A $W(E_6)$ -equivariant projective embedding of the moduli space of cubic surfaces

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Abstract

An explicit projective embedding of the moduli space of marked cubic surfaces is given. This embedding is equivariant under the Weyl group of type E_6 . The image is defined by a system of linear and cubic equations. To express the embedding in a most symmetric way, the target would be 79-dimensional, however the image lies in a 9-dimensional linear subspace.

1 The moduli space of cubic surfaces

1.1 The space M and the action of the group \underline{G}

We first fix some notation and recall a few known facts on the moduli space of marked cubic surfaces. The moduli space of marked cubic surfaces, which we denote by M, is studied for example in [5] and [8]. Since any nonsingular cubic surface can be obtained by blowing up the projective plane \mathbf{P}^2 at six points, it can be represented by a 3×6 -matrix of which columns give homogeneous coordinates of the six points. In order to get a smooth cubic surface from six points, we assume that no three points are collinear and the six points are not on a conic. On the set of 3×6 matrices, we have a cannical action of GL_3 on the left and the group \mathbf{C}^{\times} acts naturally on homogeneous coordinates. By killing such ambiguity of coordinates, we get the following expression

$$x = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & x_3 & x_4 \end{pmatrix};$$

in this paper we use local coordinates (x_1, x_2, x_3, x_4) on M. The six points represented by the matrix above produces a non-singular cubic surface if and only if the following quantity does not vanish.

$$D(x) := x_1 x_2 x_3 x_4 (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1) \times (x_1 - x_2)(x_1 - x_3)(x_2 - x_4)(x_3 - x_4) D_1 D_2 Q,$$

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where

$$D_1 := x_1 x_4 - x_2 x_3,$$

$$D_2 := x_1 x_4 - x_4 + x_2 - x_2 x_3 + x_3 - x_1,$$

$$Q := -x_2 x_3 x_1 - x_2 x_3 x_4 + x_2 x_3 + x_1 x_4 x_2 + x_1 x_4 x_3 - x_1 x_4,$$

Thus we can identify the moduli space M with the affine open set $\{x = (x_1, \ldots, x_4) \mid D(x) \neq 0\}$.

Let us define as in [8] six bi-rational transformations s_1, \ldots, s_6 in $x = (x_1, \ldots, x_4)$:

$$s_{1}: (x_{1}, x_{2}, x_{3}, x_{4}) \rightarrow \left(\frac{1}{x_{1}}, \frac{1}{x_{2}}, \frac{x_{3}}{x_{1}}, \frac{x_{4}}{x_{2}}\right),$$

$$s_{2}: (x_{1}, x_{2}, x_{3}, x_{4}) \rightarrow (x_{3}, x_{4}, x_{1}, x_{2}),$$

$$s_{3}: (x_{1}, x_{2}, x_{3}, x_{4}) \rightarrow \left(\frac{x_{1} - x_{3}}{1 - x_{3}}, \frac{x_{2} - x_{4}}{1 - x_{4}}, \frac{x_{3}}{x_{3} - 1}, \frac{x_{4}}{x_{4} - 1}\right),$$

$$s_{4}: (x_{1}, x_{2}, x_{3}, x_{4}) \rightarrow \left(\frac{1}{x_{1}}, \frac{x_{2}}{x_{1}}, \frac{1}{x_{3}}, \frac{x_{4}}{x_{3}}\right),$$

$$s_{5}: (x_{1}, x_{2}, x_{3}, x_{4}) \rightarrow (x_{2}, x_{1}, x_{4}, x_{3}),$$

$$s_{6}: (x_{1}, x_{2}, x_{3}, x_{4}) \rightarrow \left(\frac{1}{x_{1}}, \frac{1}{x_{2}}, \frac{1}{x_{3}}, \frac{1}{x_{4}}\right).$$

If M is regarded as the configuration space of six points in \mathbf{P}^2 , the transformation s_1 , for example, corresponds to the interchange of the two points represented by the first two column vectors of the matrix x. Each s_i turns out to be a bi-regular involution on M, and they form a group \underline{G} isomorphic to the Weyl group of type E_6 ; relation of the generators are given by the Coxeter graph

$$s_1 - s_2 - s_3 - s_4 - s_5$$

 $\begin{vmatrix} & & \\ & & \\ & & \\ & & s_6 \end{vmatrix}$

1.2 Root system Δ of type E_6

We review the root system Δ of type E_6 , following [6]. Consider an 8-dimensional Euclidean space \tilde{E} with a standard basis $\varepsilon_1, \dots, \varepsilon_8$. Let $\langle \cdot, \cdot \rangle$ be the inner product on \tilde{E} defined by $\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$ and let E be the linear subspace of \tilde{E} spanned by the six vectors

$$\varepsilon_1, \cdots, \varepsilon_5, \ \tilde{\varepsilon} = \varepsilon_6 - \varepsilon_7 - \varepsilon_8.$$

We introduce the 36 vectors:

$$r = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \tilde{\varepsilon}),$$

$$r_{1j} = -\varepsilon_{j-1} + r_0, \quad 2 \le j \le 6$$

$$r_{jk} = \varepsilon_{j-1} - \varepsilon_{k-1}, \quad 2 \le j < k \le 6$$

$$r_{1jk} = -\varepsilon_{j-1} - \varepsilon_{k-1}, \quad 2 \le j < k \le 6$$

$$r_{ijk} = -\varepsilon_{i-1} - \varepsilon_{j-1} - \varepsilon_{k-1} + r_0, \quad 2 \le i < j < k \le 6$$

where $r_{ij} = r_{ji}, r_{ijk} = r_{jik} = r_{ikj}$,

$$r_0 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \tilde{\varepsilon}).$$

Note that

$$r \perp r_{ij}, \quad r_{ij} \perp r_{kl}, \quad r_{ij} \perp r_{ijk}, \quad r_{ij} \perp r_{klm}, \quad r_{ijk} \perp r_{ilm}$$

The set

$$\Delta = \{\pm r, \ \pm r_{ij}, \ \pm r_{ijk}\}$$

forms a root system of type E_6 . For example,

 $r_{12}, r_{123}, r_{23}, r_{34}, r_{45}, r_{56}$

can serve as a system of positive simple roots; its extended Dynkin diagram is given as



The set $\{r, r_{jk}, r_{ijk}\}$ is the totality of positive roots of Δ .

