0). Figure 4 shows a non-König-Egerváry graph G whose G_0 has no perfect matching.

Figure 4: G is a non-König-Egervary graph with $core(G) = \{a, b\}$ and $G_0 = C_5$.

Lemma 3.4 *Let* $G = (V, E)$ *and* $G_0 = G - N[core(G)]$ *. Then the following assertions are valid:*

(i) no α -critical edge in G has an endpoint in $N[\text{core}(G)]$;

 $(iii) \alpha(G) = \alpha(G_0) + \xi(G), \Omega(G_0) = \{S \cap V(G_0) : S \in \Omega(G)\}, core(G_0) = \emptyset;$

(iii) $e = xy$ *is an* α -critical edge of G if and only if e is an α -critical edge of G_0 .

Proof. (*i*) Let $e = xy$ be an α -critical edge in G, and let $\overline{S} \in \Omega(G - e)$. Since $|\overline{S}| = \alpha(G - e) > \alpha(G)$, it follows that $x, y \in \overline{S}$ and $\overline{S} - \{x\}, \overline{S} - \{y\} \in \Omega(G)$. Now, the inclusion $N(core(G)) \subseteq \bigcap \{V - S : S \in \Omega(G)\}\)$ completes the proof that no α -critical edge in G has an endpoint in $N(core(G))$, and respectively, in $core(G)$.

(*ii*) By definition of G_0 , if $S \in \Omega(G)$, then $S - core(G) = S \cap V(G_0)$, and therefore

$$
\alpha(G) - \xi(G) = |S - core(G)| \le \alpha(G_0).
$$

For any $S_{G_0} \in \Omega(G_0)$ we have that $S_{G_0} \cup core(G)$ is stable, and hence

$$
|S_{G_0} \cup core(G)| = \alpha(G_0) + \xi(G) \le \alpha(G).
$$

Consequently, we get $\alpha(G) = \alpha(G_0) + \xi(G)$. Now it is easy to check that $\Omega(G_0)$ $\{S \cap V(G_0) : S \in \Omega(G)\}\$ and $core(G_0) = \emptyset$.

(*iii*) Let $e = xy$ be an α -critical edge of G. By (*i*), we infer that $e \in E(G_0)$, and as we saw above, there is some stable set S_{xy} such that $S_{xy} \cup \{x\}$, $S_{xy} \cup \{y\} \in \Omega(G)$ and $S_{xy} \cup \{x, y\} \in \Omega(G - e)$. Hence, *(ii)* implies that

$$
V(G_0) \cap (S_{xy} \cup \{x\}), V(G_0) \cap (S_{xy} \cup \{y\}) \in \Omega(G_0) \text{ and } V(G_0) \cap (S_{xy} \cup \{x,y\}) \in \Omega(G_0 - e),
$$

because $V(G_0) \cap (S_{xy} \cup \{x,y\})$ is stable in $G_0 - e$ and larger than $V(G_0) \cap (S_{xy} \cup \{x\})$. Therefore, e is α -critical in G_0 , as well. Similarly, we can show that any α -critical edge of G_0 is α -critical in G too.

Proposition 3.5 If G is a König-Egerváry graph, then

 $(i) \mathcal{E}(G) + \eta(G) \leq \alpha(G)$; (iii) $\sigma(G) + \eta(G) \leq \mu(G);$ *(iii)* $\xi(G) + 2\eta(G) + \sigma(G) \leq n(G)$.

Proof. For any $S \in \Omega(G)$, we have that $core(G) \subseteq S$, and by Lemma [3.4](#page-6-0)(*i*), no α -critical edge has an endpoint in $core(G)$. In addition, according to Proposition $2.10(ii)$ $2.10(ii)$, S meets each α -critical edge in exactly one vertex. Hence, it follows that $\xi(G) + \eta(G) \leq \alpha(G)$, and using Corollary [3.3](#page-5-0) we obtain *(ii)*. Clearly, *(iii)* follows from (i) and (ii) .

