ROLLE'S THEOREM IS EITHER FALSE OR TRIVIAL IN INFINITE-DIMENSIONAL BANACH SPACES

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Dedicated to Albert Galvany and Pilar Olivella, who among other residents of the Colegio de España in Paris contributed to create the mood of magic rationality in which the tubes were conceived.

Abstract. We prove the following new characterization of C^p (Lipschitz) smoothness in Banach spaces. An infinite-dimensional Banach space X has a C^p smooth (Lipschitz) bump function if and only if it has another C^p smooth (Lipschitz) bump function f such that $f'(x) \neq 0$ for every point x in the interior of the support of f (that is, f does not satisfy Rolle's theorem). Moreover, the support of this bump can be assumed to be a smooth starlike body. As a by-product of the proof of this result we also obtain other useful characterizations of C^p smoothness related to the existence of a certain kind of deleting diffeomorphisms, as well as to the failure of Brouwer's fixed point theorem even for smooth self-mappings of starlike bodies in all infinite-dimensional spaces. Finally, we study the structure of the set of gradients of bump functions in the Hilbert space ℓ_2, and as a consequence of the failure of Rolle's theorem in infinite dimensions we get the following result. The usual norm of the Hilbert space ℓ_2 can be uniformly approximated by C^1 smooth Lipschiz functions ψ so that the cones generated by the sets of derivatives $\psi'(\ell_2)$ have empty interior. This implies that there are C^1 smooth Lipschitz bumps in ℓ_2 so that the cones generated by their sets of gradients have empty interior.

1. Introduction and main results

Rolle's theorem in finite-dimensional spaces states that, for every bounded open subset U of $\mathbb{R}^n$ and for every continuous function $f : \overline{U} \longrightarrow \mathbb{R}$ such that f is differentiable in U and constant on the boundary ∂U, there exists a point $x \in U$ such that $f'(x) = 0$. Unfortunately, Rolle's theorem does not remain valid in infinite dimensions. It was S. A. Shkarin [32] that first showed the failure of Rolle's theorem in superreflexive infinite-dimensional spaces and in non-reflexive spaces which have smooth norms. The class of spaces for which Rolle's theorem fails was substantially enlarged in [6], where it was also shown that an approximate version of Rolle's theorem remains nevertheless true in all Banach spaces. In fact, as a consequence of the existence of diffeomorphisms deleting points in infinite-dimensional spaces (see [1, 5]), it is easy to see that Rolle's theorem fails in all infinite-dimensional Banach spaces which have smooth norms [7].

However, none of these results allows to characterize the spaces for which Rolle's theorem fails since, as shown by R. Haydon [25], there are Banach spaces with smooth bump functions which do not possess any equivalent smooth norms. Of

1991 *Mathematics Subject Classification.* 46B20, 58B99.

course, Rolle's theorem is trivially true in the Banach spaces which do not have any smooth bumps (if X is such a space then every function on X satisfying the hypothesis of Rolle's theorem must be a constant). Thus, in many infinite-dimensional Banach spaces, Rolle's theorem either fails or is trivial, depending on the smoothness properties of the spaces considered. In this setting, it does not seem too risky to conjecture, as it was done in [6], that Rolle's theorem should fail in an infinite-dimensional Banach space if and only if our space has a C^1 smooth bump function. In this paper we will prove this conjecture to be right, thus providing an interesting new characterization of smoothness in Banach spaces.

Our main result is the following

Theorem 1.1. *Let X be an infinite-dimensional Banach space which has a C^p smooth (Lipschitz) bump function. Then there exists another C^p smooth (Lipschitz) bump function $f : X \longrightarrow [0,1]$ with the property that $f'(x) \neq 0$ for every $x \in int(suppf)$.*

Here supp$f$ denotes the support of f, that is, $\text{supp} f = \overline{\{x \in X : f(x) \neq 0\}}$. Let us recall that $b : X \longrightarrow \mathbb{R}$ is said to be a bump function on X provided b is not constantly zero and b has a bounded support.

From this result it is easily deduced the following

Corollary 1.2. *Let X be an infinite-dimensional Banach space. The following statements are equivalent.*

(1) *X has a C^p smooth (and Lipschitz) bump function.*
(2) *There exist a bounded contractible open subset U of X and a continuous function $f : \overline{U} \longrightarrow \mathbb{R}$ such that f is C^p smooth (and Lipschitz) in U, $f = 0$ on ∂U, and yet $f'(x) \neq 0$ for all $x \in U$, that is, Rolle's theorem fails in X.*
(3) *There exist a C^p smooth (and Lipschitz) function $f : X \longrightarrow [0,1]$ and a bounded contractible open subset U of X such that $f = 0$ precisely on $X \setminus U$ and yet $f'(x) \neq 0$ for all $x \in U$.*

Just in order to complete the picture of Rolle's theorem in infinite-dimensional Banach spaces let us quote the two positive results from [6, 3] on approximate and subdifferential substitutes of Rolle's theorem, which guarantee the existence of arbitrarily small derivatives (instead of vanishing ones) for every function sayisfying (in an approximate manner) the conditions of the classic Rolle's theorem.