Let s_r, s_{ij} and s_{ijk} be the reflections on E with respect to r, r_{ij} and r_{ijk} . These reflections act on Δ as

$$s_r: r_{ij} \leftrightarrow r_{ij}, r_{ijk} \leftrightarrow r_{lmn}, \{i, j, k, l, m, n\} = \{1, \dots, 6\},$$

$$s_{ij}: \text{ permutation of the indices } i \text{ and } j,$$

$$s_{123}: r_{12} \leftrightarrow r_{12}, r_{14} \leftrightarrow r_{234}, r_{56} \leftrightarrow r_{56},$$

$$r_{145} \leftrightarrow r_{145}, r \leftrightarrow r_{456},$$

modulo signs. Let us define two reflection group

$$G_1 = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56} \rangle \cong S_6, \quad G = \langle G_1, s_{123} \rangle \cong W(E_6),$$

where S_6 is the symmetric group on six numerals $\{1, \ldots, 6\}$, $W(E_6)$ the Weyl group of type E_6 , and $\langle a, b, \ldots \rangle$ denotes the group generated by a, b, \ldots Note that

$$G_1 \subset \langle G_1, s_r \rangle = G_1 \times \langle s_r \rangle \subset G,$$

and G acts transitively on Δ .

1.3 Naruki's cross-ratio variety

A smooth compactification of M known as Naruki's cross-ratio variety \mathcal{C} ([5], [8]), embedded in (\mathbf{P}^1)⁴⁵, is the union of M and the 76 divisors. The 36 of them correspond to the positive roots of Δ (they are said to be of the first kind), the other 40 divisors (said to be of the second kind, and are isomorphic to (\mathbf{P}^1)³) can be represented as follows: Take three subsets Δ_1 , Δ_2 , Δ_3 of Δ satisfying the following conditions:

- Each of Δ_1 , Δ_2 , Δ_3 is a root system of type A_2 .
- $\Delta_1, \Delta_2, \Delta_3$ are mutually orthogonal.
- The vectors in $\Delta_1 \cup \Delta_2 \cup \Delta_3$ span E.

Note that each one of such three root systems determines the other two.

Such a triple $\{\Delta_1, \Delta_2, \Delta_3\}$ dertermines a divisor. According to the naming of the roots, we must use two different expressions: The first one is of the form

$$\{\pm r_{12}, \pm r_{23}, \pm r_{13}\}, \{\pm r_{45}, \pm r_{56}, \pm r_{46}\}, \{\pm r, \pm r_{123}, \pm r_{456}\},\$$

(the corresponding divisor is denoted by $Z_{123,456}$ in [8],) and the second one is of the form

 $\{\pm r_{12}, \pm r_{234}, \pm r_{134}\}, \{\pm r_{34}, \pm r_{356}, \pm r_{456}\}, \{\pm r_{56}, \pm r_{125}, \pm r_{126}\},\$

(the corresponding divisor is denoted by $Z_{12,34,56}$ in [8]). Note that

$$Z_{123,456} = Z_{456,123}, \quad Z_{12,34,56} = Z_{56,12,34} \neq Z_{12,56,34}.$$

Thus, permuting the indices under G_1 , we have $40 \ (= 10 + 30)$ such divisors. The group G acts transitively on these 40 divisors.

Remark: These divisors (of the second kind) are disjoint to each other, and can be blown-down to points. In fact they correspond bijectively to the cusps of the modular group studied in [1] (see also [3] and [7]).

2 Embedding $\varphi: M \to \mathbf{P}^{80-1}$

2.1 Coordinates on P^{80-1} and the action of G

Let \mathcal{A} be the set of 40 labels (123, 456) and (12, 34, 56) with the following identification

$$(123, 456) = (213, 456) = (132, 456) = (456, 123),$$

 $(12, 34, 56) = (21, 34, 56) = (56, 12, 34).$

Since G acts on the set Δ of roots, it acts also on the set of 40 divisors above, and so that it also acts on the set \mathcal{A} of 40 labels.

We introduce 80 homogeneous coordinates

$$y_{\alpha}, \quad y_{-\alpha}, \qquad \alpha \in \mathcal{A}$$

on \mathbf{P}^{80-1} . I define an action of G on \mathbf{P}^{80-1} by the following action of the generators s_{12}, \ldots, s_{56} and s_{123} on the coordinates. Let s be one of the generators and $\alpha \in \mathcal{A}$; we assign

$$\begin{aligned} s(y_{\alpha}) &= y_{\alpha}, \quad s(y_{-\alpha}) = y_{-\alpha} & \text{if } s\alpha = \alpha, \\ s(y_{\alpha}) &= y_{-\beta}, \quad s(y_{-\alpha}) = y_{\beta} & \text{if } s\alpha = \beta \neq \alpha. \end{aligned}$$

2.2 Definition of φ

In this section we define a map $M \to \mathbf{P}^{80-1}$. For a 3×6 matrix $x = (x_{ij})$, we consider 80 polynomials of degree 18 as follows:

$$y_{(123,456)}(x) = D_{123}(x)D_{456}(x)Q(x),$$

$$y_{(12,34,56)}(x) = D_{134}(x)D_{234}(x)D_{356}(x)D_{456}(x)D_{512}(x)D_{612}(x),$$

and $y_{-\alpha}(x) = -y_{\alpha}$ ($\alpha \in \mathcal{A}$), where Q(x) is the determinant of the 6 × 6-matrix with columns

$$(x_{1j}x_{2j}, x_{2j}x_{3j}, x_{3j}x_{1j}, x_{1j}^2, x_{2j}^2, x_{3j}^2)$$
 $j = 1, \dots, 6.$

