# A  $W(E_6)$ -equivariant projective embedding of the moduli space of cubic surfaces

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#### Abstract

An explicit projective embedding of the moduli space of marked cubic surfaces is given. This embedding is equivariant under the Weyl group of type  $E_6$ . The image is defined by a system of linear and cubic equations. To express the embedding in a most symmetric way, the target would be 79-dimensional, however the image lies in a 9-dimensional linear subspace.

## 1 The moduli space of cubic surfaces

### 1.1 The space M and the action of the group  $G$

We first fix some notation and recall a few known facts on the moduli space of marked cubic surfaces. The moduli space of marked cubic surfaces, which we denote by  $M$ , is studied for example in[[5](#page-14-0)] and[[8](#page-14-0)]. Since any nonsingular cubic surface can be obtained by blowing up the projective plane  $\mathbf{P}^2$  at six points, it can be represented by a  $3 \times 6$ -matrix of which columns give homogeneous coordinates of the six points. In order to get a smooth cubic surface from six points, we assume that no three points are collinear and the six points are not on a conic. On the set of  $3 \times 6$ matrices, we have a cannical action of  $GL_3$  on the left and the group  $\mathbb{C}^{\times}$  acts naturally on homogeneous coordinates. By killing such ambiguity of coordinates, we get the following expression

$$
x = \left(\begin{array}{rrrrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & x_3 & x_4 \end{array}\right);
$$

in this paper we use local coordinates  $(x_1, x_2, x_3, x_4)$  on M. The six points represented by the matrix above produces a non-singular cubic surface if and only if the following quantitiy does not vanish.

$$
D(x) := x_1 x_2 x_3 x_4 (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)
$$
  
 
$$
\times (x_1 - x_2)(x_1 - x_3)(x_2 - x_4)(x_3 - x_4) D_1 D_2 Q,
$$

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where

$$
D_1 := x_1x_4 - x_2x_3,
$$
  
\n
$$
D_2 := x_1x_4 - x_4 + x_2 - x_2x_3 + x_3 - x_1,
$$
  
\n
$$
Q := -x_2x_3x_1 - x_2x_3x_4 + x_2x_3 + x_1x_4x_2 + x_1x_4x_3 - x_1x_4,
$$

Thus we can identify the moduli space M with the affine open set  $\{x = (x_1, \ldots, x_4) \mid$  $D(x) \neq 0$ .

Let us define as in [\[8](#page-14-0)] six bi-rational transformations  $s_1, \ldots, s_6$  in  $x = (x_1, \ldots, x_4)$ :

$$
s_1: (x_1, x_2, x_3, x_4) \rightarrow \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{x_3}{x_1}, \frac{x_4}{x_2}\right),
$$
  
\n
$$
s_2: (x_1, x_2, x_3, x_4) \rightarrow (x_3, x_4, x_1, x_2),
$$
  
\n
$$
s_3: (x_1, x_2, x_3, x_4) \rightarrow \left(\frac{x_1 - x_3}{1 - x_3}, \frac{x_2 - x_4}{1 - x_4}, \frac{x_3}{x_3 - 1}, \frac{x_4}{x_4 - 1}\right),
$$
  
\n
$$
s_4: (x_1, x_2, x_3, x_4) \rightarrow \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{1}{x_3}, \frac{x_4}{x_3}\right),
$$
  
\n
$$
s_5: (x_1, x_2, x_3, x_4) \rightarrow (x_2, x_1, x_4, x_3),
$$
  
\n
$$
s_6: (x_1, x_2, x_3, x_4) \rightarrow \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}\right).
$$

If M is regarded as the configuration space of six points in  $\mathbf{P}^2$ , the transformation  $s<sub>1</sub>$ , for example, corresponds to the interchange of the two points represented by the first two column vectors of the matrix x. Each  $s_i$  turns out to be a bi-regular involution on  $M$ , and they form a group  $G$  isomorphic to the Weyl group of type  $E_6$ ; relation of the generators are given by the Coxeter graph

$$
\begin{array}{cccc} s_1 & -s_2 & -s_3 & -s_4 & -s_5 \ & & | & & \\ s_6 & & & & \end{array}
$$

### 1.2 Root system  $\Delta$  of type  $E_6$

Wereview the root system  $\Delta$  of type  $E_6$ , following [[6\]](#page-14-0). Consider an 8-dimensional Euclidean space E with a standard basis  $\varepsilon_1, \dots, \varepsilon_8$ . Let  $\langle \cdot, \cdot \rangle$  be the inner product on  $\tilde{E}$  defined by  $\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$  and let E be the linear subspace of  $\tilde{E}$  spanned by the six vectors

$$
\varepsilon_1,\cdots,\varepsilon_5,\ \tilde{\varepsilon}=\varepsilon_6-\varepsilon_7-\varepsilon_8.
$$

We introduce the 36 vectors:

$$
r = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \tilde{\varepsilon}),
$$
  
\n
$$
r_{1j} = -\varepsilon_{j-1} + r_0, \quad 2 \le j \le 6
$$
  
\n
$$
r_{jk} = \varepsilon_{j-1} - \varepsilon_{k-1}, \quad 2 \le j < k \le 6
$$

$$
r_{1jk} = -\varepsilon_{j-1} - \varepsilon_{k-1}, \quad 2 \le j < k \le 6
$$
\n
$$
r_{ijk} = -\varepsilon_{i-1} - \varepsilon_{j-1} - \varepsilon_{k-1} + r_0, \quad 2 \le i < j < k \le 6
$$

where  $r_{ij} = r_{ji}, r_{ijk} = r_{jik} = r_{ikj}$ ,

$$
r_0 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \tilde{\varepsilon}).
$$

Note that

$$
r \perp r_{ij}
$$
,  $r_{ij} \perp r_{kl}$ ,  $r_{ij} \perp r_{ijk}$ ,  $r_{ij} \perp r_{klm}$ ,  $r_{ijk} \perp r_{ilm}$ .

The set

$$
\Delta = \{\pm r, \ \pm r_{ij}, \ \pm r_{ijk}\}
$$

forms a root system of type  $E_6$ . For example,

 $r_{12}$ ,  $r_{123}$ ,  $r_{23}$ ,  $r_{34}$ ,  $r_{45}$ ,  $r_{56}$ 

can serve as a system of positive simple roots; its extended Dynkin diagram is given as



The set  $\{r, r_{jk}, r_{ijk}\}$  is the totality of positive roots of  $\Delta$ .