Notice that $\xi(K_3+e) + \eta(K_3+e) = \alpha(K_3+e)$ and also $\eta(K_3+e) + \sigma(K_3+e) =$ $\mu(K_3 + e)$, but there are König-Egerváry graphs satisfying $\xi(G) + \eta(G) < \alpha(G)$ and $\eta(G) + \sigma(G) < \mu(G)$. For instance, $G = C_6$, and also the graph W in Figure 5 is a König-Egerváry non-bipartite graph that has $\eta(W) = |\{e\}| = 1, \xi(W) = |\{a\}| = 1 =$ $\sigma(W), \alpha(W) = \mu(W) = 4.$

Figure 5: W is a non-bipartite Koenig-Egervary graph and $\xi(W) + \eta(W) < \alpha(W)$.

Observe that Proposition [3.5](#page-6-0) is not true for general graphs; e.g., the graph W_1 in Figure 6 has $\alpha(W_1) = 3, \mu(W_1) = 2, \eta(W_1) = 3, \xi(W_1) = 2, \sigma(W_1) = 1$. However, there are non-König-Egerváry graphs satisfying $\xi(G) + \eta(G) < \alpha(G)$ and $\eta(G)$ + $\sigma(G) < \mu(G)$, for example, the graph W_2 in Figure 6 has $\alpha(W_2) = 3, \eta(W_2) = 1$ $|\{ab, cd\}|$, $\xi(W_2) = \sigma(W_2) = 0$. There also exist non-König-Egerváry graphs satisfying $\xi(G) + \eta(G) = \alpha(G)$ and $\eta(G) + \sigma(G) = \mu(G)$, e.g., the graph W_3 in Figure 6. Nevertheless, $\xi(K_5-e)+\eta(K_5-e)=\alpha(K_5-e)$, but $\eta(K_5-e)+\sigma(K_5-e)>\mu(K_5-e)$.

Figure 6: Non-Koenig-Egervary graphs.

Proposition 3.6 If G is a König-Egerváry graph, then the following assertions are *equivalent:*

 $(i) \xi(G) + \eta(G) = \alpha(G);$ $(iii) \sigma(G) + \eta(G) = \mu(G);$ *(iii)* $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$ *.* **Proof.** Suppose that $\xi(G) + \eta(G) = \alpha(G)$. According to Corollary [3.3](#page-5-0), we get that $\mu(G) = \alpha(G) + \sigma(G) - \xi(G) = \xi(G) + \eta(G) + \sigma(G) - \xi(G) = \eta(G) + \sigma(G)$. The converse is proven in the same way.

Suppose $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$. Proposition [3.5](#page-6-0) claims that $\xi(G) + \eta(G) \leq$ $\alpha(G)$ and $\sigma(G) + \eta(G) \leq \mu(G)$. Together with $\alpha(G) + \mu(G) = n(G)$, which is true for König-Egerváry graphs, it gives us the two equalities needed. Conversely, if, for instance, $\xi(G) + \eta(G) = \alpha(G)$ then, as we already proved, $\sigma(G) + \eta(G) = \mu(G)$. Summing these two equalities we obtain $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$.

4 König-Egerváry Graphs for which $\xi + \eta = \alpha$

Lemma 4.1 *Let* G *be a König-Egerváry graph and* $G_0 = G - N[core(G)]$ *. If* G_0 *has a unique perfect matching then its* α*-critical edges coincide with its* µ*-critical edges.*

Proof. By Theorem [2.3,](#page-2-0) it is enough to show that all the edges of M (the unique perfect matching of G_0) are also α -critical.

According to Proposition [2.1,](#page-2-0) we may write G as $G = S * H$, where $S \in \Omega(G)$ and $H = G[V - S]$ has $n(H) = \mu(G)$. By virtue of Lemma [3.4](#page-6-0)(*ii*), G_0 has $\alpha(G_0)$ $|S-core(G)| = q$ and $core(G_0) = \emptyset$. Let $M = \{a_i b_i : 1 \leq i \leq q\}$ and suppose that $\{a_i: 1 \leq i \leq q\} = A \subseteq S$. We shall show that any $a_i b_i \in M$ is α -critical, by exhibiting a maximum stable set S_0 in G_0 that satisfies: $b_i \in S_0$ and $S_0 \cap N(a_i) = \{b_i\}$. For the sake of simplicity, let us take $i = 1$. In the sequel, if $D \subseteq V(G_0)$, then by $M(D)$ we mean the set of vertices, which D is matched onto.