Since we have

$$y_{\alpha}(gxh) = (\det g)^6 y_{\alpha}(x) (\det h)^3,$$

the correswpondence above defines a map $\varphi : M \to \mathbf{P}^{80-1}$. For later use, we present 40 polynomials $y_{\alpha}(x)$ in terms of the coordinates (x_1, x_2, x_3, x_4) introduced in §1; the remaining 40 polynomoals are given by $y_{-\alpha}(x) = -y_{\alpha}(x)$. In the following table, $y_{\alpha}(x)$ is denoted simply by α , and we number them as y_1, \ldots, y_{40} :

$$\begin{array}{rcl} y_1 = (156,234) &:= & D_1Q: & y_2 = (123,456):= D_2Q: \\ y_3 = (124,356) &:= & (x_2-x_1)Q: & y_4 = (145,236):= (x_3-x_1)Q: \\ y_5 = (146,235) &:= & (x_4-x_2)Q: & y_6 = (134,256):= (x_4-x_3)Q: \\ y_7 = (135,246) &:= & x_1(x_4-1)Q: & y_8 = (136,245):= x_2(x_3-1)Q: \\ y_9 = (125,346) &:= & x_3(x_2-1)Q: & y_{10} = (126,345):= x_4(x_1-1)Q: \\ y_{11} = (12,56,34) &:= & D_1(x_1-1)(x_2-1)(x_4-x_3): \\ y_{12} = (16,23,45) &:= & D_1(x_1-1)(x_3-1)(x_4-x_2): \\ y_{13} = (15,23,46) &:= & D_1(x_2-1)(x_4-1)(x_3-x_1): \\ y_{14} = (13,56,24) &:= & D_1(x_3-1)(x_4-1)(x_2-x_1): \\ y_{15} = (15,24,36) &:= & -D_1x_1(x_2-1)(x_3-1): \\ y_{16} = (16,24,35) &:= & -D_1x_2(x_1-1)(x_4-1): \\ y_{17} = (15,34,26) &:= & -D_1x_4(x_2-1)(x_3-1): \\ y_{18} = (16,34,25) &:= & -D_1x_4(x_2-1)(x_3-1): \\ y_{19} = (12,36,45) &:= & D_2x_2x_3(x_1-1): \\ \end{array}$$

$$\begin{array}{rcl} y_{20} &= (12,35,46) &:= & D_2 x_1 x_4 (x_2-1): \\ y_{21} &= (13,26,45) &:= & D_2 x_1 x_4 (x_3-1): \\ y_{22} &= (13,25,46) &:= & D_2 x_1 x_2 (x_4-x_3): \\ y_{23} &= (13,24,56) &:= & D_2 x_1 x_3 (x_4-x_2): \\ y_{24} &= (15,46,23) &:= & D_2 x_2 x_4 (x_3-x_1): \\ y_{25} &= (16,45,23) &:= & D_2 x_2 x_4 (x_2-x_1): \\ y_{26} &= (12,34,56) &:= & D_2 x_3 x_4 (x_2-x_1): \\ y_{27} &= (13,46,25) &:= & x_1 (x_2-1) (x_3-1) (x_4-x_2) (x_4-x_3): \\ y_{28} &= (13,45,26) &:= & x_2 (x_1-1) (x_4-1) (x_3-x_1) (x_4-x_2): \\ y_{29} &= (12,46,35) &:= & x_3 (x_1-1) (x_4-1) (x_2-x_1) (x_4-x_2): \\ y_{30} &= (12,45,36) &:= & x_4 (x_2-1) (x_3-1) (x_2-x_1) (x_3-x_1): \\ y_{31} &= (14,35,26) &:= & -x_2 (x_2-1) (x_3-x_1) (x_4-x_3): \\ y_{32} &= (14,36,25) &:= & -x_2 (x_2-1) (x_3-x_1) (x_4-x_3): \\ y_{33} &= (14,25,36) &:= & -x_4 (x_4-1) (x_2-x_1) (x_4-x_3): \\ y_{35} &= (16,25,34) &:= & -x_2 x_3 (x_1-1) (x_4-x_2) (x_4-x_3): \\ y_{36} &= (15,26,34) &:= & -x_1 x_4 (x_2-1) (x_3-x_1) (x_4-x_3): \\ y_{38} &= (15,36,24) &:= & -x_1 x_4 (x_3-1) (x_2-x_1) (x_4-x_2): \\ y_{38} &= (15,36,24) &:= & -x_2 x_3 (x_4-1) (x_2-x_1) (x_3-x_1): \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2): \\ y_{39} &= (14,25,36) &:= & -x_2 x_3 (x_4-1) (x_2-x_1) (x_4-x_2): \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2): \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2): \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) : \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) : \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{39} &= (14,25,36) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{39} &= (14,25,56) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{39} &= (14,23,56) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{40} &= (14,23,56) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{40} &= (14,23,56) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{40} &= (14,23,56) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{40} &= (14,23,56) &:= & (x_2-x_1) (x_3-x_1) (x_4-x_2) (x_4-x_3): \\ y_{40} &= ($$

2.3 *G*-Equivariance of φ

Recall that the group <u>G</u> acts on M, and that G acts on \mathbf{P}^{80-1} . Let us identify the groups <u>G</u> and G by

$$\iota: s_{12} \mapsto s_1, \ldots, s_{56} \mapsto s_5, \ s_{123} \mapsto s_6.$$

Then we have

Theorem 1 The map $\varphi: M \to \mathbf{P}^{80-1}$ is G-equivariant:

$$g(\varphi(x)) = \varphi(\iota(g)x), \quad g \in G, \ x \in M,$$

that is,

$$(gy_{\alpha})(x) = c_g y_{\alpha}(\iota(g)x), \quad g \in G, \ \alpha \in \pm \mathcal{A}, \ x \in M,$$

where c_g is a rational function in (x_1, x_2, x_3, x_4) .

Convention: Once this theorem is established, we ignore the redundant ones $y_{-\alpha}(x) = -y_{\alpha}(x)$ and regard φ as the map

$$M \ni x \longmapsto : y_{\alpha}(x) :\in \mathbf{P}^{40-1}.$$

The group G still acts on \mathbf{P}^{40-1} by the transformations given in §2.3.