Theorem 1.3 (Azagra–Gómez–Jaramillo)**.** *Let U be a bounded connected open subset of a Banach space X. Let $f : \overline{U} \longrightarrow \mathbb{R}$ be a bounded continuous function which is (Gâteaux) differentiable in U. Let $R > 0$ and $x_0 \in U$ be such that $dist(x_0, \partial U) = R$. Suppose that $f(\partial U) \subseteq [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Then there exists some $x_\varepsilon \in U$ such that $\|f'(x_\varepsilon)\| \leq \frac{\varepsilon}{R}$.*

Theorem 1.4 (Azagra–Deville)**.** *Let U be a bounded connected open subset of a Banach space X which has a C^1 smooth Lipschitz bump function. Let $f : \overline{U} \longrightarrow \mathbb{R}$ be a bounded continuous function, and let $R > 0$ and $x_0 \in U$ be such that $dist(x_0, \partial U) = R$. Suppose that $f(\partial U) \subseteq [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Then*

$$\inf\{\|p\| : p \in D^- f(x) \cup D^+ f(x), x \in U\} \leq \frac{2\varepsilon}{R}.$$

(Here $D^-f(x)$ and $D^+f(x)$ denote the subdifferential and superdifferential sets of f at x, respectively; see [16], p. 339, for the definitions).

The "twisted tube" method that we develop in section 2 in order to prove theorem 1.1 is interesting in itself and, with little more work, provides a useful characterization of C^p smoothness in infinite-dimensional Banach spaces related to the existence of a certain kind of *deleting diffeomorphisms*. Namely, we have the following

Theorem 1.5. *Let X be an infinite-dimensional Banach space. The following assertions are equivalent.*

1. *X has a C^p smooth bump function.*
2. *There exists a nonempty contractible closed subset D of the unit ball B_X and a C^p diffeomorphism $f : X \longrightarrow X \setminus D$ so that f restricts to the identity outside B_X.*

This result yields the following corollaries.

First, the celebrated Brouwer's fixed point theorem fails even for smooth self-mappings of balls or starlike bodies in all infinite-dimensional Banach spaces. Let us recall that Brouwer's theorem states that every continuous self-map of the unit ball of a finite-dimensional normed space admits a fixed point. This is the same as saying that there is no continuous retraction from the unit ball onto the unit sphere, or that the unit sphere is not contractible (the identity map on the sphere is not homotopic to a constant map). In infinite dimensions the situation is completely different and Brouwer's theorem is no longer true (see [13, 30, 9, 29, 22, 8, 2]. Theorem 1.5 yields a trivial proof that Brouwer's theorem is false in infinite dimensions even for smooth self-mappings of balls or starlike bodies; this is a particular case (the non-Lipschitz one) of the main result in [2].

Second, we deduce from the above characterization that the support of the bump functions which violate Rolle's theorem can always be assumed to be a smooth starlike body. This is all shown in section 3.

In section 2 we give the proofs of theorems 1.1 and 1.5. A much simpler proof of theorem 1.1 for the non-Lipschitz case is included in this section too.

Finally, in section 4 we study the structure of the set of gradients of bump functions in the Hilbert space ℓ_2, and as a consequence of the failure of Rolle's theorem in infinite dimensions we get the following result. The usual norm of the Hilbert space ℓ_2 can be uniformly approximated by C^1 smooth Lipschitz functions ψ so that the cones generated by the sets of derivatives $\psi'(\ell_2)$ have empty interior. This implies that there are C^1 smooth Lipschitz bumps in ℓ_2 so that the cones generated by their sets of gradients have empty interior.

2. The proofs

The idea behind the proof of theorem 1.1 is as simple as this. First we build a twisted tube T of infinite length in the interior of the unit ball B_X, with a beginning but with no end. This twisted tube can be thought of as directed by an ever-winding infinite path p that gets lost in the infinitely many dimensions of our space X. In technical words, one can construct a diffeomorphism π between a straight (unbounded) half-cilynder C and a twisted (bounded) tube T contained in B_X. The tube T is going to be the support of a smooth bump function f that does not satisfy

Rolle's theorem. In order to define such a function f we only have to make it strictly increase in the direction which is tangent to the leading path p at each point of the tube T. The graph of f would thus represent an ever-ascending stairway built upon our twisted tube, with a beginning but no end.

