On the size of Diophantine *m*-tuples

ANDREJ DUJELLA

Abstract

Let n be a nonzero integer and assume that a set S of positive integers has the property that xy + n is a perfect square whenever x and y are distinct elements of S. In this paper we find some upper bounds for the size of the set S. We prove that if $|n| \leq 400$ then $|S| \leq 32$, and if |n| > 400 then $|S| < 267.81 \log |n| (\log \log |n|)^2$. The question whether there exists an absolute bound (independent on n) for |S| still remains open.

1 Introduction

Let *n* be a nonzero integer. A set of *m* positive integers $\{a_1, a_2, \ldots, a_m\}$ is said to have the property D(n) if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Such a set is called a Diophantine *m*-tuple (with the property D(n)), or P_n -set of size *m*.

Diophantus found the quadruple $\{1, 33, 68, 105\}$ with the property D(256). The first Diophantine quadruple with the property D(1), the set $\{1, 3, 8, 120\}$, was found by Fermat (see [8, 9]). Baker and Davenport [3] proved that this Fermat's set cannot be extended to the Diophantine quintuple, and a famous conjecture is that there does not exist a Diophantine quintuple with the property D(1). The theorem of Baker and Davenport has been recently generalized to several parametric families of quadruples [12, 14, 16], but the conjecture is still unproved.

On the other hand, there are examples of Diophantine quintuples and sextuples like $\{1, 33, 105, 320, 18240\}$ with the property D(256) [11] and $\{99, 315, 9920, 32768, 44460, 19534284\}$ with the property D(2985984) [19].

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The purpose of this paper is to find some upper bounds for the numbers M_n defined by

 $M_n = \sup\{|S| : S \text{ has the property } D(n)\},\$

where |S| denotes the number of elements in the set S.

Considering congruences modulo 4, it is easy to prove that $M_{4k+2} = 3$ for all integers k (see [6, 21, 29]). In [10] we proved that if $n \not\equiv 2 \pmod{4}$ and $n \not\in \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then $M_n \ge 4$. Recently, we were able to prove that $M_1 \le 8$ (see [15]). (As we said before, the conjecture is that $M_1 = 4$.) Since a set with the property D(4) may contain at most two odd elements, this result implies $M_4 \le 10$.

Since the number of integer points on the elliptic curve

$$y^{2} = (a_{1}x + n)(a_{2}x + n)(a_{3}x + n)$$
(1)

is finite, we conclude that there does not exist an infinite set with the property D(n). However, bounds for the size [2] and for the number [33] of solutions of (1) depend not only on n but also on a_1, a_2, a_3 .

On the other hand, we may consider the hyperelliptic curve

$$y^{2} = (a_{1}x + n)(a_{2}x + n)(a_{3}x + n)(a_{4}x + n)(a_{5}x + n)$$
(2)

of genus g = 2. Caporaso, Harris and Mazur [7] proved that the Lang conjecture on varieties of general type implies that for $g \ge 2$ the number $B(g, \mathbf{K}) = \max_C |C(\mathbf{K})|$ is finite. Here C runs over all curves of genus g over a number field \mathbf{K} , and $C(\mathbf{K})$ denotes the set of all \mathbf{K} -rational points on C. However, even the question whether $B(2, \mathbf{Q}) < \infty$ is still open. An example of Keller and Kulesz [26] shows that $B(2, \mathbf{Q}) \ge 588$ (see also [17, 34]). Since $M_n \le 5 + B(2, \mathbf{Q})$ (by [23] we have also $M_n \le 4 + B(4, \mathbf{Q}(\sqrt{n}))$, we see that the Lang conjecture implies that

$$M = \sup\{M_n : n \in \mathbf{Z} \setminus \{0\}\}\$$

is finite.