Let  $s_r, s_{ij}$  and  $s_{ijk}$  be the reflections on E with respect to  $r, r_{ij}$  and  $r_{ijk}$ . These reflections act on  $\Delta$  as

$$
s_r: r_{ij} \leftrightarrow r_{ij}, r_{ijk} \leftrightarrow r_{lmn}, \quad \{i, j, k, l, m, n\} = \{1, ..., 6\},
$$
  
\n
$$
s_{ij}: \text{permutation of the indices } i \text{ and } j,
$$
  
\n
$$
s_{123}: r_{12} \leftrightarrow r_{12}, r_{14} \leftrightarrow r_{234}, r_{56} \leftrightarrow r_{56},
$$
  
\n
$$
r_{145} \leftrightarrow r_{145}, r \leftrightarrow r_{456},
$$

modulo signs. Let us define two reflection group

$$
G_1 = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56} \rangle \cong S_6, \quad G = \langle G_1, s_{123} \rangle \cong W(E_6),
$$

where  $S_6$  is the symmetric group on six numerals  $\{1, \ldots, 6\}$ ,  $W(E_6)$  the Weyl group of type  $E_6$ , and  $\langle a, b, \ldots \rangle$  denotes the group generated by  $a, b, \ldots$  Note that

$$
G_1 \subset \langle G_1, s_r \rangle = G_1 \times \langle s_r \rangle \subset G,
$$

and G acts transitively on  $\Delta$ .

### 1.3 Naruki's cross-ratio variety

A smooth compactification of M known as Naruki's cross-ratio variety  $\mathcal{C}$  ([\[5](#page-14-0)], [\[8](#page-14-0)]), embedded in  $(\mathbf{P}^1)^{45}$ , is the union of M and the 76 divisors. The 36 of them correspond to the positive roots of  $\Delta$  (they are said to be of the first kind), the other 40 divisors (said to be of the second kind, and are isomorphic to  $(\mathbf{P}^1)^3$ ) can be represented as follows: Take three subsets  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$  of  $\Delta$  satisfying the following conditions:

- Each of  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$  is a root system of type  $A_2$ .
- $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$  are mutually orthogonal.
- The vectors in  $\Delta_1 \cup \Delta_2 \cup \Delta_3$  span E.

Note that each one of such three root systems determines the other two.

Such a triple  $\{\Delta_1, \Delta_2, \Delta_3\}$  dertermines a divisor. According to the naming of the roots, we must use two different expressions: The first one is of the form

$$
\{\pm r_{12}, \pm r_{23}, \pm r_{13}\}, \quad \{\pm r_{45}, \pm r_{56}, \pm r_{46}\}, \quad \{\pm r, \pm r_{123}, \pm r_{456}\},
$$

(the corresponding divisor is denoted by  $Z_{123,456}$  in [\[8\]](#page-14-0),) and the second one is of the form

 $\{\pm r_{12}, \pm r_{234}, \pm r_{134}\}, \quad \{\pm r_{34}, \pm r_{356}, \pm r_{456}\}, \quad \{\pm r_{56}, \pm r_{125}, \pm r_{126}\},$ 

(thecorresponding divisor is denoted by  $Z_{12,34,56}$  in [[8\]](#page-14-0)). Note that

$$
Z_{123,456} = Z_{456,123}, \quad Z_{12,34,56} = Z_{56,12,34} \neq Z_{12,56,34}.
$$

Thus, permuting the indices under  $G_1$ , we have  $40 (= 10 + 30)$  such divisors. The group G acts transitively on these 40 divisors.

Remark: These divisors (of the second kind) are disjoint to each other, and can be blown-down to points. In fact they correspond bijectively to the cusps of the modular group studied in[[1\]](#page-13-0) (see also [\[3\]](#page-14-0) and [\[7\]](#page-14-0)).

# 2 Embedding  $\varphi : M \to \mathbf{P}^{80-1}$

### 2.1 Coordinates on  $\mathbf{P}^{80-1}$  and the action of G

Let A be the set of 40 labels  $(123, 456)$  and  $(12, 34, 56)$  with the following identification

$$
(123, 456) = (213, 456) = (132, 456) = (456, 123),(12, 34, 56) = (21, 34, 56) = (56, 12, 34).
$$

Since G acts on the set  $\Delta$  of roots, it acts also on the set of 40 divisors above, and so that it also acts on the set  $\mathcal A$  of 40 labels.

We introduce 80 homogeneous coordinates

$$
y_{\alpha}, \quad y_{-\alpha}, \qquad \alpha \in \mathcal{A}
$$

on  $\mathbf{P}^{80-1}$ . I define an action of G on  $\mathbf{P}^{80-1}$  by the following action of the generators  $s_{12}, \ldots, s_{56}$  and  $s_{123}$  on the coordinates. Let s be one of the generators and  $\alpha \in \mathcal{A}$ ; we assign

$$
s(y_{\alpha}) = y_{\alpha}, \quad s(y_{-\alpha}) = y_{-\alpha} \quad \text{if } s\alpha = \alpha,
$$
  

$$
s(y_{\alpha}) = y_{-\beta}, \quad s(y_{-\alpha}) = y_{\beta} \quad \text{if } s\alpha = \beta \neq \alpha.
$$

### 2.2 Definition of  $\varphi$

In this section we define a map  $M \to \mathbf{P}^{80-1}$ . For a 3 × 6 matrix  $x = (x_{ij})$ , we consider 80 polynomials of degree 18 as follows:

$$
y_{(123,456)}(x) = D_{123}(x)D_{456}(x)Q(x),
$$
  
\n
$$
y_{(12,34,56)}(x) = D_{134}(x)D_{234}(x)D_{356}(x)D_{456}(x)D_{512}(x)D_{612}(x),
$$

and  $y_{-\alpha}(x) = -y_{\alpha}$  ( $\alpha \in \mathcal{A}$ ), where  $Q(x)$  is the determinant of the 6 × 6-matrix with columns

$$
(x_{1j}x_{2j}, x_{2j}x_{3j}, x_{3j}x_{1j}, x_{1j}^2, x_{2j}^2, x_{3j}^2) \quad j = 1, \ldots, 6.
$$