In order to prove the theorem, we have only to check the identity for a set of gnerators of G. Under s_1 , the fourty polynomials are transformed as follows:

where $c_1 = (x_1 x_2)^{-3}$, under s_2 ,

$y_1 \to y_1,$	$y_2 \rightarrow y_2,$	$y_3 \rightarrow -y_6,$	$y_4 \rightarrow y_4,$	$y_5 \rightarrow y_5,$
$y_6 \rightarrow -y_3,$	$y_7 \rightarrow -y_9,$	$y_8 \rightarrow -y_{10},$	$y_9 \rightarrow -y_7,$	$y_{10} \rightarrow -y_8,$
$y_{11} \to -y_{14},$	$y_{12} \to y_{12},$	$y_{13} \to y_{13},$	$y_{14} \to -y_{11},$	$y_{15} \to -y_{17},$
$y_{16} \to -y_{18},$	$y_{17} \to -y_{15},$	$y_{18} \to -y_{16},$	$y_{19} \to -y_{21},$	$y_{20} \to -y_{22},$
$y_{21} \to -y_{19},$	$y_{22} \to -y_{20},$	$y_{23} \to -y_{26},$	$y_{24} \to y_{24},$	$y_{25} \to y_{25},$
$y_{26} \rightarrow -y_{23},$	$y_{27} \to -y_{29},$	$y_{28} \to -y_{30},$	$y_{29} \rightarrow -y_{27},$	$y_{30} \to -y_{28},$
$y_{31} \to -y_{33},$	$y_{32} \rightarrow -y_{34},$	$y_{33} \to -y_{31},$	$y_{34} \to -y_{32},$	$y_{35} \rightarrow -y_{37},$
$y_{36} \to -y_{38},$	$y_{37} \rightarrow -y_{35},$	$y_{38} \to -y_{36},$	$y_{39} \to y_{39},$	$y_{40} \to y_{40},$

under s_3 ,

$y_1 \to c_3 y_1,$	$y_2 \rightarrow -c_3 y_3,$	$y_3 \rightarrow -c_3 y_2,$	$y_4 \rightarrow -c_3 y_7,$	$y_5 \rightarrow -c_3 y_8,$
$y_6 \rightarrow c_3 y_6,$	$y_7 \rightarrow -c_3 y_4,$	$y_8 \rightarrow -c_3 y_5,$	$y_9 \rightarrow c_3 y_9,$	$y_{10} \rightarrow c_3 y_{10},$
$y_{11} \to c_3 y_{11},$	$y_{12} \to -c_3 y_{16},$	$y_{13} \to -c_3 y_{15},$	$y_{14} \rightarrow -c_3 y_{39},$	$y_{15} \to -c_3 y_{13},$
$y_{16} \rightarrow -c_3 y_{12},$	$y_{17} \rightarrow c_3 y_{17},$	$y_{18} \to c_3 y_{18},$	$y_{19} \to -c_3 y_{29},$	$y_{20} \to -c_3 y_{30},$
$y_{21} \rightarrow -c_3 y_{34},$	$y_{22} \rightarrow -c_3 y_{33},$	$y_{23} \rightarrow -c_3 y_{40},$	$y_{24} \rightarrow -c_3 y_{38},$	$y_{25} \to -c_3 y_{37},$
$y_{26} \to c_3 y_{26},$	$y_{27} \rightarrow -c_3 y_{32},$	$y_{28} \to -c_3 y_{31},$	$y_{29} \rightarrow -c_3 y_{19},$	$y_{30} \to -c_3 y_{20},$
$y_{31} \to -c_3 y_{28},$	$y_{32} \rightarrow -c_3 y_{27},$	$y_{33} \to -c_3 y_{22},$	$y_{34} \rightarrow -c_3 y_{21},$	$y_{35} \rightarrow c_3 y_{35},$
$y_{36} \rightarrow c_3 y_{36},$	$y_{37} \rightarrow -c_3 y_{25},$	$y_{38} \rightarrow -c_3 y_{24},$	$y_{39} \rightarrow -c_3 y_{14},$	$y_{40} \rightarrow -c_3 y_{23},$

where $c_3 = (1 - x_3)^{-3}(1 - x_4)^{-3}$, under s_4 ,

$y_1 \rightarrow -c_4 y_5,$	$y_2 \to c_4 y_2,$	$y_3 \rightarrow -c_4 y_9,$	$y_4 \rightarrow c_4 y_4,$	$y_5 \rightarrow -c_4 y_1,$
$y_6 \rightarrow -c_4 y_7,$	$y_7 \rightarrow -c_4 y_6,$	$y_8 \to c_4 y_8,$	$y_9 \rightarrow -c_4 y_3,$	$y_{10} \to c_4 y_{10},$
$y_{11} \rightarrow -c_4 y_{29},$	$y_{12} \rightarrow c_4 y_{12},$	$y_{13} \to -c_4 y_{40},$	$y_{14} \rightarrow -c_4 y_{27},$	$y_{15} \to -c_4 y_{33},$
$y_{16} \rightarrow -c_4 y_{35},$	$y_{17} \rightarrow -c_4 y_{31},$	$y_{18} \rightarrow -c_4 y_{37},$	$y_{19} \rightarrow c_4 y_{19},$	$y_{20} \to -c_4 y_{26},$
$y_{21} \to c_4 y_{21},$	$y_{22} \to -c_4 y_{23},$	$y_{23} \to -c_4 y_{22},$	$y_{24} \rightarrow -c_4 y_{39},$	$y_{25} \to c_4 y_{25},$
$y_{26} \rightarrow -c_4 y_{20},$	$y_{27} \rightarrow -c_4 y_{14},$	$y_{28} \to c_4 y_{28},$	$y_{29} \rightarrow -c_4 y_{11},$	$y_{30} \to c_4 y_{30},$
$y_{31} \rightarrow -c_4 y_{17},$	$y_{32} \rightarrow -c_4 y_{38},$	$y_{33} \rightarrow -c_4 y_{15},$	$y_{34} \rightarrow -c_4 y_{36},$	$y_{35} \rightarrow -c_4 y_{16},$
$y_{36} \rightarrow -c_4 y_{34},$	$y_{37} \rightarrow -c_4 y_{18},$	$y_{38} \to -c_4 y_{32},$	$y_{39} \rightarrow -c_4 y_{24},$	$y_{40} \rightarrow -c_4 y_{13},$