The spirit of the proof that (1) implies (2) in theorem 1.5 is not very different. We will make use of the diffeomorphism π between a straight (unbounded) half-cilynder C and a bounded twisted tube T contained in B_X. If we consider a straight closed half-cilinder C' contained in the interior of C and directed by the same line as C, it is elementary that there is a diffeomorphism $g : X \longrightarrow X \setminus C'$ so that g restricts to the identity outside C. In fact this is true even in the plane. Now, by composing this diffeomorphism g with the diffeomorphisms π and π^{-1} that give us an appropriate coordinate system in the twisted tube $T = \pi(C)$, we get a diffeomorphism $f : X \longrightarrow X \setminus T'$, where $T' = \pi(C')$ is a smaller closed twisted tube inside T, and f restricts to the identity outside the unit ball. The precise definition of f would be $f(x) = \pi(g(\pi^{-1}(x)))$ if $x \in T$, and $f(x) = x$ if $x \in X \setminus T$. If we take $D = T'$ we are done.

In the rest of his section we will be involved in the task of formalizing these ideas.

The following theorem guarantees the existence of bounded infinite twisted tubes in all infinite-dimensional Banach spaces.

Theorem 2.1. *There are universal constants $M > 0$ (large) and $\varepsilon > 0$ (small) such that, for every infinite-dimensional Banach space X, if we consider the decomposition $X = H \oplus [z]$ (where $H = Ker\, z^*$ for some $z^* \in X^*$ with $z^*(z) = \|z^*\| = \|z\| = 1$) and the open half-cilynder C of diameter ε, directed by z, and with base on H, $C = \{x + tz \in X : \|x\| < \varepsilon, t > 0\}$, then there exists an injection $\pi : C \longrightarrow B_X$ which is a C^∞ diffeomorphism onto its image. The image $T = \pi(C)$ is thus a bounded open set which we will call a bounded open infinitely twisted tube in X. Moreover, the derivatives of the mappings $\pi : C \longrightarrow T$ and $\pi^{-1} : T \longrightarrow C$ are both uniformly bounded by M.*

Assume for a while that theorem 2.1 is already established and let us explain how theorems 1.1 and 1.5 can be deduced.

Proof of theorem 1.1.

Consider the diffeomorphism $\pi : C \longrightarrow T \subset B_X$ from theorem 2.1. Take a C^p smooth (Lipschitz) non-negative bump function φ on H so that the support of φ is contained in the base of C, that is, $\varphi(x) = 0$ whenever $\|x\| \geq \frac{\varepsilon}{2}$, for instance. Pick a C^∞ smooth real function $\mu : \mathbb{R} \longrightarrow [0, 1]$ such that $\mu(t) = 0$ for $t \leq 1$, $0 < \mu(t) < 1$ for $t > 1$ and $0 < \mu'(t) < 1$ for all $t > 1$. Then define $g : X = H \oplus [z] \longrightarrow \mathbb{R}$ by

$$g(x, t) = \varphi(x)\mu(t).$$

It is plain that g is a C^p smooth (Lipschitz) function such that $g'(x, t) \neq 0$ for every $x \in \text{int}(\text{supp}f)$, that is, for every x such that $g(x, t) \neq 0$ (take into account that the interior of the support of g coincides in this case with the open support of g, that is the set of points at which g does not vanish). Indeed,

$$g'(x, t)(0, 1) = \frac{\partial g}{\partial t}(x, t) = \varphi(x)\mu'(t)$$

and therefore $g'(x,t)(0,1) = 0$ if and only if $\varphi(x) = 0$ or $\mu'(t) = 0$, which happens if and only if $\varphi(x) = 0$ or $\mu(t) = 0$, that is to say, $g(x,t) = 0$. Now let us define $f : X \longrightarrow \mathbb{R}$ by

$$f(y) = \begin{cases} g(\pi^{-1}(y)) & \text{if } y \in T; \\ 0 & \text{if } y \notin T \end{cases}$$

It is clear that f is a well defined C^p smooth (Lipschitz) function, and $\text{supp}(f) = \pi(\text{supp}(g)) \subset T$, from which it follows that f has a bounded support. We claim that $f'(y) \neq 0$ whenever $y \in \text{int}(\text{supp} f)$, that is, f does not satisfy Rolle's theorem. Indeed, if $y \in \text{int}(\text{supp} f)$ then $\pi^{-1}(y) = (x,t) \in \text{int}(\text{supp} g)$ and therefore $g'(x,t)(0,1) \neq 0$. But then

$$f'(y) = g'(x,t) \circ D\pi^{-1}(y) \neq 0,$$

because $D\pi^{-1}(y)$ is a linear isomorphism. This concludes the proof of theorem 1.1.

Now we will turn our attention to the proof of theorem 1.5. Before proceeding with the proof, let us fix some standard terminology and notation used throughout this section and the following one. A closed subset A of a Banach space X is said to be a starlike body provided A has a non-empty interior and there exists a point $x_0 \in \text{int} A$ such that each ray emanating from x_0 meets the boundary of A at most once. In this case we will say that A is *starlike with respect to* x_0. When dealing with starlike bodies, we can always assume that they are starlike with respect to the origin (up to a suitable translation), and we will do so unless otherwise stated.