Since we have

$$
y_{\alpha}(gxh) = (\det g)^{6} y_{\alpha}(x) (\det h)^{3},
$$

the correswpondence above defines a map  $\varphi : M \to \mathbf{P}^{80-1}$ . For later use, we present 40 polynomials  $y_{\alpha}(x)$  in terms of the coordinates  $(x_1, x_2, x_3, x_4)$  introduced in §1; the remaining 40 polynomoals are given by  $y_{-\alpha}(x) = -y_{\alpha}(x)$ . In the following table,  $y_{\alpha}(x)$  is denoted simply by  $\alpha$ , and we number them as  $y_1, \ldots, y_{40}$ :

$$
y_1 = (156, 234) := D_1Q: y_2 = (123, 456) := D_2Q:
$$
  
\n
$$
y_3 = (124, 356) := (x_2 - x_1)Q: y_4 = (145, 236) := (x_3 - x_1)Q:
$$
  
\n
$$
y_5 = (146, 235) := (x_4 - x_2)Q: y_6 = (134, 256) := (x_4 - x_3)Q:
$$
  
\n
$$
y_7 = (135, 246) := x_1(x_4 - 1)Q: y_8 = (136, 245) := x_2(x_3 - 1)Q:
$$
  
\n
$$
y_9 = (125, 346) := x_3(x_2 - 1)Q: y_{10} = (126, 345) := x_4(x_1 - 1)Q:
$$
  
\n
$$
y_{11} = (12, 56, 34) := D_1(x_1 - 1)(x_2 - 1)(x_4 - x_3):
$$
  
\n
$$
y_{12} = (16, 23, 45) := D_1(x_1 - 1)(x_3 - 1)(x_4 - x_2):
$$
  
\n
$$
y_{13} = (15, 23, 46) := D_1(x_2 - 1)(x_4 - 1)(x_3 - x_1):
$$
  
\n
$$
y_{14} = (13, 56, 24) := D_1(x_3 - 1)(x_4 - 1)(x_2 - x_1):
$$
  
\n
$$
y_{15} = (15, 24, 36) := -D_1x_1(x_2 - 1)(x_3 - 1):
$$
  
\n
$$
y_{16} = (16, 24, 35) := -D_1x_2(x_1 - 1)(x_4 - 1):
$$
  
\n
$$
y_{17} = (15, 34, 26) := -D_1x_3(x_1 - 1)(x_4 - 1):
$$
  
\n
$$
y_{18} = (16, 34, 25) := -D_1x_4(x_2 - 1)(x_3 - 1):
$$
  
\n
$$
y_{19} = (12, 36, 45)
$$

$$
y_{20} = (12, 35, 46) := D_2x_1x_4(x_2 - 1):
$$
  
\n
$$
y_{21} = (13, 26, 45) := D_2x_1x_4(x_3 - 1):
$$
  
\n
$$
y_{22} = (13, 25, 46) := D_2x_2x_3(x_4 - 1):
$$
  
\n
$$
y_{23} = (13, 24, 56) := D_2x_1x_2(x_4 - x_3):
$$
  
\n
$$
y_{24} = (15, 46, 23) := D_2x_1x_3(x_4 - x_2):
$$
  
\n
$$
y_{25} = (16, 45, 23) := D_2x_2x_4(x_3 - x_1):
$$
  
\n
$$
y_{26} = (12, 34, 56) := D_2x_3x_4(x_2 - x_1):
$$
  
\n
$$
y_{27} = (13, 46, 25) := x_1(x_2 - 1)(x_3 - 1)(x_4 - x_2)(x_4 - x_3):
$$
  
\n
$$
y_{28} = (13, 45, 26) := x_2(x_1 - 1)(x_4 - 1)(x_3 - x_1)(x_4 - x_3):
$$
  
\n
$$
y_{29} = (12, 46, 35) := x_3(x_1 - 1)(x_4 - 1)(x_2 - x_1)(x_4 - x_2):
$$
  
\n
$$
y_{30} = (12, 45, 36) := x_4(x_2 - 1)(x_3 - 1)(x_2 - x_1)(x_3 - x_1):
$$
  
\n
$$
y_{31} = (14, 35, 26) := -x_1(x_1 - 1)(x_4 - x_2)(x_4 - x_3):
$$
  
\n
$$
y_{32} = (14, 36, 25) := -x_2(x_2 - 1)(x_3 - x_1)(x_4 - x_2):
$$
  
\n
$$
y_{33} = (14, 25, 36) := -x_3(x_3 - 1)(x_2 - x_1)(x_4 - x_2):
$$
  
\n

# 2.3 G-Equivariance of  $\varphi$

Recall that the group  $G$  acts on  $M$ , and that G acts on  $\mathbf{P}^{80-1}$ . Let us identify the groups  $G$  and  $G$  by

$$
\iota:s_{12}\mapsto s_1,\ldots,s_{56}\mapsto s_5,\ s_{123}\mapsto s_6.
$$

Then we have

**Theorem 1** The map  $\varphi : M \to \mathbf{P}^{80-1}$  is G-equivariant:

$$
g(\varphi(x)) = \varphi(\iota(g)x), \quad g \in G, \ x \in M,
$$

that is,

$$
(gy_{\alpha})(x) = c_g y_{\alpha}(\iota(g)x), \quad g \in G, \ \alpha \in \pm \mathcal{A}, \ x \in M,
$$

where  $c_g$  is a rational function in  $(x_1, x_2, x_3, x_4)$ .

Convention: Once this theorem is established, we ignore the redundant ones  $y_{-\alpha}(x) = -y_\alpha(x)$  and regard  $\varphi$  as the map

$$
M \ni x \longmapsto : y_{\alpha}(x) : \in \mathbf{P}^{40-1}.
$$

The group G still acts on  $\mathbf{P}^{40-1}$  by the transformations given in§2.3.

In order to prove the theorem, we have only to check the identity for a set of gnerators of  $G$ . Under  $s_1$ , the fourty polynomials are transformed as follows:

 $y_1 \rightarrow -c_1y_6$ ,  $y_2 \rightarrow c_1y_2$ ,  $y_3 \rightarrow c_1y_3$ ,  $y_4 \rightarrow -c_1y_8$ ,  $y_5 \rightarrow -c_1y_7$ ,  $y_6 \to -c_1y_1$ ,  $y_7 \to -c_1y_5$ ,  $y_8 \to -c_1y_4$ ,  $y_9 \to c_1y_9$ ,  $y_{10} \to c_1y_{10}$ ,  $y_{11} \rightarrow c_1 y_{11}$ ,  $y_{12} \rightarrow -c_1 y_{28}$ ,  $y_{13} \rightarrow -c_1 y_{27}$ ,  $y_{14} \rightarrow -c_1 y_{40}$ ,  $y_{15} \rightarrow -c_1 y_{32}$ ,  $y_{16} \rightarrow -c_1y_{31}, y_{17} \rightarrow -c_1y_{35}, y_{18} \rightarrow -c_1y_{36}, y_{19} \rightarrow c_1y_{19}, y_{20} \rightarrow c_1y_{20},$  $y_{21} \rightarrow -c_1y_{25}$ ,  $y_{22} \rightarrow -c_1y_{24}$ ,  $y_{23} \rightarrow -c_1y_{39}$ ,  $y_{24} \rightarrow -c_1y_{22}$ ,  $y_{25} \rightarrow -c_1y_{21}$ ,  $y_{26} \rightarrow c_1y_{26}$ ,  $y_{27} \rightarrow -c_1y_{13}$ ,  $y_{28} \rightarrow -c_1y_{12}$ ,  $y_{29} \rightarrow c_1y_{29}$ ,  $y_{30} \rightarrow c_1y_{30}$ ,  $y_{31} \rightarrow -c_1y_{16}, y_{32} \rightarrow -c_1y_{15}, y_{33} \rightarrow -c_1y_{38}, y_{34} \rightarrow -c_1y_{37}, y_{35} \rightarrow -c_1y_{17}$  $y_{36} \rightarrow -c_1y_{18}, y_{37} \rightarrow -c_1y_{34}, y_{38} \rightarrow -c_1y_{33}, y_{39} \rightarrow -c_1y_{23}, y_{40} \rightarrow -c_1y_{14},$ 

where  $c_1 = (x_1 x_2)^{-3}$ , under  $s_2$ ,

 $y_1 \rightarrow y_1$ ,  $y_2 \rightarrow y_2$ ,  $y_3 \rightarrow -y_6$ ,  $y_4 \rightarrow y_4$ ,  $y_5 \rightarrow y_5$ ,  $y_6 \rightarrow -y_3$ ,  $y_7 \rightarrow -y_9$ ,  $y_8 \rightarrow -y_{10}$ ,  $y_9 \rightarrow -y_7$ ,  $y_{10} \rightarrow -y_8$ ,  $y_{11} \rightarrow -y_{14}, y_{12} \rightarrow y_{12}, y_{13} \rightarrow y_{13}, y_{14} \rightarrow -y_{11}, y_{15} \rightarrow -y_{17},$  $y_{16} \rightarrow -y_{18}, y_{17} \rightarrow -y_{15}, y_{18} \rightarrow -y_{16}, y_{19} \rightarrow -y_{21}, y_{20} \rightarrow -y_{22},$  $y_{21} \rightarrow -y_{19}, \ y_{22} \rightarrow -y_{20}, \ y_{23} \rightarrow -y_{26}, \ y_{24} \rightarrow y_{24}, \ y_{25} \rightarrow y_{25},$  $y_{26} \rightarrow -y_{23}$ ,  $y_{27} \rightarrow -y_{29}$ ,  $y_{28} \rightarrow -y_{30}$ ,  $y_{29} \rightarrow -y_{27}$ ,  $y_{30} \rightarrow -y_{28}$ ,  $y_{31} \rightarrow -y_{33}$ ,  $y_{32} \rightarrow -y_{34}$ ,  $y_{33} \rightarrow -y_{31}$ ,  $y_{34} \rightarrow -y_{32}$ ,  $y_{35} \rightarrow -y_{37}$ ,  $y_{36} \rightarrow -y_{38}, \ y_{37} \rightarrow -y_{35}, \ y_{38} \rightarrow -y_{36}, \ y_{39} \rightarrow y_{39}, \ y_{40} \rightarrow y_{40},$ 

under  $s_3$ ,



where  $c_3 = (1 - x_3)^{-3} (1 - x_4)^{-3}$ , under  $s_4$ ,



where  $c_4 = (x_1x_3)^{-3}$ , under  $s_5$ ,

$$
y_1 \rightarrow y_1, \t y_2 \rightarrow y_2, \t y_3 \rightarrow y_3, \t y_4 \rightarrow -y_5, \t y_5 \rightarrow -y_4, \n y_6 \rightarrow y_6, \t y_7 \rightarrow -y_8, \t y_8 \rightarrow -y_7, \t y_9 \rightarrow -y_{10}, \t y_{10} \rightarrow -y_9, \n y_{11} \rightarrow y_{11}, \t y_{12} \rightarrow -y_{13}, \t y_{13} \rightarrow -y_{12}, \t y_{14} \rightarrow y_{14}, \t y_{15} \rightarrow -y_{16}, \n y_{16} \rightarrow -y_{15}, \t y_{17} \rightarrow -y_{18}, \t y_{18} \rightarrow -y_{17}, \t y_{19} \rightarrow -y_{20}, \t y_{20} \rightarrow -y_{19}, \n y_{21} \rightarrow -y_{22}, \t y_{22} \rightarrow -y_{21}, \t y_{23} \rightarrow y_{23}, \t y_{24} \rightarrow -y_{25}, \t y_{25} \rightarrow -y_{24}, \n y_{26} \rightarrow y_{26}, \t y_{27} \rightarrow -y_{28}, \t y_{28} \rightarrow -y_{27}, \t y_{29} \rightarrow -y_{30}, \t y_{30} \rightarrow -y_{29}, \n y_{31} \rightarrow -y_{32}, \t y_{32} \rightarrow -y_{31}, \t y_{33} \rightarrow -y_{34}, \t y_{34} \rightarrow -y_{33}, \t y_{35} \rightarrow -y_{36}, \n y_{36} \rightarrow -y_{35}, \t y_{37} \rightarrow -y_{38}, \t y_{38} \rightarrow -y_{37}, \t y_{39} \rightarrow y_{39}, \t y_{40} \rightarrow y_{40},
$$

and under  $s_6$ ,

$$
y_1 \rightarrow -c_6y_{39}, \quad y_2 \rightarrow c_6y_2, \quad y_3 \rightarrow -c_6y_{26}, \quad y_4 \rightarrow -c_6y_{25}, \quad y_5 \rightarrow -c_6y_{24},
$$
  
\n
$$
y_6 \rightarrow -c_6y_{23}, \quad y_7 \rightarrow -c_6y_{22}, \quad y_8 \rightarrow -c_6y_{21}, \quad y_9 \rightarrow -c_6y_{20}, \quad y_{10} \rightarrow -c_6y_{19},
$$
  