where $c_4 = (x_1 x_3)^{-3}$, under s_5 ,

$y_1 \to y_1,$	$y_2 \rightarrow y_2,$	$y_3 \rightarrow y_3,$	$y_4 \rightarrow -y_5,$	$y_5 \rightarrow -y_4,$
$y_6 \rightarrow y_6,$	$y_7 \rightarrow -y_8,$	$y_8 \rightarrow -y_7,$	$y_9 \rightarrow -y_{10},$	$y_{10} \rightarrow -y_9,$
$y_{11} \to y_{11},$	$y_{12} \to -y_{13},$	$y_{13} \to -y_{12},$	$y_{14} \to y_{14},$	$y_{15} \to -y_{16},$
$y_{16} \to -y_{15},$	$y_{17} \to -y_{18},$	$y_{18} \to -y_{17},$	$y_{19} \to -y_{20},$	$y_{20} \rightarrow -y_{19},$
$y_{21} \to -y_{22},$	$y_{22} \to -y_{21},$	$y_{23} \to y_{23},$	$y_{24} \to -y_{25},$	$y_{25} \to -y_{24},$
$y_{26} \to y_{26},$	$y_{27} \rightarrow -y_{28},$	$y_{28} \rightarrow -y_{27},$	$y_{29} \rightarrow -y_{30},$	$y_{30} \to -y_{29},$
$y_{31} \to -y_{32},$	$y_{32} \rightarrow -y_{31},$	$y_{33} \to -y_{34},$	$y_{34} \rightarrow -y_{33},$	$y_{35} \rightarrow -y_{36},$
$y_{36} \rightarrow -y_{35},$	$y_{37} \rightarrow -y_{38},$	$y_{38} \rightarrow -y_{37},$	$y_{39} \rightarrow y_{39},$	$y_{40} \rightarrow y_{40},$

and under s_6 ,

$$\begin{array}{ll} y_1 \to -c_6 y_{39}, & y_2 \to c_6 y_2, & y_3 \to -c_6 y_{26}, & y_4 \to -c_6 y_{25}, & y_5 \to -c_6 y_{24}, \\ y_6 \to -c_6 y_{23}, & y_7 \to -c_6 y_{22}, & y_8 \to -c_6 y_{21}, & y_9 \to -c_6 y_{20}, & y_{10} \to -c_6 y_{19}, \\ y_{11} \to c_6 y_{11}, & y_{12} \to c_6 y_{12}, & y_{13} \to c_6 y_{13}, & y_{14} \to c_6 y_{14}, & y_{15} \to -c_6 y_{18}, \\ y_{16} \to -c_6 y_{17}, & y_{17} \to -c_6 y_{16}, & y_{18} \to -c_6 y_{15}, & y_{19} \to -c_6 y_{10}, & y_{20} \to -c_6 y_{9}, \\ y_{21} \to -c_6 y_8, & y_{22} \to -c_6 y_7, & y_{23} \to -c_6 y_6, & y_{24} \to -c_6 y_5, & y_{25} \to -c_6 y_4, \\ y_{26} \to -c_6 y_3, & y_{27} \to c_6 y_{27}, & y_{28} \to c_6 y_{28}, & y_{29} \to c_6 y_{29}, & y_{30} \to c_6 y_{30}, \\ y_{31} \to -c_6 y_{35}, & y_{32} \to -c_6 y_{36}, & y_{33} \to -c_6 y_{37}, & y_{34} \to -c_6 y_{38}, & y_{35} \to -c_6 y_{31}, \\ y_{36} \to -c_6 y_{32}, & y_{37} \to -c_6 y_{33}, & y_{38} \to -c_6 y_{34}, & y_{39} \to -c_6 y_1, & y_{40} \to c_6 y_{40}, \end{array}$$

where $c_6 = (x_1 x_2 x_3 x_4)^{-2}$. Maybe it is interesting to see what happens under the operation of the involution s_r (classically called the association) which sends (x_1, x_2, x_3, x_4) to

$$\left(\frac{(x_4-1)D_1}{(x_4-x_2)(x_4-x_3)},\frac{(x_3-1)D_1}{(x_3-x_1)(x_4-x_3)},\frac{(x_2-1)D_1}{(x_4-x_2)(x_2-x_1)},\frac{(x_1-1)D_1}{(x_3-x_1)/(x_2-x_1)}\right):$$

$$\begin{array}{lll} y_1 \to c_r y_1, & y_2 \to c_r y_2, & y_3 \to c_r y_3, & y_4 \to c_r y_4, & y_5 \to c_r y_5, \\ y_6 \to c_r y_6, & y_7 \to c_r y_7, & y_8 \to c_r y_8, & y_9 \to c_r y_9, & y_{10} \to c_r y_{10}, \\ y_{11} \to -c_r y_{26}, & y_{12} \to -c_r y_{25}, & y_{13} \to -c_r y_{24}, & y_{14} \to -c_r y_{23}, & y_{15} \to -c_r y_{38}, \\ y_{16} \to -c_r y_{37}, & y_{17} \to -c_r y_{36}, & y_{18} \to -c_r y_{35}, & y_{19} \to -c_r y_{30}, & y_{20} \to -c_r y_{29}, \\ y_{21} \to -c_r y_{28}, & y_{22} \to -c_r y_{27}, & y_{23} \to -c_r y_{14}, & y_{24} \to -c_r y_{13}, & y_{25} \to -c_r y_{12}, \\ y_{26} \to -c_r y_{11}, & y_{27} \to -c_r y_{22}, & y_{28} \to -c_r y_{21}, & y_{29} \to -c_r y_{20}, & y_{30} \to -c_r y_{19}, \\ y_{31} \to -c_r y_{34}, & y_{32} \to -c_r y_{33}, & y_{33} \to -c_r y_{32}, & y_{34} \to -c_r y_{31}, & y_{35} \to -c_r y_{18}, \\ y_{36} \to -c_r y_{17}, & y_{37} \to -c_r y_{16}, & y_{38} \to -c_r y_{15}, & y_{39} \to -c_r y_{40}, & y_{40} \to -c_r y_{39}, \end{array}$$

where

$$c_r = \left(\frac{D_1 D_2}{(-x_4 + x_2)(x_4 - x_3)(x_1 - x_3)(x_1 - x_2)}\right)^3.$$

2.4 φ embeds M

It is known in [10] that the map

$$M \ni x \longrightarrow y_1(x) : y_3(x) : y_4(x) : y_5(x) : y_7(x) \in \mathbf{P}^4$$

is two-to-one, and induces an embedding of the quotient space $M/\langle s_r \rangle$. Thus the composite of φ and the projection

$$M \longrightarrow \mathbf{P}^{40-1} \longrightarrow \mathbf{P}^4$$

is a two-to-one map. This fact together with the equivariance of φ under the involution s_r shown just above implies

Theorem 2 φ embeds M into \mathbf{P}^{40-1} .