For a starlike body A, the characteristic cone of A is defined as

$$ccA = \{x \in X \mid rx \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A as

$$q_A(x) = \inf\{\lambda > 0 \mid \frac{1}{\lambda} x \in A\}$$

for all $x \in X$. It is easily seen that for every starlike body A its Minkowski functional q_A is a continuous function which satisfies $q_A(rx) = r q_A(x)$ for every $r \geq 0$ and $q_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X \mid q_A(x) \leq 1\}$, and $\partial A = \{x \in X \mid q_A(x) = 1\}$, where ∂A stands for the boundary of A. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(\lambda x) = \lambda \psi(x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$ is a starlike body. Convex bodies (that is, closed convex sets with nonempty interior) are an important kind of starlike bodies. We will say that A is a C^p smooth (Lipschitz) starlike body provided its Minkowski functional q_A is C^p smooth (and Lipschitz) on the set $X \setminus q_A^{-1}(0)$.

It is worth noting that for every Banach space $(X, \|.\|)$ with a C^p smooth (Lipschitz) bump function there exist a functional ψ and constants $a, b > 0$ such that ψ is C^p smooth (Lipschitz) away from the origin, $\psi(\lambda x) = |\lambda| \psi(x)$ for every $x \in X$ and $\lambda \in \mathbb{R}$, and $a\|x\| \leq \psi(x) \leq b\|x\|$ for every $x \in X$ (see [16], proposition II.5.1). The level sets of this function are precisely the boundaries of the smooth bounded starlike bodies $A_c = \{x \in X \mid \psi(x) \leq c\}$, $c \in \mathbb{R}$. This shows in particular that every Banach space having a C^p smooth (Lipschitz) bump function has a C^p smooth (Lipschitz) bounded starlike body as well. The converse is obviously true too.

Proof of theorem 1.5.

First of all let us choose a number $\varepsilon > 0$, a cilynder C, a bounded twisted tube T, and a diffeomorphism $\pi : C \longrightarrow T$ from theorem 2.1.

Let B be a C^∞ smooth convex body in the plane $\mathbb{R}^2$ whose boundary contains the set

$$\{(s,t) : t = -1, |s| \leq \frac{\varepsilon}{4}\} \cup \{(s,t) : |s| = \frac{\varepsilon}{2}, t \geq -1 + \frac{\varepsilon}{4}\},$$

and let q_B be the Minkowski functional of B. Define $B' = \frac{1}{2}B = \{(s,t) : q_B(s,t) \leq \frac{1}{2}\}$. Let $\theta : (\frac{1}{2}, \infty) \longrightarrow [0, \infty)$ be a C^∞ smooth real function so that $\theta'(t) < 0$ for $\frac{1}{2} < t < 1$, $\theta(t) = 0$ for $t \geq 1$, and $\lim_{t \to \frac{1}{2}^+} \theta(t) = +\infty$. Now define $\varphi : \mathbb{R}^2 \backslash B' \longrightarrow \mathbb{R}^2$ by

$$\varphi(s,t) = (\varphi_1(s,t), \varphi_2(s,t)) = (s, t + \theta(q_B(s,t))).$$

It is elementary to check that φ is a C^∞ diffeomorphism from $\mathbb{R}^2 \setminus B'$ onto $\mathbb{R}^2$ so that φ restricts to the identity outside the band B.

Next, recall that since X has a C^p smooth bump then it has a C^p bounded starlike body A as well. If $X = H \oplus [z]$, take $W = A \cap H$, which is a C^p bounded starlike body in H, and denote by q_W its Minkowski functional. We can assume that $W \subseteq B(0,1)$, that is, $\|x\| \leq q_W(x)$ for all $x \in H$. Let us define

$$\psi(x,t) = q_B(q_W(x), t)$$

for all $(x,t) \in X$. It is clear that ψ is a continuous function which is positive-homogeneous and C^p smooth away from the half-line $L = \{(x,t) \in X : x = 0, t \geq 0\}$. Then the sets

$$U = \{(x,t) \in X : \psi(x,t) \leq 1\}, \quad U' = \{(x,t) \in X : \psi(x,t) \leq \frac{1}{2}\}$$

are cilyndrical C^p starlike bodies whose characteristic cones are the half-line L. If we define

$$h(x,t) = (x, \varphi_2^{-1}(q_W(x), t))$$

for $(x,t) \in X$, it is not difficult to realize that h is a C^p diffeomorphism from X onto $X \setminus U'$ so that h restricts to the identity outside U. The inverse of h is given by

$$h^{-1}(x,t) = (x, t + \theta(\psi(x,t))).$$

Now consider the cilyndrical bodies $V := (0,2) + U$ and $V' := (0,2) + U'$, and put $g(x,t) = h(x, t-2)$. Then $g : X \longrightarrow X \setminus V'$ is a C^p diffeomorphism such that g is the identity outside V. Note that, since $W \subseteq B(0,1)$, we have that $V' \subset V \subset C = \{(x,t) \in X : \|x\| < \varepsilon, t > 0\}$. Let us define

$$f(x) = \begin{cases} \pi(g(\pi^{-1}(x))) & \text{if } x \in T; \\ x & \text{otherwise.} \end{cases}$$