\n
$$
y_{11} \rightarrow c_6y_{11}, \quad y_{12} \rightarrow c_6y_{12}, \quad y_{13} \rightarrow c_6y_{13}, \quad y_{14} \rightarrow c_6y_{14}, \quad y_{15} \rightarrow -c_6y_{18},
$$
  
\n
$$
y_{16} \rightarrow -c_6y_{17}, \quad y_{17} \rightarrow -c_6y_{16}, \quad y_{18} \rightarrow -c_6y_{15}, \quad y_{19} \rightarrow -c_6y_{10}, \quad y_{20} \rightarrow -c_6y_{9},
$$
  
\n
$$
y_{21} \rightarrow -c_6y_{8}, \quad y_{22} \rightarrow -c_6y_{7}, \quad y_{23} \rightarrow -c_6y_{6}, \quad y_{24} \rightarrow -c_6y_{5}, \quad y_{25} \rightarrow -c_6y_{4},
$$
  
\n
$$
y_{26} \rightarrow -c_6y_{3}, \quad y_{27} \rightarrow c_6y_{27}, \quad y_{28} \rightarrow c_6y_{28}, \quad y_{29} \rightarrow c_6y_{29}, \quad y_{30} \rightarrow c_6y_{30},
$$
  
\n
$$
y_{31} \rightarrow -c_6y_{35}, \quad y_{32} \rightarrow -c_6y_{36}, \quad y_{33} \rightarrow -c_6y_{37}, \quad y_{34} \rightarrow -c_6y_{38}, \quad y_{35} \rightarrow -c_6y_{31},
$$
  
\n
$$
y_{36} \rightarrow -c_6y_{32}, \quad y_{37} \rightarrow -c_6y_{33}, \quad y_{38} \rightarrow -c_6y_{34}, \quad y_{39} \rightarrow -c_6y_{1}, \quad y_{40} \rightarrow c_6y_{40},
$$

where  $c_6 = (x_1x_2x_3x_4)^{-2}$ . Maybe it is interesting to see what happens under the operation of the involution  $s_r$  (classically called the association) which sends  $(x_1, x_2, x_3, x_4)$  to

$$
\left(\frac{(x_4-1)D_1}{(x_4-x_2)(x_4-x_3)},\frac{(x_3-1)D_1}{(x_3-x_1)(x_4-x_3)},\frac{(x_2-1)D_1}{(x_4-x_2)(x_2-x_1)},\frac{(x_1-1)D_1}{(x_3-x_1)/(x_2-x_1)}\right):
$$

 $y_1 \rightarrow c_r y_1$ ,  $y_2 \rightarrow c_r y_2$ ,  $y_3 \rightarrow c_r y_3$ ,  $y_4 \rightarrow c_r y_4$ ,  $y_5 \rightarrow c_r y_5$ ,  $y_6 \rightarrow c_r y_6$ ,  $y_7 \rightarrow c_r y_7$ ,  $y_8 \rightarrow c_r y_8$ ,  $y_9 \rightarrow c_r y_9$ ,  $y_{10} \rightarrow c_r y_{10}$ ,  $y_{11} \rightarrow -c_r y_{26}$ ,  $y_{12} \rightarrow -c_r y_{25}$ ,  $y_{13} \rightarrow -c_r y_{24}$ ,  $y_{14} \rightarrow -c_r y_{23}$ ,  $y_{15} \rightarrow -c_r y_{38}$ ,  $y_{16} \rightarrow -c_r y_{37}$ ,  $y_{17} \rightarrow -c_r y_{36}$ ,  $y_{18} \rightarrow -c_r y_{35}$ ,  $y_{19} \rightarrow -c_r y_{30}$ ,  $y_{20} \rightarrow -c_r y_{29}$ ,  $y_{21} \rightarrow -c_r y_{28}$ ,  $y_{22} \rightarrow -c_r y_{27}$ ,  $y_{23} \rightarrow -c_r y_{14}$ ,  $y_{24} \rightarrow -c_r y_{13}$ ,  $y_{25} \rightarrow -c_r y_{12}$ ,  $y_{26} \rightarrow -c_r y_{11}, y_{27} \rightarrow -c_r y_{22}, y_{28} \rightarrow -c_r y_{21}, y_{29} \rightarrow -c_r y_{20}, y_{30} \rightarrow -c_r y_{19},$  $y_{31} \rightarrow -c_r y_{34}$ ,  $y_{32} \rightarrow -c_r y_{33}$ ,  $y_{33} \rightarrow -c_r y_{32}$ ,  $y_{34} \rightarrow -c_r y_{31}$ ,  $y_{35} \rightarrow -c_r y_{18}$ ,  $y_{36} \rightarrow -c_r y_{17}, y_{37} \rightarrow -c_r y_{16}, y_{38} \rightarrow -c_r y_{15}, y_{39} \rightarrow -c_r y_{40}, y_{40} \rightarrow -c_r y_{39},$ 

where

$$
c_r = \left(\frac{D_1 D_2}{(-x_4 + x_2)(x_4 - x_3)(x_1 - x_3)(x_1 - x_2)}\right)^3.
$$

#### 2.4  $\varphi$  embeds M

It is known in[[10](#page-14-0)] that the map

$$
M \ni x \longrightarrow y_1(x) : y_3(x) : y_4(x) : y_5(x) : y_7(x) \in \mathbf{P}^4
$$

is two-to-one, and induces an embedding of the quotient space  $M/(s_r)$ . Thus the composite of  $\varphi$  and the projection

$$
M \longrightarrow \mathbf{P}^{40-1} \longrightarrow \mathbf{P}^4
$$

is a two-to-one map. This fact together with the equivariance of  $\varphi$  under the involution  $s_r$  shown just above implies

**Theorem 2**  $\varphi$  embeds M into  $\mathbf{P}^{40-1}$ .

### 2.5 Prolongation of  $\varphi$  to degenerate arrangements

Let us consider degenerate arrangements of six points on the plane. Since arrangements with three collinear points can be transformed under  $G$  to those with six points on a conic, we assume, Without loss of generality, that our arrangements represented by  $x = (x_1, \ldots, x_4)$  satisfies  $Q = 0$ , that is, the six points are on a conic. Since a (nonsingular) conic is isomorphic to a line, such arrangements form the configuration space

$$
X(2,6) = GL(2) \setminus \{Mat(2,6) \mid \text{any } 2 \times 2 \text{ minor } \neq 0\} / (\mathbb{C}^{\times})^6
$$

of six points on the projective line: if we represent a point of  $X(2, 6)$  by a matrix of the form

$$
z = \left(\begin{array}{rrrrr} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & z_1 & z_2 & z_3 \end{array}\right),
$$

where

$$
\prod_{i=1}^{3} z_i (z_i - 1) \prod_{1 \le i < j \le 3} (z_i - z_j) \neq 0,
$$

then the degenerate arrangements in question can be parametrized by  $z = (z_1, z_2, z_3)$ as

$$
x_1 = (1 - z_1)/(1 - z_2), \quad x_2 = (1 - z_1)/(1 - z_3), \quad x_3 = z_1/z_2, \quad x_4 = z_1/z_3.
$$