At present we are able to prove only the weaker result that M_n is finite for all $n \in \mathbb{Z} \setminus \{0\}$. In the proof of this result we will try to estimate the number of "large" (greater than $|n|^3$), "small" (between n^2 and $|n|^3$) and "very small" (less that n^2) elements of a set with the property D(n). Let us introduce the following notation:

$$A_n = \sup\{|S \cap [|n|^3, +\infty\rangle| : S \text{ has the property } D(n)\},\$$

$$B_n = \sup\{|S \cap \langle n^2, |n|^3\rangle| : S \text{ has the property } D(n)\},\$$

$$C_n = \sup\{|S \cap [1, n^2]| : S \text{ has the property } D(n)\}.$$

In estimating the number of "large" elements, we used a theorem of Bennett [4] on simultaneous approximations of algebraic numbers and a very useful gap principle. We proved

Theorem 1 $A_n \leq 21$ for all nonzero integers n.

For the estimate of the number of "small" elements we used a "weak" variant of the gap principle and we proved

Theorem 2 $B_n < 0.65 \log |n| + 2.24$ for all nonzero integers n.

Finally, in the estimate of the number of "very small" elements we used a large sieve method due to Gallagher [18] and we proved

Theorem 3 $C_n < 265.55 \log |n| (\log \log |n|)^2 + 9.01 \log \log |n|$ for $|n| \ge 400$.

Since we checked that $C_n \leq 5$ for $|n| \leq 400$, we may combine Theorems 1, 2 and 3 to obtain

Theorem 4

$$M_n \leq 32 \quad for \ |n| \leq 400,$$

$$M_n < 267.81 \ \log |n| \ (\log \log |n|)^2 \quad for \ |n| \geq 400.$$

2 Large elements

Assume that the set $\{a, b, c, d\}$ has the property D(n). Let $ab + n = r^2$, $ac + n = s^2$, $bc + n = t^2$, where r, s, t are nonegative integers. Eliminating d from the system

$$ad + 1 = x^2$$
, $bd + 1 = y^2$, $cd + 1 = z^2$

we obtain the following system of Pellian equations

$$az^2 - cx^2 = n(a - c), (3)$$

$$bz^2 - cy^2 = n(b - c). (4)$$

We will apply the following theorem of Bennett [4] on simultaneous approximations of square roots of two rationals which are very close to 1.

Theorem 5 ([4]) If c_i , p_i , q and L are integers for $0 \le i \le 2$, with $c_0 < c_1 < c_2$, $c_j = 0$ for some $0 \le j \le 2$, q nonzero and $L > M^9$, where

$$M = \max\{|c_0|, |c_1|, |c_2|\},\$$

then we have

$$\max_{0 \le i \le 2} \left\{ \left| \sqrt{1 + \frac{c_i}{L}} - \frac{p_i}{q} \right| \right\} > (130L\gamma)^{-1} q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33L\gamma)}{\log(1.7L^2 \prod_{0 \le i < j \le 2} (c_i - c_j)^{-2})}$$

and

$$\gamma = \begin{cases} \frac{(c_2 - c_0)^2 (c_2 - c_1)^2}{2c_2 - c_0 - c_1} & \text{if } c_2 - c_1 \ge c_1 - c_0, \\ \frac{(c_2 - c_0)^2 (c_1 - c_0)^2}{c_1 + c_2 - 2c_0} & \text{if } c_2 - c_1 < c_1 - c_0. \end{cases}$$

We will apply Theorem 5 to the numbers

$$\theta_1 = \frac{s}{a}\sqrt{\frac{a}{c}} = \sqrt{\frac{ac+n}{ac}} = \sqrt{1+\frac{n}{ac}} = \sqrt{1+\frac{nb}{abc}},$$

$$\theta_2 = \frac{t}{b}\sqrt{\frac{b}{c}} = \sqrt{\frac{bc+n}{bc}} = \sqrt{1+\frac{n}{bc}} = \sqrt{1+\frac{na}{abc}}.$$

Lemma 1 Assume that a < b < c and ac > n. Then all positive integer solutions x, y, z of the system (3) and (4) satisfy

$$\max\left(|\theta_1 - \frac{sbx}{abz}|, |\theta_2 - \frac{zay}{abz}|\right) < \frac{c \cdot |n|}{a}z^{-2}$$

PROOF. We have

$$\left|\frac{s}{a}\sqrt{\frac{a}{c}} - \frac{sbx}{abz}\right| = \frac{s}{az\sqrt{c}}|z\sqrt{a} - x\sqrt{c}| = \frac{s}{az\sqrt{c}} \cdot \frac{|n(c-a)|}{z\sqrt{a} + x\sqrt{c}}.$$