It is then clear that f is a C^p diffeomorphism from X onto $X \setminus T'$, where $T' = \pi(V')$ is a smaller closed twisted tube inside $\pi(V) \subseteq T$, and f restricts to the identity outside the larger tube $\pi(V) \subset T$, which is contained in B_X. This completes the proof that (1) implies (2).

Conversely, if there is such an f as in (2), we can assume that $f(0) \neq 0$ and take $T \in X^*$ so that $T(f(0)) \neq 0$; then the function $b : X \longrightarrow \mathbb{R}$ defined by $b(x) = T(x - f(x))$ is a C^p smooth bump on X.

Now we proceed with the proof of theorem 2.1. We will make use of the following lemma, which guarantees the existence of an appropiate path of linear isomorphisms. Here Isom(X) stands for the set of linear isomorphisms of X, which is regarded as a subset of $\mathcal{L}(X,X)$, the linear continuous mappings of X into X.

Lemma 2.2. *There is a universal constant $K > 0$ such that for every infinite-dimensional Banach space X there are paths $\beta : [0,\infty) \longrightarrow Isom(X)$ and $p : [0,\infty) \longrightarrow X$ with the following properties:*

1. *Both β and p are C^∞ smooth, as well as the path of inverse isomorphisms $\beta^{-1} : [0,\infty) \longrightarrow Isom(X)$, $\beta^{-1}(t) = [\beta(t)]^{-1}$.*
2. *$1 \leq \|\beta(t)\| \leq K$ and $1 \leq \|\beta^{-1}(t)\| \leq K$ for all $t \in [0,\infty)$.*
3. *$\sup_{t\geq 0} \|\beta'(t)\| \leq K$ and $\sup_{t\geq 0} \|(\beta^{-1})'(t)\| \leq K$.*
4. *There exists a certain $v \in X$, with $1 \geq \|v\| \geq \frac{1}{K}$, such that $p'(t) = \beta(t)(v)$ for all $t \geq 0$.*
5. *For every $t, s \in [0,\infty)$ we have that $\|p(t) - p(s)\| \geq \frac{1}{K}\min\{1, |t-s|\}$.*

Proof. Let $(x_n)_{n=0}^\infty$ be a normalized basic sequence in X with biorthogonal functionals $(x_n^*)_{n=0}^\infty \subset X^*$ (that is, $x_n^*(x_k) = \delta_{n,k} = 1$ if $n = k$, and 0 otherwise) satisfying $\|x_n^*\| \leq 3$ (one can always take such sequences, see [15], p. 93, or [17], p. 39). For $n \geq 1$ set $v_n = x_n - x_{n-1}$. Let $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ function with the following properties:

(i) $\theta(t) = 0$ whenever $t \leq -\frac{1}{2}$ or $t \geq 1$;
(ii) $\theta(t) = 1$ for $t \in [0, \frac{1}{2}]$;
(iii) $\theta'(t) > 0$ for $t \in (-\frac{1}{2}, 0)$;
(iv) $\theta(t) = 1 - \theta(t-1)$ for $t \in [\frac{1}{2}, 1]$;
(v) $\sup_{t\in\mathbb{R}} |\theta'(t)| \leq 4$.

For $n \geq 1$ let us define $\theta_n : \mathbb{R} \longrightarrow \mathbb{R}$ by $\theta_n(t) = \theta(t - n + 1)$. It is clear that the functions θ_n are all C^∞ smooth and have Lipschitz constant less than or equal to 4, $\theta_n = 0$ on $(-\infty, n-1-\frac{1}{2}] \cup [n,\infty)$, $\theta_n = 1$ on $[n-1, n-\frac{1}{2}]$, and $\theta_n(t) = 1 - \theta_{n+1}(t)$ for all $t \in [n-1, n+\frac{1}{2}]$.