Among the two  $S_6$ -equivariant projective embedding of  $X(2, 6)$  presented in [\[9](#page-14-0)], let us recall the following one given by the fifteen polynomials

$$
D_{ij}(z)D_{kl}(z)D_{mn}(z), \quad \{i,j,k,l,m\} = \{1,\ldots,6\},\
$$

where  $D_{ij}(z)$  is the  $(i, j)$ -minor of the 2 × 6-matrix z. Their actual forms are given by

$$
(z_i-1)(z_j-z_k), z_j-z_k, z_i(z_j-z_k), z_i(z_j-1).
$$

It is known and easy to show that the image is projectively equivalent to the so-called Segre cubic defined by

$$
t_0 + \dots, t_5 = 0, \quad (t_0)^3 + \dots + (t_5)^3 = 0.
$$

On the other hand, let us prolong the domain of definition of the map  $\varphi$  on these degenerate arrangements by the same forty polynomials. Then the map  $\varphi$  in z-coodinates is given by  $y_1 = \cdots y_{10} = 0$  and

$$
cy_{11} = -z_1(z_2 - z_3), \t cy_{12} = -z_2 + z_1, \t cy_{13} = z_1 - z_3, \ncy_{14} = -(-1 + z_1)(z_2 - z_3), \t cy_{15} = (-1 + z_1)z_3, \t cy_{16} = (-1 + z_1)z_2, \ncy_{17} = z_1(-1 + z_3), \t cy_{18} = z_1(-1 + z_2), \t cy_{19} = -(-z_2 + z_1)z_3, \ncy_{20} = -(z_1 - z_3)z_2, \t cy_{21} = -(-z_2 + z_1)(-1 + z_3), \t cy_{22} = -(z_1 - z_3)(-1 + z_2), \ncy_{23} = -(-1 + z_1)(z_2 - z_3), \t cy_{24} = z_1 - z_3, \t cy_{25} = -z_2 + z_1, \ncy_{26} = -z_1(z_2 - z_3), \t cy_{27} = -(z_1 - z_3)(-1 + z_2), \t cy_{28} = -(-z_2 + z_1)(-1 + z_3), \ncy_{39} = -(z_1 - z_3)z_2, \t cy_{30} = -(-z_2 + z_1)z_3, \t cy_{31} = (-1 + z_3)z_2, \ncy_{32} = (-1 + z_2)z_3, \t cy_{33} = (-1 + z_2)z_3, \t cy_{34} = (-1 + z_3)z_2, \ncy_{35} = z_1(-1 + z_1)z_3, \t cy_{36} = z_1(-1 + z_3), \t cy_{37} = (-1 + z_1)z_2, \ncy_{38} = (-1 + z_1)z_3, \t cy_{39} = z_2 - z_3, \t cy_{40} = z_2 - z_3,
$$

where

$$
c = \frac{(z_1 - 1)(z_1 - z_3)(z_2 - z_3)(z_1 - z_2)z_1}{(1 - z_2)^2 z_3^2 (1 - z_3)^2 z_2^2}.
$$

This shows that the prolonged  $\varphi$  gives exactly the embedding of the arranged arragements given by  $\{x \mid Q(x) = 0\}$ , isomorphic to  $X(2, 6)$ , given above.

Further put

$$
x_1 = t\xi_1
$$
,  $x_2 = t\xi_2$ ,  $x_3 = t\xi_3$ ,  $x_4 = 1 + t\xi_4$ ,

and let t tends to zero in  $\varphi(x)$ . We can easily see that the limit is a point whchi is independ of  $(\xi_1, \ldots, \xi_4)$ .

These results together with the facts on Naruki's cross-ratio  $\mathcal C$  variety reviewd in §1.3, we can readily show

**Theorem 3** The closure of the image of M under  $\varphi$  is isomorphic to the variety obtained from Naruki's cross-ratio  $\mathcal C$  by blowing down the 40 exceptional divisors of the second kind to points.

Remark: This variety is isomorphic to the Stake compactification of the modular variety obtained in[[1](#page-13-0)]. Note that the 40 cusps correspond to the 40 points obtained by blowing down the divisors of the second kind.

# 3 Defining equations

We will find a set of generators of the ideal defining the closure  $\overline{\varphi(M)}$  of the image of M in  $\mathbf{P}^{40-1}$ . Among the forty polynomials  $y_1(x), \ldots, y_{40}(x)$ , we can find some relations. The Prücker relations ([\[10](#page-14-0)]) of the  $3 \times 3$ -minors of a  $3 \times 6$ -matrix yield linear relations among  $y_1(x), \ldots, y_{10}(x)$ ; for example,

 $(124, 356) - (145, 236) + (146, 235) - (134, 256) = y_3(x) - y_4(x) + y_5(x) - y_6(x) = 0.$ 

On the other hand, it is easy to check the cubic relation

$$
(124, 356)(15, 23, 46)(13, 26, 45) - (145, 236)(13, 56, 24)(12, 35, 46)
$$
  
=  $y_3(x)y_{13}(x)y_{21}(x) - y_4(x)y_{14}(x)y_{20}(x) = 0.$ 

Let V be the subvariety of  $\mathbf{P}^{40-1}$ , coordinatized by  $y_1, \ldots, y_{40}$ , defined by the Gorbits of the linear and the cubic equations:

$$
y_3 - y_4 + y_5 - y_6 = 0, \quad y_3 y_{13} y_{21} - y_4 y_{14} y_{20} = 0.
$$

Theorem 4  $\overline{\varphi(M)} = V$ .