2.5 Prolongation of φ to degenerate arrangements

Let us consider degenerate arrangements of six points on the plane. Since arrangements with three collinear points can be transformed under G to those with six points on a conic, we assume, Without loss of generality, that our arrangements represented by $x = (x_1, \ldots, x_4)$ satisfies Q = 0, that is, the six points are on a conic. Since a (nonsingular) conic is isomorphic to a line, such arrangements form the configuration space

$$X(2,6) = GL(2) \setminus \{Mat(2,6) \mid any \ 2 \times 2 \text{ minor } \neq 0\} / (\mathbf{C}^{\times})^{6}$$

of six points on the projective line: if we represent a point of X(2,6) by a matrix of the form

where

$$\prod_{i=1}^{3} z_i(z_i - 1) \prod_{1 \le i < j \le 3} (z_i - z_j) \neq 0,$$

then the degenerate arrangements in question can be parametrized by $z = (z_1, z_2, z_3)$ as

$$x_1 = (1 - z_1)/(1 - z_2), \quad x_2 = (1 - z_1)/(1 - z_3), \quad x_3 = z_1/z_2, \quad x_4 = z_1/z_3$$

Among the two S_6 -equivariant projective embedding of X(2, 6) presented in [9], let us recall the following one given by the fifteen polynomials

$$D_{ij}(z)D_{kl}(z)D_{mn}(z), \quad \{i, j, k, l, m\} = \{1, \dots, 6\},\$$

where $D_{ij}(z)$ is the (i, j)-minor of the 2 × 6-matrix z. Their actual forms are given by

$$(z_i - 1)(z_j - z_k), \quad z_j - z_k, \quad z_i(z_j - z_k), \quad z_i(z_j - 1).$$

It is known and easy to show that the image is projectively equivalent to the so-called Segre cubic defined by

$$t_0 + \cdots, t_5 = 0, \quad (t_0)^3 + \cdots + (t_5)^3 = 0.$$

On the other hand, let us prolong the domain of definition of the map φ on these degenerate arrangements by the same forty polynomials. Then the map φ in z-coordinates is given by $y_1 = \cdots y_{10} = 0$ and

$$\begin{array}{ll} cy_{11}=-z_1(z_2-z_3), & cy_{12}=-z_2+z_1, & cy_{13}=z_1-z_3, \\ cy_{14}=-(-1+z_1)(z_2-z_3), & cy_{15}=(-1+z_1)z_3, & cy_{16}=(-1+z_1)z_2, \\ cy_{17}=z_1(-1+z_3), & cy_{18}=z_1(-1+z_2), & cy_{19}=-(-z_2+z_1)z_3, \\ cy_{20}=-(z_1-z_3)z_2, & cy_{21}=-(-z_2+z_1)(-1+z_3), & cy_{22}=-(z_1-z_3)(-1+z_2), \\ cy_{23}=-(-1+z_1)(z_2-z_3), & cy_{24}=z_1-z_3, & cy_{25}=-z_2+z_1, \\ cy_{26}=-z_1(z_2-z_3), & cy_{27}=-(z_1-z_3)(-1+z_2), & cy_{28}=-(-z_2+z_1)(-1+z_3) \\ cy_{29}=-(z_1-z_3)z_2, & cy_{30}=-(-z_2+z_1)z_3, & cy_{31}=(-1+z_3)z_2, \\ cy_{32}=(-1+z_2)z_3, & cy_{33}=(-1+z_2)z_3, & cy_{34}=(-1+z_3)z_2, \\ cy_{35}=z_1(-1+z_2), & cy_{36}=z_1(-1+z_3), & cy_{37}=(-1+z_1)z_2, \\ cy_{38}=(-1+z_1)z_3, & cy_{39}=z_2-z_3, & cy_{40}=z_2-z_3, \end{array}$$

where

$$c = \frac{(z_1 - 1)(z_1 - z_3)(z_2 - z_3)(z_1 - z_2)z_1}{(1 - z_2)^2 z_3^2 (1 - z_3)^2 z_2^2}.$$

This shows that the prolonged φ gives exactly the embedding of the arranged arragements given by $\{x \mid Q(x) = 0\}$, isomorphic to X(2, 6), given above.

Further put

$$x_1 = t\xi_1, \quad x_2 = t\xi_2, \quad x_3 = t\xi_3, \quad x_4 = 1 + t\xi_4,$$

and let t tends to zero in $\varphi(x)$. We can easily see that the limit is a point which is independ of (ξ_1, \ldots, ξ_4) .

These results together with the facts on Naruki's cross-ratio C variety reviewd in §1.3, we can readily show

Theorem 3 The closure of the image of M under φ is isomorphic to the variety obtained from Naruki's cross-ratio C by blowing down the 40 exceptional divisors of the second kind to points.

Remark: This variety is isomorphic to the Stake compactification of the modular variety obtained in [1]. Note that the 40 cusps correspond to the 40 points obtained by blowing down the divisors of the second kind.

3 Defining equations

We will find a set of generators of the ideal defining the closure $\overline{\varphi(M)}$ of the image of M in \mathbf{P}^{40-1} . Among the forty polynomials $y_1(x), \ldots, y_{40}(x)$, we can find some relations. The Prücker relations ([10]) of the 3×3 -minors of a 3×6 -matrix yield linear relations among $y_1(x), \ldots, y_{10}(x)$; for example,

 $(124, 356) - (145, 236) + (146, 235) - (134, 256) = y_3(x) - y_4(x) + y_5(x) - y_6(x) = 0.$

On the other hand, it is easy to check the cubic relation

$$(124, 356)(15, 23, 46)(13, 26, 45) - (145, 236)(13, 56, 24)(12, 35, 46) = y_3(x)y_{13}(x)y_{21}(x) - y_4(x)y_{14}(x)y_{20}(x) = 0.$$

Let V be the subvariety of \mathbf{P}^{40-1} , coordinatized by y_1, \ldots, y_{40} , defined by the G-orbits of the linear and the cubic equations:

 $y_3 - y_4 + y_5 - y_6 = 0, \quad y_3 y_{13} y_{21} - y_4 y_{14} y_{20} = 0.$

Theorem 4 $\overline{\varphi(M)} = V$.