Our path β of linear isomorphisms is going to be of the form

$$\beta(t) = \sum_{n=1}^\infty \theta_n(t) S_n,$$

where each $S_n \in$ Isom(X) takes the vector v_1 into v_n and for every $\lambda \in [0,1]$ the mapping $L_{n,\lambda} = (1-\lambda)S_n + \lambda S_{n+1}$ is still a linear isomorphism and, moreover, the families of isomorphisms $\{L_{n,\lambda}\}_{n\in\mathbb{N},\lambda\in[0,1]}$ and $\{L_{n,\lambda}^{-1}\}_{n\in\mathbb{N},\lambda\in[0,1]}$ are uniformly bounded. Let us define the isomorphisms S_n. They are going to be of the form

$$S_n(x) = x + f_n(x)(v_n - v_1),$$

where $f_n \in X^*$ satisfies $f_n(v_1) = 1 = f_n(v_n)$, and $\|f_n\| \leq 18$ (the exact definition of f_n will be given later). Their inverses S_n^{-1} will be

$$S_n^{-1}(y) = y - f_n(y)(v_n - v_1).$$

We want the linear mappings $L_{n,\lambda} = (1-\lambda)S_n + \lambda S_{n+1}$ to be linear isomorphisms. We have

$$y = L_{n,\lambda}(x) = x + (1-\lambda)f_n(x)(v_n - v_1) + \lambda f_{n+1}(x)(v_{n+1} - v_1), \qquad (1)$$

from which

$$x = y - [(1-\lambda)f_n(x)(v_n - v_1) + \lambda f_{n+1}(x)(v_{n+1} - v_1)], \tag{2}$$

and we need to write $f_n(x)$ and $f_{n+1}(x)$ as linear functions of y. If we apply the functionals f_n and f_{n+1} successively to equation (1), we denote $A_n = f_n(x)$, $B_n = f_{n+1}(x)$, $C_n = f_n(y)$, $D_n = f_{n+1}(y)$, and we take into account that $1 = f_n(v_1) = f_n(v_n) = f_{n+1}(v_1)$, then we obtain the system

$$\begin{cases} A_n + \lambda[f_n(v_{n+1}) - 1]B_n = C_n \\ (1-\lambda)[f_{n+1}(v_n) - 1]A_n + B_n = D_n, \end{cases} \tag{3}$$

which we want to have a unique solution for A_n, B_n. The determinant of this system is

$$\Delta_{n,\lambda} = 1 - \lambda(1-\lambda)[f_{n+1}(v_n) - 1][f_n(v_{n+1}) - 1],$$

and we want $\Delta_{n,\lambda}$ to be bounded below by a strictly positive number, and this bound has to be uniform in n, λ. For $n \geq 3$ this can easily be done by setting

$$f_n = x_1^* - x_{n-1}^*$$

(so that $f_n(v_n) = 1 = f_n(v_1)$, $f_n(v_{n+1}) = 0$, $f_{n+1}(v_n) = -1$, and therefore $\Delta_{n,\lambda} = (1-\lambda)^2 + \lambda^2 \geq \frac{1}{2}$ for all $\lambda \in [0,1]$). For $n = 1, 2$, put

$$f_2 = x_1^* + 2x_2^* + \frac{7}{3}x_3^*, \quad \text{and} \quad f_1 = x_1^*;$$

then $f_2(v_3) = \frac{1}{3}$, $f_2(v_2) = 1$, $f_2(v_1) = 1$, $f_3(v_2) = -2$, $f_1(v_2) = -1$, $f_1(v_1) = 1$, and everything is fine (indeed, $\Delta_{1,\lambda} = 1$ and $\Delta_{2,\lambda} = (1-\lambda)^2 + \lambda^2 \geq \frac{1}{2}$ for all $\lambda \in [0,1]$).

Therefore, with these definitions, the linear system (3) has a unique solution for A_n, B_n, which can be easily calculated and estimated by Cramer's rule, of the form

$$\begin{aligned} A_n(y) &= \tfrac{1}{\Delta_{n,\lambda}}\big(f_n(y) - \lambda[f_n(v_{n+1}) - 1]f_{n+1}(y)\big) \\ B_n(y) &= \tfrac{1}{\Delta_{n,\lambda}}\big(f_{n+1}(y) - (1-\lambda)[f_{n+1}(v_n) - 1]f_n(y)\big). \end{aligned}$$

The linear forms $y \mapsto A_n(y)$, $y \mapsto B_n(y)$ satisfy that $\|A_n\| \leq 144 \geq \|B_n\|$ for all n, as is easily checked. Now, by substituting $f_n(x) = A_n(y)$ and $f_{n+1}(x) = B_n(y)$ in (2) we get the expression for the inverse of $L_{n,\lambda}$, that is,

$$x = L_{n,\lambda}^{-1}(y) = y - [(1-\lambda)A_n(y)(v_n - v_1) + \lambda B_n(y)(v_{n+1} - v_1)]. \tag{4}$$

By taking into account that $\|A_n\| \leq 144 \geq \|B_n\|$, $\|f_n\| \leq 18$ and $\|v_n - v_1\| \leq 4$ for all n, one can estimate that $1 \leq \|L_{n,\lambda}\| \leq 73$ and $1 \leq \|L_{n,\lambda}^{-1}\| \leq 577$ for all $n \in \mathbb{N}, \lambda \in [0,1]$.