### 3.1 An outline of the proof

By operating the group  $G$  to the linear equation, we get many linear relations among  $y_1, \ldots, y_{40}$ . These relations form a system of rank 30, that is, all y's can be expressed linearly in terms of ten chosen ones. For example, in terms of

y1, y3, y4, y5, y7, y11, y12, y13, y15, y19,

the remaining thirty ones are expressed as

$$
y_2 = y_1 - y_5 + y_4, y_6 = -y_4 + y_5 + y_3, y_8 = -y_3 - y_1 + y_7,
$$
  
\n
$$
y_9 = -y_1 + y_7 - y_4, y_{10} = y_7 - y_5 - y_3, y_{14} = y_{11} + y_{13} - y_{12},
$$
  
\n
$$
y_{16} = -y_1 + y_{12} - y_{13} - y_{11} + y_{15}, y_{17} = y_{15} - y_1 - y_{13},
$$
  
\n
$$
y_{18} = -y_{13} - y_{11} + y_{15}, y_{20} = y_{19} + y_3 + y_{11}, y_{21} = y_4 + y_{19} + y_{12},
$$
  
\n
$$
y_{22} = y_{19} + y_3 + y_{11} + y_{13} + y_5, y_{23} = y_1 - y_{12} + y_{13} + y_3 + y_{11},
$$
  
\n
$$
y_{24} = y_4 + y_1 + y_{13}, y_{25} = y_{12} + y_5 - y_1,
$$
  
\n
$$
y_{26} = -y_4 + y_5 - y_1 + y_3 + y_{11}, y_{27} = y_{19} + y_4 + y_1 + y_3 - y_7 + y_{11} + y_{13},
$$
  
\n
$$
y_{28} = -y_7 + y_{19} + y_{12} + y_5 + y_3, y_{29} = y_3 - y_7 + y_{11} + y_{19} + y_5,
$$
  
\n
$$
y_{30} = y_3 + y_1 - y_7 + y_{19} + y_4, y_{31} = -y_1 - y_{13} + y_{19} + y_{15} + y_{12},
$$
  
\n
$$
y_{32} = y_{19} + y_3 + y_{15}, y_{33} = y_{19} + y_4 + y_{15},
$$
  
\n
$$
y_{34} = y_{19} - y_{13} + y_{15} + y_{12} + y_5 - y_1 + y_3,
$$
  
\n
$$
y_{35} = -y_5 - y_1 - y_3 + y_7 - y_{1
$$

By operating the group  $G$  to the cubic equation, we get many such relations. Substituting the expressions of the  $y$ 's obtained above, we get cubic relations in terms of the chosen ten  $y$ 's; let us here rename the ten coordinates as:

$$
g_1 = y_1
$$
,  $g_2 = y_3$ ,  $g_3 = y_4$ ,  $g_4 = y_5$ ,  $g_5 = y_7$ ,  
\n $g_6 = y_{11}$ ,  $g_7 = y_{12}$ ,  $g_8 = y_{13}$ ,  $g_9 = y_{15}$ ,  $g_0 = y_{19}$ .

Among these cubic equations we can find exactly thirty linearly independent ones. Therefore the variety V is isomorphic to a subvariety of  $\mathbf{P}^9$ , coodinatized by  $g_1, \ldots, g_9, g_0$ , defined by thirty cubic equations, say  $cub_1, \ldots, cub_{30}$ . Some of them which are relatively simple are shown below:

$$
\begin{array}{rcl} cub_1&=&g_2g_8g_0+g_2g_3g_7-g_3g_6g_0-g_2g_3g_6-g_3g_6^2-g_8g_0g_3-g_3g_6g_8\\&+g_3g_7g_0+g_2g_3g_7+g_6g_7g_3,\\ cub_2&=&g_6^2g_1+g_3g_1g_0+g_1^2g_0+g_2g_0g_1-g_5g_1g_0+g_0g_1g_6+g_5g_1g_0+g_2g_3g_6\\&+g_2g_6g_1+g_2g_8g_6+g_3g_1g_6+g_1^2g_6+g_8g_1g_6-g_5g_3g_6-g_5g_1g_6\\ cub_3&:=&-g_8g_0g_3+g_8g_0g_4+g_2g_8g_7+g_2g_7g_4-g_0g_4g_6-g_3g_6g_8-g_3g_6g_4,\\ cub_4&:=g_2g_0g_7+g_2^2g_7+g_2g_8g_7+g_2g_3g_6+g_2g_0g_1+g_2g_8g_6+g_3g_3g_7\\&+g_2g_1g_7+g_2g_8g_7+g_2g_3g_4+g_2g_3g_4+g_2^2g_3+g_2^2g_3+g_2^2g_8+g_5g_4g_0\\&-g_5g_3g_0-g_5g_1g_0-g_5g_3g_0+g_5g_7g_0,\\ cub_6&:=g_5g_8g_1-g_5g_3g_0+g_5g_7g_0,\\&+g_5g_8^2-g_8^2g_4-g_3g_3g_4+g_2g_1g_4+g_2^2g_8+g_3g_3g_7\\&+g_5g_8^2-g_8^2g_4-g_3g_3g_6-g_3g_3g_6-g_3g_3g_6-g_3g_3g_6-g_3g_3g_6-g_3g_3g_6\\&+g_3g_3g_6+g_3g_3g_6+g_3g_3g_6-g_3g_3g_6-g_3g_3g_6-g_3g_3g_4+2
$$

We first study this system over the field  $K := \mathbf{C}(g_1, \ldots, g_5)$ . Geometrically speaking, we project the variety  $V$  onto the 4-dimensional projective space coordinatized by  $g_1 : \cdots : g_5$ , and study the generic fibre of the projection

$$
\pi: \mathbf{P}^9 \supset V \ni g_1: \cdots : g_0 \longmapsto g_1: \cdots : g_5 \in \mathbf{P}^4.
$$

We shall prove that  $\pi$  is generically two-to-one. This implies  $\pi$  is two-to-one on

$$
V^{\circ} := V - V \cap \cup_{j=1}^{40} \{y_j = 0\}.
$$

Thus the argument in §2.4 shows that  $\varphi : M \to V^{\circ}$  is an isomorphism.

We next study the intersection  $V \cap \{y_1 = 0\}$ , and prove that  $V \cap \bigcup_{j=1}^{40} \{y_j = 0\}$  is the totality of the G-orbit of the closure of the image of  $X(2, 6)$  under the prolonged  $\varphi$ , stated in §2.5.

