An automorphic form related to cubic surfaces. First draft. 1997.

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(Remark added 2000: See the paper "Cubic Surfaces and Borcherds Products" by Allcock and Freitag, math.AG/0002066)

In the paper [A-C-T] the authors showed that the moduli space of cubic surfaces was  $(CH^4 \setminus H)/G$ , where H is the union of the reflection hyperplanes of the reflection group G. The purpose of this note is to construct an automorphic form, called the discriminant, on complex hyperbolic space  $CH^4$  whose zeros are exactly the reflection hyperplanes of G, each with multiplicity 1.

The complex hyperbolic space of  $E^{1,4}$  embeds naturally in the Grassmannian of 2 dimensional positive definite subspaces of the underlying even integral lattice M of  $E^{1,4}$ . We will construct the automorphic form on complex hyperbolic space by constructing an automorphic form  $\Psi$  on the Grassmannian and restricting it to complex hyperbolic space. The construction of  $\Psi$  is similar to that of example 13.7 of [B]. The lattice M is isomorphic to  $A_2 \oplus A_2^4(-1)$ , so it has dimension 10 and determinant  $3^5$ . We let the elements  $e_{\gamma}$  for  $\gamma \in M'/M$  stand for the obvious basis of the group ring C[M'/M]. We will construct a modular form of weight -3 and type  $\rho_M$  for  $SL_2(Z)$  which is holomorphic on the upper half plane and meromorphic at cusps, by which we mean a holomorphic vector valued function  $f = \sum_{\gamma \in M'/M} e_{\gamma} f_{\gamma}(\tau)$  such that

$$f_{\gamma}(\tau+1) = e^{\pi i \gamma^2} f_{\gamma}(\tau), \qquad f_{\gamma}(-1/\tau) = -i3^{-5/2} \tau^{-3} \sum_{\delta \in M'/M} e^{-2\pi i (\delta,\gamma)} f_{\delta}(\tau)$$

Recall that if  $\gamma, \delta \in M'/M$  then  $\gamma^2$  is well defined mod 2 and  $(\gamma, \delta)$  is well defined mod 1. There are 4 orbits of vectors  $\gamma \in M'/M$  under Aut(M), which we name as follows: a nonzero vector  $\gamma$  with  $\gamma^2/2 \equiv n/3 \mod 1$  will be called a vector of type n (n = 0, 1, 2), and the zero vector will be called a vector of type 00.

We will construct a modular form f such that the components  $f_{\gamma}$  of f are given by functions  $f_{00}$ ,  $f_0$ ,  $f_1$ , or  $f_2$  depending only on the type of  $\gamma$ . We now work out the conditions that these functions have to satisfy for f to transform correctly. In order to check that f is a modular form under  $\tau \mapsto -1/\tau$  we need to know, given some fixed vector u, how many vectors v there are with given type and given inner product with u. These numbers are given in the following table.

Using this table we see that  $f_{00}$ ,  $f_0$ ,  $f_1$ , and  $f_2$  have to satisfy the equations

$$\begin{split} f_{00}(\tau+1) &= f_{00}(\tau), \quad f_{00}(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau)+80f_0(\tau)+90f_1(\tau)+72f_2(\tau)) \\ f_0(\tau+1) &= f_0(\tau), \quad f_0(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau)-f_0(\tau)+9f_1(\tau)-9f_2(\tau)) \\ f_1(\tau+1) &= e^{2\pi i/3}f_1(\tau), \quad f_1(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau)+8f_0(\tau)-9f_1(\tau)) \\ f_2(\tau+1) &= e^{4\pi i/3}f_2(\tau), \quad f_2(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau)-10f_0(\tau)+9f_2(\tau)) \end{split}$$

One solution of these equations is given as follows.

$$f_{00}(\tau) = 24\eta(3\tau)^{3}\eta(\tau)^{-9} = 24(1+9q+54q^{2}+O(q^{3}))$$
  

$$f_{0}(\tau) = -3\eta(3\tau)^{3}\eta(\tau)^{-9} = -3+O(q)$$
  

$$f_{1}(\tau) = 0$$
  

$$f_{2}(\tau) = \eta(\tau/3)^{3}\eta(\tau)^{-9} + 3\eta(3\tau)^{3}\eta(\tau)^{-9} = q^{-1/3} + 14q^{2/3} + 92q^{5/3} + O(q^{8/3})$$

Most of the transformations follow formally from the functional equations  $\eta(\tau + 1) = e^{2\pi i/24}\eta(\tau)$  and  $\eta(-1/\tau) = \sqrt{\tau/i\eta(\tau)}$  of  $\eta$ . The only one which takes slightly more work is the transformation of  $f_2$  under  $\tau \mapsto \tau + 1$ , and this follows from the identity  $\eta(\tau)^3 = \sum_{n \in \mathbb{Z}} (4n+1)q^{(4n+1)^2/8}$  and its consequence

$$\eta(\tau/3)^3\eta(\tau)^{-9} + \eta((\tau+1)/3)^3\eta(\tau+1)^{-9} + \eta((\tau+2)/3)^3\eta(\tau+2)^{-9} = -9\eta(3\tau)^3\eta(\tau)^{-9}.$$

By theorem 13.3 of [B] there is an automorphic form  $\Psi$  on the symmetric space of Mwith the following properties. It has weight  $12 = (\text{coefficient of } q^0 \text{ in } f_{00})/2$ . The zeros of  $\Psi$  correspond to the negative powers of q in f, so are zeros of order 1 orthogonal to all the norm -2/3 vectors of M'.  $\Psi$  is holomorphic on the symmetric space, and therefore holomorphic at cusps as well by the Koecher boundedness principle.  $\Psi$  is an automorphic form for some one dimensional representation of Aut(M).

By restricting  $\Psi$  to complex hyperbolic space we get an automorphic form which has zeros of order 3 along the reflection hyperplanes (because f has zeros of order 1, but every reflection hyperplane is the restriction of 3 hyperplanes of the symmetric space of M). So by taking the cube root of the restriction of  $\Psi$  we get an automorphic form for a one dimensional character of G whose zeros are exactly the reflection hyperplanes with multiplicity 1.

References.

[A-C-T] D. J. Allcock, J. Carlson, D. Toledo, A Complex Hyperbolic Structure for Moduli of Cubic Surfaces, alg-geom/9709016

[B] R. E. Borcherds, Automorphic forms with singularities on Grassmannians, alggeom/9609022, to appear in Invent. Math.