3.1 An outline of the proof

By operating the group G to the linear equation, we get many linear relations among y_1, \ldots, y_{40} . These relations form a system of rank 30, that is, all y's can be expressed linearly in terms of ten chosen ones. For example, in terms of

 $y_1, y_3, y_4, y_5, y_7, y_{11}, y_{12}, y_{13}, y_{15}, y_{19},$

the remaining thirty ones are expressed as

$$\begin{array}{rcl} y_2 &=& y_1 - y_5 + y_4, & y_6 = -y_4 + y_5 + y_3, & y_8 = -y_3 - y_1 + y_7, \\ y_9 &=& -y_1 + y_7 - y_4, & y_{10} = y_7 - y_5 - y_3, & y_{14} = y_{11} + y_{13} - y_{12}, \\ y_{16} &=& -y_1 + y_{12} - y_{13} - y_{11} + y_{15}, & y_{17} = y_{15} - y_1 - y_{13}, \\ y_{18} &=& -y_{13} - y_{11} + y_{15}, & y_{20} = y_{19} + y_3 + y_{11}, & y_{21} = y_4 + y_{19} + y_{12}, \\ y_{22} &=& y_{19} + y_3 + y_{11} + y_{13} + y_5, & y_{23} = y_1 - y_{12} + y_{13} + y_3 + y_{11}, \\ y_{24} &=& y_4 + y_1 + y_{13}, & y_{25} = y_{12} + y_5 - y_1, \\ y_{26} &=& -y_4 + y_5 - y_1 + y_3 + y_{11}, & y_{27} = y_{19} + y_4 + y_1 + y_3 - y_7 + y_{11} + y_{13}, \\ y_{28} &=& -y_7 + y_{19} + y_{12} + y_5 + y_3, & y_{29} = y_3 - y_7 + y_{11} + y_{19} + y_5, \\ y_{30} &=& y_3 + y_1 - y_7 + y_{19} + y_4, & y_{31} = -y_1 - y_{13} + y_{19} + y_{15} + y_{12}, \\ y_{32} &=& y_{19} + y_3 + y_{15}, & y_{33} = y_{19} + y_4 + y_{15}, \\ y_{34} &=& y_{19} - y_{13} + y_{15} + y_{12} + y_5 - y_1 + y_3, \\ y_{35} &=& -y_5 - y_1 - y_3 + y_7 - y_{11} + y_{15} - y_{13}, \\ y_{36} &=& -y_1 + y_7 - y_4 - y_{13} + y_{15}, \\ y_{37} &=& -y_3 + y_7 - y_{11} + y_{15} - y_1 + y_{12} - y_{13}, \\ y_{38} &=& -y_1 + y_7 + y_{15}, & y_{39} = y_1 - y_{12} + y_{13}, \\ y_{40} &=& -y_5 + y_4 + y_1 + y_{13} - y_{12}. \end{array}$$

By operating the group G to the cubic equation, we get many such relations. Substituting the expressions of the y's obtained above, we get cubic relations in terms of the chosen ten y's; let us here rename the ten coordinates as:

$$g_1 = y_1, \ g_2 = y_3, \ g_3 = y_4, \ g_4 = y_5, \ g_5 = y_7, \ g_6 = y_{11}, \ g_7 = y_{12}, \ g_8 = y_{13}, \ g_9 = y_{15}, \ g_0 = y_{19}.$$

Among these cubic equations we can find exactly thirty linearly independent ones. Therefore the variety V is isomorphic to a subvariety of \mathbf{P}^9 , coordinatized by g_1, \ldots, g_9, g_0 , defined by thirty cubic equations, say cub_1, \ldots, cub_{30} . Some of them which are relatively simple are shown below:

We first study this system over the field $K := \mathbf{C}(g_1, \ldots, g_5)$. Geometrically speaking, we project the variety V onto the 4-dimensional projective space coordi-

natized by $g_1 : \cdots : g_5$, and study the generic fibre of the projection

$$\pi: \mathbf{P}^9 \supset V \ni g_1: \dots: g_0 \longmapsto g_1: \dots: g_5 \in \mathbf{P}^4.$$

We shall prove that π is generically two-to-one. This implies π is two-to-one on

$$V^{\circ} := V - V \cap \bigcup_{j=1}^{40} \{ y_j = 0 \}.$$

Thus the argument in §2.4 shows that $\varphi: M \to V^{\circ}$ is an isomorphism.

We next study the intersection $V \cap \{y_1 = 0\}$, and prove that $V \cap \bigcup_{j=1}^{40} \{y_j = 0\}$ is the totality of the *G*-orbit of the closure of the image of X(2, 6) under the prolonged φ , stated in §2.5.

3.2 Computation over K

From $cub_3 = 0$ and $cub_7 = 0$, we can solve g_8 and g_9 as:

$$g_8 = g_3g_6(g_2 + g_6 + g_0 + g_4)/(g_0g_2 - g_0g_3 + g_0g_4 - g_3g_6),$$

$$g_9 = g_5g_6(g_0 + g_3)/(-g_2g_3 + g_5g_3 - g_3g_6 - g_0g_3 - g_4g_3 + 2g_2g_4 - g_5g_4 + g_4g_6 + g_0g_4 + g_4^2 + g_2^2 - g_5g_2 + g_2g_6 + g_0g_2 - g_5g_6).$$

Substituting these into cub_{19} , we can solve g_6 :

$$g_6 = -\frac{g_1(-g_5g_0 - g_5g_2 + g_0^2 + 2g_0g_2 + g_0g_4 + g_7g_0 + g_2g_7 + g_2^2 + g_2g_4)}{g_0g_1 + g_1g_4 + g_1g_7 + g_2g_1 - g_5g_7 - g_5g_4}$$