#### 3.2 Computation over K

From  $cub_3 = 0$  and  $cub_7 = 0$ , we can solve  $g_8$  and  $g_9$  as:

$$
g_8 = g_3g_6(g_2 + g_6 + g_0 + g_4)/(g_0g_2 - g_0g_3 + g_0g_4 - g_3g_6),
$$
  
\n
$$
g_9 = g_5g_6(g_0 + g_3)/(-g_2g_3 + g_5g_3 - g_3g_6 - g_0g_3 - g_4g_3 + 2g_2g_4 - g_5g_4 + g_4g_6 + g_0g_4
$$
  
\n
$$
+g_4^2 + g_2^2 - g_5g_2 + g_2g_6 + g_0g_2 - g_5g_6).
$$

Substituting these into  $cub_{19}$ , we can solve  $g_6$ :

$$
g_6 = -\frac{g_1(-g_5g_0 - g_5g_2 + g_0^2 + 2g_0g_2 + g_0g_4 + g_7g_0 + g_2g_7 + g_2^2 + g_2g_4)}{g_0g_1 + g_1g_4 + g_1g_7 + g_2g_1 - g_5g_7 - g_5g_4}.
$$

Substituting these expressions into  $cub_1, \ldots, cub_{30}$ , we have  $cub_3 = cub_7 = cub_{19} = 0$ of course and  $cub_{10} = 0$ ; though most of the remaining ones are complicated,  $cub_8$ is relatively simple:

$$
cub_8 = -(g_1 - g_5)qq/(g_0g_1 + g_1g_4 + g_1g_7 + g_2g_1 - g_5g_7 - g_5g_4),
$$

where

$$
qq = g_1g_4t^2 + (-g_2g_5 - g_5g_4 + g_1g_4 + g_5g_3)s^2 + 2g_1g_4st
$$
  
+ 
$$
(-g_5g_3g_1 + g_2g_1g_5 + g_1g_4g_3 - g_4^2g_5 + g_1g_2g_4 + g_5g_3g_4 - g_2g_5g_4 + g_4^2g_1)s
$$
  
+ 
$$
(-g_4g_5g_1 + g_4^2g_1 + g_1g_4g_3 + g_1g_2g_4)t + g_3g_2g_1g_4 - g_5g_3g_1g_4 + g_1g_4^2g_3
$$

is a quadratic form in

$$
t:=g_0, \quad s:=g_7.
$$

By this quadratic equation, we reduce the degree with respect to  $t$  of the numerators of the *cub*'s to 1. In this way we get 26 (=  $30 - 4$ ) equations of the form

$$
cube_j : a_j t + b_j = 0, \quad j = 1, \dots, 30, \quad j \neq 3, 7, 10, 19,
$$

where  $a_j, b_j \in K[s]$ . The polynomials

$$
D_{jk} = a_j b_k - a_k b_j \in K[s], \quad j < k
$$

<span id="page-13-0"></span>have a unique common factor dd, which is quadratic in s; its actual form is given by

$$
dd := (g_1^2g_4^2 + 2g_2g_1^2g_4 + g_3^2g_5^2 + g_2^2g_3^2 + g_2^2g_1^2 + g_5^2g_4^2 + 2g_3g_2g_1g_4 - 2g_2g_1g_5g_4
$$
  
\n
$$
-2g_3g_5^2g_4 - 2g_1g_5g_4^2 + 2g_2^2g_3g_1 - 2g_3^2g_5g_2 + 2g_5g_3g_2g_4 + 2g_5g_3g_1g_4 - 2g_5g_3g_1g_2)s^2
$$
  
\n
$$
+ (-g_2^2g_1^3 - 2g_1^3g_2g_4 + 2g_1^2g_2g_4^2 + 2g_1^2g_5g_4^2 + g_1^2g_4^3 - g_1^3g_4^2 - 2g_1g_5g_4^3
$$
  
\n
$$
-2g_3g_5^2g_4^2 - 2g_3g_2^2g_1^2 + 2g_3g_2g_1g_4^2 + g_3^2g_2^2g_4 + g_5^2g_4^3 + 2g_1^2g_2g_5g_4
$$
  
\n
$$
+2g_2^2g_3g_1g_4 - 4g_3g_5g_1g_2g_4 + g_5^2g_3^2g_4 + g_2^2g_1^2g_4 - g_1g_5^2g_4^2 - g_2^2g_3^2g_1
$$
  
\n
$$
-g_3^2g_5^2g_1 + 2g_5g_3g_2g_4^2 + 2g_5g_1^2g_2g_3 + 2g_3^2g_5g_1g_2 - 2g_1g_2g_5g_4^2
$$
  
\n
$$
+2g_3g_5^2g_1g_4 + 2g_3g_1g_5g_4^2 - 2g_3g_5g_1^2g_4 - 2g_3^2g_3g_2g_4 - 2g_1^2g_2g_3g_4
$$
  
\n
$$
+g_1^2g_3^2g_4^2 + g_3g_1^2g_3g_4
$$

So  $dd(s) = 0$  is the compatibility condition of the system of equations  $cube_j$ :  $a_i(s)t + b_i(s) = 0$ , among which  $cube_{11} : a_{11}(s)t + b_{11}(s) = 0$  has the minimal degree of coefficients; actually, degrees of  $a_{11}$  and  $b_{11}$  with respect to s are 3 and 4.

Now we get three equations  $dd = 0, qq = 0$  and  $cube_{11} = 0$ . Substituting the expression  $t = -b_{11}/a_{11}$  into qq, we get a polynomial in s. We can prove that this polynomial has dd as a factor.

We have proved that for a given generic point  $g_0 : \cdots : g_5 \in \mathbf{P}^4$ , the inverse under  $\pi$  consists of two points. They are obtained as follows: solve the quadratic equation  $dd = 0$  with respect to  $s(= g_7)$ . For each root,  $t(= g_0)$  is uniquely determined by the linear equation cube<sub>11</sub>. And  $g_6, g_8, g_9$  are uniquely determined by cub3, cub7, cub19 as stated above. Then all the relations are satisfied.

### 3.3 Intersection of V and  $\{y_1 = 0\}$

We should better work on V living in  $\mathbf{P}^{40-1}$ . In terms of the forty coordinates  $y_1, \ldots, y_{40}$ , the cubic equations are 2-term equations. Thus the vanishing of  $y_1$ implies that of some other coordinates. Thanks to G-action, we can assume that  $y_{10} = 0$ . The vanishing of these two coordinates forces the vanishing of other coordinates. Tedious case-by-case study shows that every component of  $V \cap \{y_1 = 0\}$  is included in the G-orbit of

$$
V \cap \{y_1 = \cdots = y_9 = y_0 = 0\},\
$$

which is the closure of the image of  $\{x \mid Q(x) = 0\}$  under the prolonged  $\varphi$ .

### References

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