If n < 0, then $s = \sqrt{ac - |n|} < \sqrt{ac}$ and we obtain

$$|\theta_1 - \frac{sbx}{abz}| < \frac{\sqrt{ac} \cdot |n| \cdot c}{a\sqrt{acz^2}} = \frac{c|n|}{a}z^{-2}.$$

If n > 0, then $x\sqrt{c} > z\sqrt{a}$ and we obtain

$$|\theta_1 - \frac{sbx}{abz}| < \frac{\sqrt{ac+n} \cdot n \cdot c}{2a\sqrt{acz^2}} = \sqrt{1 + \frac{n}{ac}} \cdot \frac{cn}{2a} z^{-2} < \frac{cn}{a} z^{-2}.$$

In the same manner, we obtain $|\theta_2 - \frac{tay}{abz}| < \frac{c|n|}{b}z^{-2} < \frac{c|n|}{a}z^{-2}$.

Lemma 2 Let $\{a, b, c, d\}$, a < b < c < d, be a Diophantine quadruple with the property D(n). If $c > b^{11}|n|^{11}$, then $d \le c^{131}$.

PROOF. Let r, s, t, x, y, z be defined as in the beginning of this section. We will apply Theorem 5 with $\{c_0, c_1, c_2\} = \{0, na, nb\}, L = abc, M = |nb|, q = abz, p_1 = sbx, p_2 = tay.$ Since $abc > |n|^9 b^9$, the condition $L > M^9$ is satisfied. For the quantity γ from Theorem 5 we have $\gamma = \frac{b^2(b-a)^2}{2b-a}|n|^3$ if $b \ge 2a$ and $\gamma = \frac{a^2b^2}{a+b}|n|^3$ if $a < b \le 2a$. In both cases we have

$$\frac{b^3}{6}|n|^3 \le \gamma < \frac{b^3}{2}|n|^3.$$

For the quantity λ from Theorem 5 we have

$$\lambda = 1 + \frac{\log(33abc\gamma)}{\log(1.7c^2(b-a)^{-2}n^{-6})} = 2 - \lambda_1,$$

where

$$\lambda_1 = \frac{\log \frac{1.7c}{33ab(b-a)^2 n^6 \gamma}}{\log(1.7c^2(b-a)^{-2}n^{-6})}$$

Theorem 5 and Lemma 1 imply

$$\frac{c|n|}{az^2} > (130abc\gamma)^{-1}(abz)^{\lambda_1 - 2} > (130abc\gamma)^{-1}a^{-2}b^{-2}z^{\lambda_1 - 2}$$

This implies

$$z^{\lambda_1} < 130a^2b^3c^2|n|\gamma$$

and

$$\log z < \frac{\log \left(130a^2b^3c^2|n|\gamma\right)\log\left(1.7c^2(b-a)^{-2}n^{-6}\right)}{\log\left(\frac{1.7c}{33ab(b-a)^2n^6\gamma}\right)}.$$
(5)

Let us estimate the right hand side of (5). We have

$$130a^2b^3c^2|n|\gamma < 65a^2b^6c^2n^4 < c^3 \cdot \frac{65a^2}{b^5|n|^7} < c^3,$$

unless n = -1, a = 1, b = 2. However, in [13] it was proved that the Diophantine pair $\{1, 2\}$ with the property D(-1) cannot be extended to a Diophantine quadruple.

The same result implies also that if |n| = 1, then b - a > 1. Therefore

$$1.7c^2(b-a)^{-2}n^{-6} < c^2.$$

Finally,

$$\frac{1.7c}{33ab(b-a)^2n^6\gamma} > 0.103a^{-1}b^{-6}cn^{-9} > c^{\frac{1}{11}} \cdot \frac{b^4|n|}{9.71a} > c^{\frac{1}{11}} \, .$$

The last estimate shows that $\lambda_1 > 0$, what we implicitly used in (5).