Claim 1. There exists some $S_0 \in \Omega(G_0)$ with $b_1 \in S_0$.

Otherwise, any $W \in \Omega(G_0)$ contains a_1 , because $|M| = |W|$ and $|W \cap \{a_i, b_i\}| = 1$ holds for every $j \in \{1, 2, ..., q\}$. Hence, it follows that $a_1 \in core(G_0)$, in contradiction with $core(G_0) = \emptyset$.

Claim 2. The following procedure gives rise to some $S_0 \in \Omega(G_0)$ that contains b_1 .

Input: G_0 , $A = \{a_1, a_2, ..., a_q\}$, $b_1 \in B = \{b_1, b_2, ..., b_q\} = M(A);$ **Output:** $b_1 \in S_0 \in \Omega(G_0);$ $S_0 := \{b_1\};$ $D := \{b_1\};$ while $(N(D) \cap A) - M(S_0) \neq \emptyset$ do begin Step 1. $S_1 := S_0$; Step 2. $S_0 := S_0 \cup M((N(D) \cap A) - M(S_0));$ Step 3. $D := S_0 - S_1$; end Step 4. $S_0 := S_0 \cup M(B - S_0)$.

Clearly, $|S_0| = q$ and no edge of G_0 joins some $a_l \in S_0$ to any $b_i \in S_0$, according to building procedure of S_0 . Any maximum stable set $W \in \Omega(G_0)$ that contains b_1 must contain also all $b_j \in S_0$, because $|W| = |M|$ and $|W \cap \{a_j, b_j\}| = 1$ holds for every $j \in \{1, 2, ..., q\}$. Hence, the set $\{b_j : b_j \in S_0\}$ is stable, and consequently, we

obtain that $S_0 \in \Omega(G_0)$. An example of $S_0 \in \Omega(G_0)$ obtained by this procedure is illustrated in Figure 7.

Figure 7: The graph G_0 has a unique perfect matching and $\xi(G_0) = 0$.

Claim 3. $S_0 \cup \{a_1\} \in \Omega(G_0 - a_1b_1)$, and hence, the edge a_1b_1 is α -critical in G_0 . Firstly, no $a_i \in S_0$ is adjacent to a_1 , because $a_i, a_1 \in A$. Secondly, no $b_j \in S_0 - \{b_1\}$ is adjacent to a_1 , otherwise there exists an even cycle C, with half of its edges belonging to M, which means that $(M - E(C)) \cup (E(C) - M)$ is another perfect matching in G_0 , in contradiction with the premises on G_0 . Therefore, $S_0 \cup \{a_1\} \in \Omega(G_0 - a_1b_1)$ and this implies that the edge a_1b_1 is α -critical in G_0 . Since a_1b_1 is an arbitrary edge of M, we may conclude that all the edges of M are α -critical in G_0 .

It is interesting to notice that if G_0 were bipartite for every König-Egerváry graph G, then it would be possible to prove Lemma [4.1](#page-8-0) using only Proposition [2.6](#page-3-0). Figures [2,](#page-3-0) 7 show that Proposition [2.6](#page-3-0) is not enough for our purposes, because there exist non-bipartite König-Egerváry graphs G with nonempty cores and whose $G_0 = G N[core(G)]$ have a unique perfect matching.

Theorem 4.2 *Let* G *be a König-Egerváry graph and* $G_0 = G - N[core(G)]$ *. Then the following assertions are equivalent:*

 (i) G_0 *has a unique perfect matching;*

(ii) α -critical edges of G_0 form a maximal matching in G_0 ;

 $(iii) \xi(G) + \eta(G) = \alpha(G);$ $(iv) \sigma(G) + \eta(G) = \mu(G);$

 $(v) \xi(G) + 2\eta(G) + \sigma(G) = n(G)$.

Proof. According to Proposition [3.2](#page-5-0), G_0 is also a König-Egerváry graph and has a perfect matching, say M_0 .

 $(i) \Leftrightarrow (ii)$ If M_0 is the unique perfect matching of G_0 , all its edges are μ -critical and, by Lemma [4.1,](#page-8-0) α -critical, as well. In other words, the α -critical edges of G_0 form a maximal matching. The converse is true according to Proposition [2.10\(](#page-4-0)*iii*).