So let us define $\beta : [0,\infty) \longrightarrow \mathrm{Isom}(X)$ by

$$\beta(t) = \sum_{n=1}^{\infty} \theta_n(t) S_n. \tag{5}$$

This path is well defined because the sum is locally finite; in fact, from the definition of θ_n it is clear that, for a given $t_0 \in [0,\infty)$ there exist some $\delta > 0$ and $N = N(t_0) \in \mathbb{N}$ such that $\beta(t) = \theta_N(t)S_N + \theta_{N+1}(t)S_{N+1}$ for all $t \in (t_0 - \delta, t_0 + \delta)$, that is, β is locally of the form $\beta(t) = L_{n,\lambda(t)}$, where $\lambda(t) = \theta_n(t)$. This implies that the $\beta(t)$ are really linear isomorphisms and that the path is C^∞ smooth.

On the other hand, the path $\beta^{-1}(t) = [\beta(t)]^{-1} \in \mathrm{Isom}(X)$ is C^∞ smooth as well, because it is the composition of our path β with the mapping $\varphi : \mathrm{Isom}(X) \longrightarrow$

$\mathrm{Isom}(X)$, $\varphi(U) = U^{-1}$, which is C^∞ smooth and whose derivative is given by $\varphi'(U)(S) = -U^{-1} \circ S \circ U^{-1}$ for every $S \in \mathcal{L}(X, X)$ (see [14], theorem 5.4.3). This proves condition (1) of the lemma.

Next, by bearing in mind the local expression of β and the above estimations for $\|L_{n,\lambda}\|$ and $\|L_{n,\lambda}^{-1}\|$, we deduce that

$$1 \leq \|\beta(t)\| \leq R \geq \|\beta^{-1}(t)\| \geq 1$$

for all $t \in [0, \infty)$, where $R \geq 577$ will be fixed later. This shows condition (2). Now, if $t_0 \in [0, \infty)$ and we write $\beta(t) = \theta_N(t) S_N + \theta_{N+1}(t) S_{N+1}$ for $t \in (t_0 - \delta, t_0 + \delta)$ as above, then it is clear that β' is locally of the form

$$\beta'(t) = \theta_N'(t) S_N + \theta_{N+1}'(t) S_{N+1}$$

and therefore

$$\|\beta'(t)\| \leq |\theta_N'(t)| \|S_N\| + |\theta_{N+1}'(t)| \|S_{N+1}\| \leq 4(73 + 73) = 584,$$

from which we get $\sup_{t \geq 0} \|\beta'(t)\| \leq 584 \leq R$. Moreover, we have

$$(\beta^{-1})'(t) = -(\beta(t))^{-1} \circ \beta'(t) \circ (\beta(t))^{-1}$$

and therefore

$$\|(\beta^{-1})'(t)\| \leq \|\beta(t)^{-1}\|^2 \|\beta'(t)\| \leq (577)^2 584,$$

from which $\sup_{t \geq 0} \|(\beta^{-1})'(t)\| \leq R$ and condition (3) is satisfied as well provided we fix $R = (577)^2 584$.

Now let us define the path $p : [0, \infty) \longrightarrow X$ by

$$p(t) = \int_{-\infty}^{t} \beta(s)(v_1) ds = \int_{-\infty}^{t} \big(\sum_{n=1}^{\infty} \theta_n(s) S_n(v_1) \big) ds.$$

It is clear that p is a C^∞ smooth path in X, and $p'(t) = \beta(t)(v_1)$ for all $t \geq 0$ (from which it follows that p is Lipschitz). Let us see that p is bounded. For a given $t > 0$ there exists $N = N(t) \in \mathbb{N}$ so that $N - 1 - \frac{1}{2} \leq t \leq N - \frac{1}{2}$ and therefore, taking into account the definition of θ_n and the fact that $S_n(v_1) = v_n = x_n - x_{n-1}$ for all n, we have that

$$\begin{aligned}
\|p(t)\| &= \| \int_{-\infty}^{t} \sum_{n=1}^{\infty} \theta_n(s) S_n(v_1) ds \| = \| \sum_{n=1}^{\infty} \big(\int_{-\infty}^{t} \theta_n(s) ds \big) v_n \| \\
&= \| \big(\int_{-\infty}^{\infty} \theta(s) ds \big) \sum_{n=1}^{N-1} v_n + \big(\int_{-\infty}^{t} \theta_N(s) ds \big) v_N \| \\
&\leq \big(\int_{-\infty}^{\infty} \theta(s) ds \big) \| \sum_{n=1}^{N-1} v_n \| + \big(\int_{-\infty}^{\infty} \theta(s) ds \big) \|v_N\| \\
&= \big(\int_{-\infty}^{\infty} \theta(s) ds \big) \big(\|x_{N-1} - x_0\| + \|x_N - x_{N-1}\| \big) \leq \frac{3}{2}(2 + 2) = 6.
\end{aligned}$$

This shows that the image of p is contained in the ball $B(0, 6)$ and p is bounded. Let us also remark that $2 \geq \|v_1\| \geq \frac{x_1^*(x_1 - x_0)}{\|x_1^*\|} \geq \frac{1}{4}$.