Putting these three estimates in (5), we obtain

$$\log z < \frac{3\log c \cdot 2\log c}{\frac{1}{11}\log c} = 66\log c.$$

Hence, $z < c^{66}$ and

$$d = \frac{z^2 - n}{c} \le \frac{z^2 + |n|}{c} < \frac{c^{132} + c^{\frac{1}{11}}}{c} < c^{131} + 1.$$

Now we will develop a very useful gap principle for the elements of a Diophantine *m*-tuple. The principle is based on the following construction which generalizes the constructions of Arkin, Hoggatt and Strauss [1] and Jones [25] for the case n = 1.

Lemma 3 If $\{a, b, c\}$ is a Diophantine triple with the property D(n) and $ab + n = r^2$, $ac + n = s^2$, $bc + n = t^2$, then there exist integers e, x, y, z such that

$$ae + n^2 = x^2$$
, $be + n^2 = y^2$, $ce + n^2 = z^2$

and

$$c = a + b + \frac{e}{n} + \frac{2}{n^2}(abe + rxy).$$

PROOF. Define

$$e = n(a+b+c) + 2abc - 2rst.$$

Then

$$(ae + n2) - (at - rs)2 = an(a + b + c) + 2a2bc - 2arst + n2 - a2(bc + n) + 2arst - (ab + n)(ac + n) = 0.$$

Hence we may take x = at - rs, and analogously y = bs - rt, z = cr - st. We have

$$abe + rxy = abn(a + b + c) + 2a^2b^2c - 2abrst$$

+ $abrst - a(ab + n)(bc + n) - b(ab + n)(ac + n) + rst(ab + n)$
= $-abcn - n^2(a + b) + rstn$,

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and finally

$$a+b+\frac{e}{n}+\frac{2}{n^2}(abe+rxy) = 2a+2b+c+\frac{2abc}{n}-\frac{2rst}{n}-\frac{2abc}{n}-2a-2b+\frac{2rst}{n} = c.$$

Lemma 4 If $\{a, b, c, d\}$ is a Diophantine quadruple with the property D(n)and $|n|^3 \le a < b < c < d$, then

$$d > \frac{3.847 \, bc}{n^2}$$
 .

PROOF. We apply Lemma 3 to the triple $\{a, c, d\}$. Since $ce + n^2$ is a perfect square, we have that $ce + n^2 \ge 0$. On the other hand, the assumption is that $c > |n|^3$. Hence, if $e \le -1$, then $ce + n^2 < -|n|^3 + n^2 < 0$, a contradiction. Since e is an integer, we have $e \ge 0$. If e = 0, then d = a + c + 2s. If $e \ge 1$, then

$$d > a + c + \frac{2ac}{n^2} + \frac{2s\sqrt{ac}}{n^2} > \frac{2ac}{n^2}.$$
 (6)

(Note that if n > 0 then x < 0, y < 0, and if n < 0 and b > |n| then x > 0, y > 0.)

Analogously, applying Lemma 3 to the triple $\{b, c, d\}$ we obtain that d = b + c + 2t or $d > b + c + \frac{2bc}{n^2} + \frac{2t\sqrt{bc}}{n^2}$. However, d = b + c + 2t is impossible since b + c + 2t > a + c + 2s and

$$b + c + 2t \le b + c + 2\sqrt{c(c-1) + n} < 4c \le \frac{2ac}{n^2},$$

unless $a < 2n^2$. But if $|n|^3 \le a < 2n^2$, then |n| = 1, a = 1, and in that case we have

$$a + c + \frac{2ac}{n^2} + \frac{2s\sqrt{ac}}{n^2} > 3c + 2\sqrt{c(c-1)} > 4c.$$

Hence we proved that

$$d > b + c + \frac{2bc}{n^2} + \frac{2t\sqrt{bc}}{n^2}.$$
 (7)

From [30] we know that the triples $\{1, 2, 3\}$ and $\{1, 2, 4\}$ cannot be extended to Diophantine quadruples. Thus $bc \geq 10$ and it implies

$$t^2 = bc + n \ge bc - |n| > bc - \sqrt[6]{bc} > 0.853 \, bc.$$

If we put this in (7), we obtain $d > \frac{3.847 bc}{n^2}$.