 $(i) \Rightarrow (iii)$ Assume that M_0 is the unique perfect matching of G_0 . By Lemma [3.4\(](#page-6-0)*ii*), it follows that $\alpha(G_0) = \alpha(G) - \xi(G)$. Lemma [3.4](#page-6-0)(*iii*) and the uniqueness of M imply that $\alpha(G_0) = \eta(G_0) = \eta(G)$. Hence, it results in $\xi(G) + \eta(G) = \alpha(G)$.

 $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ It is the claim of Proposition [3.6.](#page-7-0)

 $(v) \Rightarrow (ii)$ By Proposition [3.1,](#page-5-0)

 $|N(core(G))| = |\bigcap \{V - S : S \in \Omega(G)\}| = \sigma(G).$

Hence, $n(G_0) = n(G) - \xi(G) - \sigma(G)$. Now, our premise claims that $2\eta(G) = n(G_0)$. By Lemma 3.4(*iii*) we obtain $2\eta(G_0) = n(G_0)$. According to Theorem [2.3\(](#page-2-0)*iii*) the set of α -critical edges of G form a matching, say M. Applying again Lemma 3.4(*iii*), we see that $M_0 = M$ and it consists of α -critical edges of G_0 .

Notice that Theorem [4.2](#page-9-0) fails for non-König-Egerváry graphs. In Figure 8 is presented a non-König-Egerváry graph G having $\xi(G) = |\{v\}| = 1, \eta(G) = 10$, (all the edges of the two C_5 are α -critical), $\alpha(G) = 5 < \mu(G) = 6$, but $G_0 = G - N[core(G)]$ owns a unique perfect matching.

Figure 8: A non-Koenig-Egervary graph satisfying $\xi(G) + \eta(G) < \alpha(G)$.

Now using Theorem [4.2](#page-9-0) we are giving a new characterization of the bipartite graphs that have a unique perfect matching (see some previous discussions of this topic in [\[3](#page-12-0)] and [\[15](#page-13-0)]). This result generalizes Corollary [2.9](#page-4-0).

Corollary 4.3 *Let* G *be a bipartite graph. Then the following assertions are equivalent:*

(i) G *has a unique perfect matching;* (ii) α -critical edges of *G* form a maximal matching; (iii) $\eta(G) = \alpha(G)$; *(iv)* $\eta(G) = \mu(G)$ *;* (v) 2 $\eta(G) = n(G)$.

Proof. (*i*) \Leftrightarrow (*ii*) If M is the unique perfect matching of G, all its edges are μ -critical and, by Proposition [2.6,](#page-3-0) α -critical, as well. In other words, the α -critical edges of G form a maximal matching. The converse is true according to Proposition [2.10](#page-4-0)(*iii*).

The other equivalences follow from Theorem [4.2,](#page-9-0) and the observation that if a bipartite graph has a perfect matching, then the two stable sets of its standard partition are maximum, and, consequently, $\xi(G) = 0$.

It is interesting to notice that the equality $2\alpha(G) = n(G)$ mentioned in Corollary [2.9](#page-4-0) follows from Corollary 4.3, but it can not join the above series of equivalences (see, for example, C_4).

Let us also observe that for the bipartite graph G in Figure [9,](#page-11-0) the subgraph $G_0 = G - N[core(G)]$ has more than one perfect matching.

Proposition 4.4 *If* G *is a König-Egervary graph and there is some* $S \in \Omega(G)$ *such that the set* $W = (S, V(G) - S)$ generates a forest, then

$$
\xi(G) + \eta(G) = \alpha(G), \sigma(G) + \eta(G) = \mu(G), \text{ and } \xi(G) + 2\eta(G) + \sigma(G) = n(G).
$$

Proof. If $G_0 = G - N[core(G)], A = S - core(G), B = V(G) - S - N(core(G)),$ then Proposition $3.2(iii)$ implies that G_0 is also a König-Egerváry graph and has a perfect matching, say M. Let G_1 be the partial graph of G_0 having $W \cap E(G_0)$ as edge set. Then, M is a perfect matching in G_1 , as well. Since G_1 is a forest, M is unique. By Lemma [2.2](#page-2-0), any maximum matching of G_0 is contained in (A, B) , and since the edges from (A, B) yield a unique perfect matching, namely M, it follows that M is the unique perfect matching of G_0 itself. Hence, according to Theorem [4.2,](#page-9-0) we obtain that $\xi(G) + \eta(G) = \alpha(G)$. By Proposition [3.5\(](#page-6-0)*iii*), it implies $\sigma(G) + \eta(G) = \mu(G)$, and immediately $\xi(G) + 2\eta(G) + \sigma(G) = n(G)$.

