An automorphic form related to cubic surfaces. First draft. 1997.

R. E. Borcherds.

(Remark added 2000: See the paper "Cubic Surfaces and Borcherds Products" by Allcock and Freitag, [math.AG/0002066](http://arxiv.org/abs/math/0002066))

In the paper [A-C-T] the authors showed that the moduli space of cubic surfaces was $(CH⁴_H)/G$, where H is the union of the reflection hyperplanes of the reflection group G. The purpose of this note is to construct an automorphic form, called the discriminant, on complex hyperbolic space CH^4 whose zeros are exactly the reflection hyperplanes of G, each with multiplicity 1.

The complex hyperbolic space of $E^{1,4}$ embeds naturally in the Grassmannian of 2 dimensional positive definite subspaces of the underlying even integral lattice M of $E^{1,4}$. We will construct the automorphic form on complex hyperbolic space by constructing an automorphic form Ψ on the Grassmannian and restricting it to complex hyperbolic space. The construction of Ψ is similar to that of example 13.7 of [B]. The lattice M is isomorphic to $A_2 \oplus A_2^4(-1)$, so it has dimension 10 and determinant 3^5 . We let the elements e_γ for $\gamma \in M'/M$ stand for the obvious basis of the group ring $C[M'/M]$. We will construct a modular form of weight -3 and type ρ_M for $SL_2(Z)$ which is holomorphic on the upper half plane and meromorphic at cusps, by which we mean a holomorphic vector valued function $f = \sum_{\gamma \in M'/M} e_{\gamma} f_{\gamma}(\tau)$ such that

$$
f_{\gamma}(\tau+1) = e^{\pi i \gamma^2} f_{\gamma}(\tau), \qquad f_{\gamma}(-1/\tau) = -i3^{-5/2} \tau^{-3} \sum_{\delta \in M'/M} e^{-2\pi i (\delta, \gamma)} f_{\delta}(\tau)
$$

Recall that if $\gamma, \delta \in M'/M$ then γ^2 is well defined mod 2 and (γ, δ) is well defined mod 1. There are 4 orbits of vectors $\gamma \in M'/M$ under Aut (M) , which we name as follows: a nonzero vector γ with $\gamma^2/2 \equiv n/3 \mod 1$ will be called a vector of type $n (n = 0, 1, 2)$, and the zero vector will be called a vector of type 00.

We will construct a modular form f such that the components f_{γ} of f are given by functions f_{00} , f_0 , f_1 , or f_2 depending only on the type of γ . We now work out the conditions that these functions have to satisfy for f to transform correctly. In order to check that f is a modular form under $\tau \mapsto -1/\tau$ we need to know, given some fixed vector u, how many vectors v there are with given type and given inner product with u . These numbers are given in the following table.

type of u	00	00	00	00	0	0	0	0	1	1	1	1	2	2	2	2
type of v	00	0	1	2	00	0	1	2	00	0	1	2	00	0	1	2
$(u, v) \equiv 0$	1	80	90	72	1	26	36	18	1	32	24	24	1	20	30	30
$(u, v) \equiv 1/3$	0	0	0	0	27	27	27	0	24	33	24	0	30	30	21	
$(u, v) \equiv 2/3$	0	0	0	0	27	27	27	0	24	33	24	0	30	30	21	

Using this table we see that f_{00} , f_0 , f_1 , and f_2 have to satisfy the equations

$$
f_{00}(\tau + 1) = f_{00}(\tau), \quad f_{00}(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) + 80f_0(\tau) + 90f_1(\tau) + 72f_2(\tau))
$$

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$$
f_0(\tau + 1) = f_0(\tau), \quad f_0(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) - f_0(\tau) + 9f_1(\tau) - 9f_2(\tau))
$$

\n
$$
f_1(\tau + 1) = e^{2\pi i/3}f_1(\tau), \quad f_1(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) + 8f_0(\tau) - 9f_1(\tau))
$$

\n
$$
f_2(\tau + 1) = e^{4\pi i/3}f_2(\tau), \quad f_2(-1/\tau) = -i3^{-5/2}\tau^{-3}(f_{00}(\tau) - 10f_0(\tau) + 9f_2(\tau))
$$

One solution of these equations is given as follows.

$$
f_{00}(\tau) = 24\eta (3\tau)^3 \eta(\tau)^{-9} = 24(1 + 9q + 54q^2 + O(q^3))
$$

\n
$$
f_0(\tau) = -3\eta (3\tau)^3 \eta(\tau)^{-9} = -3 + O(q)
$$

\n
$$
f_1(\tau) = 0
$$

\n
$$
f_2(\tau) = \eta(\tau/3)^3 \eta(\tau)^{-9} + 3\eta (3\tau)^3 \eta(\tau)^{-9} = q^{-1/3} + 14q^{2/3} + 92q^{5/3} + O(q^{8/3})
$$

Most of the transformations follow formally from the functional equations $\eta(\tau + 1)$ $e^{2\pi i/24}\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{\tau/i\eta(\tau)}$ of η . The only one which takes slightly more work is the transformation of f_2 under $\tau \mapsto \tau + 1$, and this follows from the identity $\eta(\tau)^3 =$ $\sum_{n\in\mathbb{Z}}(4n+1)q^{(4n+1)^2/8}$ and its consequence

$$
\eta(\tau/3)^3 \eta(\tau)^{-9} + \eta((\tau+1)/3)^3 \eta(\tau+1)^{-9} + \eta((\tau+2)/3)^3 \eta(\tau+2)^{-9} = -9\eta(3\tau)^3 \eta(\tau)^{-9}.
$$

By theorem 13.3 of [B] there is an automorphic form Ψ on the symmetric space of M with the following properties. It has weight $12 = (\text{coefficient of } q^0 \text{ in } f_{00})/2$. The zeros of Ψ correspond to the negative powers of q in f, so are zeros of order 1 orthogonal to all the norm $-2/3$ vectors of M'. Ψ is holomorphic on the symmetric space, and therefore holomorphic at cusps as well by the Koecher boundedness principle. Ψ is an automorphic form for some one dimensional representation of $\text{Aut}(M)$.

By restricting Ψ to complex hyperbolic space we get an automorphic form which has zeros of order 3 along the reflection hyperplanes (because f has zeros of order 1, but every reflection hyperplane is the restriction of 3 hyperplanes of the symmetric space of M). So by taking the cube root of the restriction of Ψ we get an automorphic form for a one dimensional character of G whose zeros are exactly the reflection hyperplanes with multiplicity 1.

References.

[A-C-T] D. J. Allcock, J. Carlson, D. Toledo, A Complex Hyperbolic Structure for Moduli of Cubic Surfaces, [alg-geom/9709016](http://arxiv.org/abs/alg-geom/9709016)

[B] R. E. Borcherds, Automorphic forms with singularities on Grassmannians, [alg](http://arxiv.org/abs/alg-geom/9609022)[geom/9609022](http://arxiv.org/abs/alg-geom/9609022), to appear in Invent. Math.