PROOF OF THEOREM 1. Assume that $\{a_1, a_2, \ldots, a_{22}\}$ has the property D(n) and $|n|^3 \le a_1 < a_2 < \cdots < a_{22}$. By Lemma 4 we find that

$$a_4 > \frac{a_2^2}{n^2}, \quad a_5 > \frac{a_2^3}{n^4}, \quad a_6 > \frac{a_2^5}{n^8}, \quad a_7 > \frac{a_2^8}{n^{14}},$$
$$a_8 > \frac{a_2^{13}}{n^{24}}, \quad a_9 > \frac{a_2^{21}}{n^{40}}, \quad a_{10} > \frac{a_2^{34}}{n^{66}}, \quad a_{11} > \frac{a_2^{55}}{n^{108}},$$

Since $a_2 > |n|^3$, we have $\frac{a_2^{55}}{n^{108}} > a_2^{11} |n|^{11}$, and we may apply Lemma 2 with $a = a_1, b = a_2, c = a_{11}$. We conclude that $a_{22} \le a_{11}^{131}$. However, Lemma 4 implies

$$a_{12} > |n|a_{11}, \quad a_{13} > \frac{a_{11}^2}{|n|}, \quad a_{14} > \frac{a_{11}^3}{n^2}, \quad a_{15} > \frac{a_{11}^5}{|n|^5},$$

$$a_{16} > \frac{a_{11}^8}{|n|^9}, \quad a_{17} > \frac{a_{11}^{13}}{n^{16}}, \quad a_{18} > \frac{a_{11}^{21}}{|n|^{27}}, \quad a_{19} > \frac{a_{11}^{34}}{|n|^{45}},$$

$$a_{20} > \frac{a_{11}^{55}}{n^{74}}, \quad a_{21} > \frac{a_{11}^{89}}{|n|^{121}}, \quad a_{22} > \frac{a_{11}^{144}}{|n|^{197}}.$$

Since $a_{11} > a_2^{11} |n|^{11} > n^{44}$, we obtain

$$a_{22} > \frac{a_{11}^{144}}{|n|^{197}} \ge a_{11}^{144 - \frac{197}{44}} > a_{11}^{139} > a_{11}^{131},$$

a contradiction.

3 Small elements

Lemma 5 If $\{a, b, c, d\}$ is a Diophantine quadruple with the property D(n), $|n| \neq 1$, and $n^2 \leq a < b < c < d$, then c > 3.88 a and d > 4.89 c.

PROOF. We will apply Lemma 3. Since $b > n^2$, we have $e \ge 0$. Thus Lemma 3 implies that

$$c \ge a + b + 2r.$$

Since $|n| \neq 1$ we have $ab \geq 20$ and $r^2 \geq ab - \sqrt[4]{ab} > 0.89ab > 0.89a^2$. Hence, c > 3.88 a.

Since $d \ge b + c + 2t > a + c + 2s$, from (6) we conclude that

$$d > a + c + \frac{2ac}{n^2} + \frac{2s\sqrt{ac}}{n^2}.$$

We have $ac \ge 24$ and $s^2 \ge ac - \sqrt[4]{ac} > 0.9 ac$. Therefore

$$d > a + c + \frac{3.89 \, ac}{n^2} > 4.89 \, c \,.$$

PROOF OF THEOREM 2. We may assume that $|n| \ge 2$ since $B_1 = B_{-1} = 0$. Let $\{a_1, a_2, \ldots, a_m\}$ be a Diophantine *m*-tuple with the property D(n) and $n^2 < a_1 < a_2 < \cdots < |n|^3$. By Lemma 5 we have

$$a_3 > 3.88a_1, \quad a_4 > 3.88 \cdot 4.89a_1, \quad \dots, \quad a_m > 3.88 \cdot 4.89^{m-3}a_1.$$

Therefore

$$3.88 \cdot 4.89^{m-3} \cdot n^2 < |n|^3$$

and from $m - 3 < \frac{\log \frac{|n|}{3.88}}{\log 4.89}$ we obtain $m < 0.65 \log |n| + 2.24$.