It is worth observing that if $(S, V(G) - S)$ generates a forest for some $S \in \Omega(G)$, this is not necessarily true for all maximum stable sets of G . For example, the graph G presented in Figure 10(*i*) and Figure 10(*ii*) has the partition $\{S_1 = \{a, b, c, d\} \in$ $\Omega(G), V(G) - S_1$ such that $(S_1, V(G) - S_1)$ does not generate a forest, (see Figure 10(*i*)), while for the partition $\{S_2 = \{a, b, y, z\} \in \Omega(G), V(G) - S_2\}$ the set $(S_2, V(G) - S_2)$ S_2) generates a forest (see Figure 10(*ii*)). Let us also remark that the converse of Proposition [4.4](#page-10-0) is not generally true. For instance, the graph in Figure 10(*iii*) is a counterexample.

Figure 10: König-Egervary graphs satisfying $\xi(G) + \eta(G) = \alpha(G)$.

Corollary 4.5 *If* T *is a tree, then*

 $\xi(T) + \eta(T) = \alpha(T), \sigma(T) + \eta(T) = \mu(T), \text{ and } \xi(T) + 2\eta(T) + \sigma(T) = n(T).$

As a consequence of Corollary 4.5, we obtain:

Corollary 4.6 [\[21](#page-14-0)] If T is a tree, then a vertex $v \in V(T)$ is in some but not in all *maximum stable sets of* T *if and only if* v *is an endpoint of an* α*-critical edge.*

Proof. If $v \in V(T)$ is in some but not in all maximum stable sets of T, then there exists $S \in \Omega(T)$ such that $v \in S - core(T)$. By Theorem [2.3](#page-2-0), α -critical edges of T form a matching. Proposition [2.6](#page-3-0) ensures that they are also μ -critical, because T is bipartite. Consequently, these edges belong to any maximum matching, which, according to Lemma [2.2,](#page-2-0) is included in $(S, V(T) - S)$. Since, by Lemma [3.4\(](#page-6-0)*i*), no α -critical edge has an endpoint in $N[core(T)]$, and Corollary [4.5](#page-11-0) ensures that $\eta(T) = \alpha(T) - \xi(T) = |S - core(T)|$, we infer that v must be an endpoint of an α -critical edge.

Conversely, let $e = vw$ be an α -critical edge in T and $\overline{S} \in \Omega(T - e)$. Since $\overline{S} = \alpha(T - e) > \alpha(T)$, it follows that $v, w \in \overline{S}$ and therefore, $\overline{S} - \{v\}, \overline{S} - \{w\} \in \Omega(T)$. Hence, v is in some, namely, in $\overline{S} - \{w\}$, but not in all maximum stable sets of T, namely, not in $\overline{S} - \{v\}$. ■

Notice that Corollary [4.5](#page-11-0) and Corollary [4.6](#page-11-0) are not valid for general bipartite graphs (see, for instance, the graph in Figure [9](#page-11-0)).

5 Conclusions

In this paper we state several properties of α -critical and μ -critical edges belonging to König-Egerváry graphs. These findings generalize some previously known results for trees and bipartite graphs. We have proved that for bipartite graphs and for some special König-Egerváry graphs, their sets of α -critical edges and μ -critical edges coincide. It seems to be interesting to characterize all the graphs having this property. From the other point of view, since the α -critical edges of a König-Egerváry graph span disjoint cliques of order two, one may be interested in describing the type of graphs where their α -critical edges span disjoint cliques of order larger than two. Another challenging problem is to describe classes of non-König-Egerváry graphs G satisfying $\xi(G) + \eta(G) = \alpha(G), \xi(G) + \eta(G) \leq \alpha(G), \text{ and/or } \alpha(G) + \sigma(G) = \mu(G) + \xi(G).$

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