Substituting these expressions into cub_1, \ldots, cub_{30} , we have $cub_3 = cub_7 = cub_{19} = 0$ of course and $cub_{10} = 0$; though most of the remaining ones are complicated, cub_8 is relatively simple:

$$cub_8 = -(g_1 - g_5)qq/(g_0g_1 + g_1g_4 + g_1g_7 + g_2g_1 - g_5g_7 - g_5g_4),$$

where

$$qq = g_1g_4t^2 + (-g_2g_5 - g_5g_4 + g_1g_4 + g_5g_3)s^2 + 2g_1g_4st + (-g_5g_3g_1 + g_2g_1g_5 + g_1g_4g_3 - g_4^2g_5 + g_1g_2g_4 + g_5g_3g_4 - g_2g_5g_4 + g_4^2g_1)s + (-g_4g_5g_1 + g_4^2g_1 + g_1g_4g_3 + g_1g_2g_4)t + g_3g_2g_1g_4 - g_5g_3g_1g_4 + g_1g_4^2g_3$$

is a quadratic form in

$$t := g_0, \quad s := g_7.$$

By this quadratic equation, we reduce the degree with respect to t of the numerators of the *cub*'s to 1. In this way we get 26 (= 30 - 4) equations of the form

$$cube_j: a_jt + b_j = 0, \quad j = 1, \dots, 30, \quad j \neq 3, 7, 10, 19,$$

where $a_j, b_j \in K[s]$. The polynomials

$$D_{jk} = a_j b_k - a_k b_j \in K[s], \quad j < k$$

have a unique common factor dd, which is quadratic in s; its actual form is given by

$$\begin{aligned} dd &:= (g_1^2 g_4^2 + 2g_2 g_1^2 g_4 + g_3^2 g_5^2 + g_2^2 g_3^2 + g_2^2 g_1^2 + g_5^2 g_4^2 + 2g_3 g_2 g_1 g_4 - 2g_2 g_1 g_5 g_4 \\ &- 2g_3 g_5^2 g_4 - 2g_1 g_5 g_4^2 + 2g_2^2 g_3 g_1 - 2g_3^2 g_5 g_2 + 2g_5 g_3 g_2 g_4 + 2g_5 g_3 g_1 g_4 - 2g_5 g_3 g_1 g_2) s^2 \\ &+ (-g_2^2 g_1^3 - 2g_1^3 g_2 g_4 + 2g_1^2 g_2 g_4^2 + 2g_1^2 g_5 g_4^2 + g_1^2 g_4^3 - g_1^3 g_4^2 - 2g_1 g_5 g_4^3 \\ &- 2g_3 g_5^2 g_4^2 - 2g_3 g_2^2 g_1^2 + 2g_3 g_2 g_1 g_4^2 + g_3^2 g_2^2 g_4 + g_5^2 g_4^3 + 2g_1^2 g_2 g_5 g_4 \\ &+ 2g_2^2 g_3 g_1 g_4 - 4g_3 g_5 g_1 g_2 g_4 + g_5^2 g_3^2 g_4 + g_2^2 g_1^2 g_4 - g_1 g_5^2 g_4^2 - g_2^2 g_3^2 g_1 \\ &- g_3^2 g_5^2 g_1 + 2g_5 g_3 g_2 g_4^2 + 2g_5 g_1^2 g_2 g_3 + 2g_3^2 g_5 g_1 g_2 - 2g_1 g_2 g_5 g_4^2 \\ &+ 2g_3 g_5^2 g_1 g_4 + 2g_3 g_1 g_5 g_4^2 - 2g_3 g_5 g_1^2 g_4 - 2g_3^2 g_5 g_2 g_4 - 2g_1^2 g_2 g_3 g_4) s \\ &- g_1^2 g_3^2 g_4^2 + g_3 g_1^2 g_4^3 - g_1^2 g_2 g_3^2 g_4 - g_1^2 g_3 g_2 g_4 - g_3^2 g_1 g_2^2 g_4 \\ &- g_3^2 g_1 g_2 g_4^2 + g_3 g_1 g_2 g_4^3 + g_2^2 g_3 g_1 g_4^2 - g_3 g_5^2 g_1^2 g_4 + g_1^3 g_3 g_5 g_4 \\ &+ g_3^2 g_5 g_1 g_4^2 - g_3^2 g_5^2 g_1 g_4 + g_3 g_5^2 g_1 g_4^2 - g_3 g_5 g_1 g_4^3 - g_3 g_1^2 g_2^2 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_1 g_2^2 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_1 g_2^2 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_1 g_2 g_2 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_1 g_2 g_5 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_3 g_1 g_4^2 - g_3^2 g_3^2 g_1 g_4 g_3 g_3 g_2^2 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_3 g_1 g_2 g_5 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_3 g_1 g_2 g_5 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g_2 g_5 g_4^2 + 2g_3 g_1^2 g_2 g_5 g_4 + 2g_3^2 g_3 g_1 g_2 g_5 g_4 \\ &+ g_3^2 g_5 g_1^2 g_4 - 2g_3 g_1 g$$

So dd(s) = 0 is the compatibility condition of the system of equations $cube_j$: $a_j(s)t + b_j(s) = 0$, among which $cube_{11} : a_{11}(s)t + b_{11}(s) = 0$ has the minimal degree of coefficients; actually, degrees of a_{11} and b_{11} with respect to s are 3 and 4.

Now we get three equations dd = 0, qq = 0 and $cube_{11} = 0$. Substituting the expression $t = -b_{11}/a_{11}$ into qq, we get a polynomial in s. We can prove that this polynomial has dd as a factor.

We have proved that for a given generic point $g_0 : \cdots : g_5 \in \mathbf{P}^4$, the inverse under π consists of two points. They are obtained as follows: solve the quadratic equation dd = 0 with respect to $s(=g_7)$. For each root, $t(=g_0)$ is uniquely determined by the linear equation $cube_{11}$. And g_6, g_8, g_9 are uniquely determined by cub3, cub7, cub19 as stated above. Then all the relations are satisfied.

3.3 Intersection of V and $\{y_1 = 0\}$

We should better work on V living in \mathbf{P}^{40-1} . In terms of the forty coordinates y_1, \ldots, y_{40} , the cubic equations are 2-term equations. Thus the vanishing of y_1 implies that of some other coordinates. Thanks to G-action, we can assume that $y_{10} = 0$. The vanishing of these two coordinates forces the vanishing of other coordinates. Tedious case-by-case study shows that every component of $V \cap \{y_1 = 0\}$ is included in the G-orbit of

$$V \cap \{y_1 = \dots = y_9 = y_0 = 0\},\$$

which is the closure of the image of $\{x \mid Q(x) = 0\}$ under the prolonged φ .

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