4 Very small elements

We are left with the task to estimate the number of "very small" elements in a Diophantine *m*-tuple. Let $\{a_1, a_2, \ldots, a_m\}$ be a Diophantine *m*-tuple with the property D(n) and assume that $a_1 < a_2 < \cdots < a_m \leq N$, where N is a positive integer. Let $1 \leq k < m$. Then $x = a_{k+1}, \ldots, x = a_m$ satisfy the system

$$a_1x + n = \Box, \quad a_2x + n = \Box, \quad \dots, \quad a_kx + n = \Box, \tag{8}$$

where \Box denotes a square of an integer. Denote by $Z_k(N)$ the number of solutions of system (8) satisfying $1 \le x \le N$.

Motivated by the observations from the introduction of [5], we will apply a sieve method based on the following theorem of Gallagher [18] (see also [24, p.29]):

Theorem 6 ([18]) If all but g(q) residue classes (mod q) are removed for each prime power q in a finite set S, then the number of integers which remain in any interval of length N is at most

$$\left(\sum_{q\in\mathcal{S}}\Lambda(q) - \log N\right) / \left(\sum_{q\in\mathcal{S}}\frac{\Lambda(q)}{g(q)} - \log N\right)$$

provided the denominator is positive. Here $\Lambda(q) = \log p$ for $q = p^{\alpha}$.

We will use Theorem 6 to estimate the number $Z_k(N)$. For this purpose, we will take

$$\mathcal{S} = \{ p : p \text{ is prime}, 83 \le p \le Q, \gcd(a_1 a_2 \cdots a_k, p) = 1 \},\$$

where Q is sufficiently large. For a prime $p \in S$ we may remove all residue classes (mod p) such that $\left(\frac{a_i x+n}{p}\right) = -1$ for some $i \in \{1, \ldots, k\}$. Here $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Let $1 \leq l \leq k$. Then

$$g(p) \leq |\{x \in \mathbf{F}_p : \left(\frac{a_i x + n}{p}\right) = 0 \text{ or } 1, \text{ for } i = 1, \dots, l\}|$$

$$\leq l + |\{x \in \mathbf{F}_p : \left(\frac{x + n\overline{a_i}}{p}\right) = \left(\frac{\overline{a_i}}{p}\right), \text{ for } i = 1, \dots, l\}|.$$

Here $a_i \overline{a_i} \equiv 1 \pmod{p}$. Using estimates for character sums (see [28, p.325]), we obtain

$$g(p) \le l + \frac{p}{2^l} + \left(\frac{l-2}{l} + \frac{1}{2^l}\right)\sqrt{p} + \frac{l}{2}.$$

Assume that $k = \lfloor \log_2 Q \rfloor$. We may take $l = \lfloor \log_2 p \rfloor$. Then we have

$$\frac{p}{2^l} + \frac{\sqrt{p}}{2^l} + \frac{3l}{2} < 2 + \frac{2}{\sqrt{p}} + \frac{3\log_2 p}{2} < \sqrt{p}$$

for $p \ge 179$. Hence

$$l + \frac{p}{2^l} + \left(\frac{l-2}{2} + \frac{1}{2^l}\right)\sqrt{p} + \frac{l}{2} < \frac{l}{2}\sqrt{p} < \frac{\log_2 p}{2}\sqrt{p} < 0.722\sqrt{p}\log p$$

for $p \geq 179$, and we may check directly that $l + \frac{p}{2^l} + \left(\frac{l-2}{2} + \frac{1}{2^l}\right) + \frac{l}{2} < 0.722 \sqrt{p} \log p$ for $83 \leq p \leq 173$. Therefore we proved that

$$g(p) < 0.722\sqrt{p}\log p.$$

By Theorem 6, we have $Z_k(N) \leq \frac{E}{F}$, where

$$E = \sum_{p \in \mathcal{S}} \log p - \log N, \quad F = \sum_{p \in \mathcal{S}} \frac{1}{0.722\sqrt{p}} - \log N.$$

By [32, Theorem 9], we have $E < \sum_{83 \le p \le Q} \log p < \theta(Q) < 1.01624 Q$.

Assume that at least $\frac{4}{5}\pi(Q)$ primes less than Q satisfy the condition $gcd(a_1a_2\cdots a_k, p) = 1$. Then we have

$$F \geq \frac{1}{0.722\sqrt{Q}} |\mathcal{S}| - \log N \geq \frac{1}{0.722\sqrt{Q}} \left(\frac{4}{5}\pi(Q) - 23\right) - \log N$$

> $1.108 \frac{\sqrt{Q}}{\log Q} - \frac{31.86}{\sqrt{Q}} - \log N.$ (9)

Since F must be positive in the applications of Theorem 6, we will choose Q of the following form

$$Q = c_1 \cdot \log^2 N \cdot (\log \log N)^2, \tag{10}$$

where c_1 is a constant.

We have to check whether our assumption is correct. Suppose that $a = a_1 a_2 \cdots a_k$ is divisible by at least one fifth of the primes $\leq Q$. Then $a \ge p_1 p_2 \cdots p_{\lceil \frac{1}{5}\pi(Q) \rceil}$, where p_i denotes the *i*th prime. By [32, p.69], we have

$$p_{\lceil \frac{1}{5}\pi(Q)\rceil} > \frac{1}{5}\pi(Q)\log(\frac{1}{5}\pi(Q)) > \frac{1}{5}\frac{Q}{\log Q}\log\left(\frac{1}{5}\frac{Q}{\log Q}\right) := R.$$

Therefore, by [32, p.70],

$$\log a > \sum_{p \le R} \log p > R \left(1 - \frac{1}{\log R} \right).$$

Assume that $Q \ge 2 \cdot 10^4$. Then $\frac{1}{5} \frac{Q}{\log Q} > Q^{0.605}$ and R > 0.128 Q. Furthermore, $\log R > 7.793$ and therefore

$$\log a > 0.105 Q.$$

On the other hand, $a < N^k$ and $\log a < k \log N \le \log_2 Q \log N$. Assume that $N \ge 1.6 \cdot 10^5$ and $c_1 \le 80$. Then $Q \le \log^{4.498} N$. In order to obtain a contradiction, it suffices to check that

$$0.105 c_1 \, \log^2 N \, (\log \log N)^2 > \frac{4.498}{\log 2} \log N \cdot \log \log N$$

or

$$c_1 \log N \log \log N > 61.81,$$

and this is certainly true for $N \ge 1.6 \cdot 10^5$ if we choose $c_1 \ge 2.08$.

Thus we may continue with estimating the quantity F. We are working under assumptions that (10) holds with $2.08 \leq c_1 \leq 80$, $Q \geq 2 \cdot 10^4$ and $N \geq 1.6 \cdot 10^5$. We would like to have the estimate of the form

$$F > \frac{\sqrt{Q}}{c_2 \log Q} \,. \tag{11}$$

This estimate will lead to

$$Z_k(N) < 1.01624 c_2 \sqrt{Q} \log Q < 4.572 c_2 \sqrt{c_1} \log N (\log \log N)^2.$$
(12)

In order to fulfill (11), it suffice to check

$$\frac{31.86}{\sqrt{Q}} + \log N < \frac{\sqrt{Q}}{\log Q} \Big(1.108 - \frac{1}{c_2} \Big).$$

Since $Q > 2 \cdot 10^4$ we have $\frac{31.86}{\sqrt{Q}} < 0.016 \frac{\sqrt{Q}}{\log Q}$. Furthermore,

$$\frac{\log N \log Q}{\sqrt{Q}} < \frac{4.498 \, \log N \log \log N}{\sqrt{c_1} \log N \log \log N} = \frac{4.498}{\sqrt{c_1}} \,.$$

Hence $c_2 > 1/(1.092 - \frac{4.498}{\sqrt{c_1}})$. Thus if we choose $c_1 = 68$, then we may take $c_2 = 1.83$ and from (12) we obtain

$$Z_k(N) < 69 \log N (\log \log N)^2.$$
(13)

Note that with this choice of c_1 , $N \ge 1.6 \cdot 10^5$ implies $Q > 60222 > 2 \cdot 10^4$.

PROOF OF THEOREM 3. Let $\{a_1, a_2, \ldots, a_m\}$ be a Diophantine *m*tuple with the property D(n) and $a_1 < a_2 < \cdots < a_m \leq n^2$. Then for any $k \in \{1, 2, \ldots, m\}$ we have

$$m \le k + Z_k(n^2)$$

Let $k = \lfloor \log_2 Q \rfloor$, where $Q = 68 \log^2 n^2 (\log \log n^2)^2$. Since $|n| \ge 400$, we have $n^2 \ge 1.6 \cdot 10^5$ and we may apply formula (13) to obtain

$$Z_k(n^2) < 69 \log n^2 (\log \log n^2)^2 < 265.55 \log |n| (\log \log |n|)^2,$$
(14)

Furthermore,

$$k < \frac{1}{\log_2} \log(\log^{4.489} n^2) < 9.01 \log\log|n|, \tag{15}$$

and combining (14) and (15) we finally obtain

 $m < 265.55 \log |n| (\log \log |n|)^2 + 9.01 \log \log |n|.$

Remark 1 In [22] Katalin Gyarmati recently considered the more general problem. She estimated min{ $|\mathcal{A}|, |\mathcal{B}|$ }, where $\mathcal{A}, \mathcal{B} \subseteq \{1, 2, ..., N\}$ satisfy the condition that ab + n is a perfect square for all $a \in \mathcal{A}, b \in \mathcal{B}$. Using her approach, it can be deduced that if $\{a_1, a_2, ..., a_m\}$ has the property D(n), where n > 0 and $a_1 < a_2 < \cdots < a_m \leq N$, then $m \leq 2n \log N$. This yields $C_n \leq 4n \log n$ for $n \geq 2$.

Remark 2 Let us mention that Rivat, Sárközy and Stewart [31] recently used Gallagher's "larger sieve" method is estimating the size of a set Z of integers such that z + z' is a perfect square whenever z and z' are distinct elements of Z. They proved that if $Z \subset \{1, 2, ..., N\}$, where N is greater that an effectively computable constant, then $|Z| < 37 \log N$.

Largest known set with the above property is a set with six elements found by J. Lagrange [27]. Maybe this may be compared with our situation where the largest known Diophantine m-tuples are Diophantine sextuples found by Gibbs [19, 20].

PROOF OF THEOREM 4. Since $M_n \leq A_n + B_n + C_n$, the second part of the theorem follows directly from Theorems 1, 2 and 3.

For $|n| \leq 400$, Theorem 2 gives $B_n \leq 6$. It is easy to verify with a computer that for $|n| \leq 400$ it holds $C_n \leq 5$. More precisely, $C_n = 5$ if and only if $n \in \{-299, -255, 256, 400\}$. These two estimates together with Theorem 1 imply $M_n \leq 32$.

5 Concluding remarks

It is not surprising that in Theorem 4 the main contribution comes from C_n . Namely, if we define $C = \sup\{C_n : n \in \mathbb{Z} \setminus \{0\}\}$, then we have M = C. Indeed, if $\{a_1, a_2, \ldots, a_m\}$ is a Diophantine *m*-tuple with the property D(n), then $\{a_1c, a_2c, \ldots, a_mc\}$ has the property $D(nc^2)$ and for sufficiently large c we have $a_i c \leq (nc^2)^2$, $i = 1, 2, \ldots, m$. It means that in order to prove $M < \infty$, it suffices to prove $C < \infty$. The above argumentation shows that it suffice to prove that for some $\varepsilon > 0$ it holds

 $\sup_{n \neq 0} \sup \{ |S \cap [1, n^{0.5 + \varepsilon}] | : S \text{ has the property } D(n) \} < \infty.$

We may define also $A = \sup\{A_n : n \in \mathbb{Z} \setminus \{0\}\}$ and $B = \sup\{B_n : n \in \mathbb{Z} \setminus \{0\}\}$. Gibbs' example mentioned in introduction shows that $C \ge 6$ and

 $M \ge 6$. If $n = a^2$, $a \ge 5$, then $B_n \ge 3$ since $\{a^2 + 1, a^2 + 2a + 1, 4a^2 + 4a + 4\}$ has the property $D(a^2)$. Hence $B \ge 3$. Finally, since $\{k, k+2, 4k+4, 16k^3 + 48k^2 + 44k + 12\}$ has the property D(1) we have $A \ge A_1 \ge 4$.

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Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia

E-mail address: duje@math.hr