Finally, let us check that p satisfies the separation condition (5). Let $0 \leq t < r$ and take $N \in \mathbb{N}$ so that $N-1-\frac{1}{2} < r \leq N - \frac{1}{2}$; then we have

$$p(r)-p(t) = \sum_{n=1}^{\infty} \big(\int_t^r \theta_n(s) ds \big) v_n = \sum_{n=1}^{\infty} \big(\int_t^r \theta_n(s) ds \big)(x_n - x_{n-1}) =$$

$$-\big(\int_t^r \theta_1(s) ds \big) x_0 + \sum_{k=1}^{N-1} \big(\int_t^r \theta_k(s) ds - \int_t^r \theta_{k+1}(s) ds \big) x_k + \big(\int_t^r \theta_N(s) ds \big) x_N.$$

By observing that $\max\{1-s, 2s-1\} \geq \frac{1}{3}$ for all $s \in \mathbb{R}$ and taking into account the definition of the θ_n, it is not difficult to see that

$$\max\{ \int_t^r \theta_N(s) ds, \int_t^r \theta_{N-1}(s) ds - \int_t^r \theta_N(s) ds \} \geq \min\{ \frac{1}{3}|t-r|, a \}, \qquad (6)$$

where $a = \int_{-\frac{1}{2}}^{0} \theta(s) ds > 0$. Then, by applying either x_N^* or x_{N-1}^* to the expression for $p(r)-p(t)$ above, depending on which the maximum in (6) is, and bearing in mind that $x_n^*(x_k) = \delta_{n,k}$ and $\|x_n^*\| \leq 4$ for all n, k, we get that

$$\max\{ x_N^*(p(r)-p(t)), x_{N-1}^*(p(r)-p(t)) \} \geq \min\{ \frac{1}{3}|t-r|, a \},$$

and it follows that $\|p(r)-p(t)\| \geq \min\{ \frac{1}{12}|t-r|, \frac{a}{4} \}$. This shows that if $R > 0$ is large enough then

$$\|p(r)-p(t)\| \geq \frac{1}{R} \min\{1, |t-r|\}$$

for all $t, r \geq 0$.

In order to get paths β and p and a vector v with properties (1)–(5) and such that p is contained in the unit ball, it is enough to multiply them all by $\frac{1}{6}$. □

Proof of theorem 2.1.

Consider $X = H \oplus [z] = H \times \mathbb{R}$ and $C_\varepsilon = \{x + tz \in X : \|x\| < \varepsilon, t > 0\}$, where $H = \text{Ker} z^*$ for some $z^* \in X^*$ with $z^*(z) = \|z^*\| = \|z\| = 1$, and $\varepsilon > 0$ is to be fixed later. Let β and p be the paths from lemma 2.2. There is no loss of generality if we assume that $v \in [z]$, $z^*(v) \geq \frac{1}{K}$. Let us define $\pi : C_\varepsilon \longrightarrow X$ by

$$\pi(x,t) = \beta(t)(x) + p(t).$$

It is clear that π is C^∞ smooth and has a bounded derivative. We are going to show that π is a diffeomorphim onto its image, T_ε, and $\pi^{-1} : T_\varepsilon \longrightarrow C_\varepsilon$ has a bounded derivative as well. To this end let us define the path $\alpha : [0, \infty) \longrightarrow X^*$ by

$$\alpha(t) = f_t = z^* \circ \beta^{-1}(t).$$

This is a C^∞ smooth and Lipschitz path in X^*, and $\alpha(t) = f_t$ satisfies that $\text{Ker} f_t = \beta(t)(H)$. It is clear from this definition and the properties of β and p that

(i) $\|\alpha'(t)\| \leq K$, and
(ii) $\alpha(t)(p'(t)) = z^*(v) \geq \frac{1}{K}$

for all $t \geq 0$. Now, for a fixed (but arbitrary) $y \in T_\varepsilon = \pi(C_\varepsilon)$, let us introduce the auxiliary function $F = F_y : [0, \infty) \longrightarrow \mathbb{R}$ defined by

$$F(t) = \alpha(t)(y - p(t)).$$

We have that

$$\begin{aligned}F_y'(r) &= \alpha'(r)(y - p(r)) - \alpha(r)(p'(r))\\ &\leq \|\alpha'(r)\|\|y - p(r)\| - \alpha(r)(p'(r))\\ &\leq K\|y - p(r)\| - \frac{1}{K}\end{aligned}$$

for all $r \geq 0$. If we choose $\varepsilon > 0$ smaller than $\frac{1}{6K^5}$ this implies that π is a C^∞ diffeomorphism onto its image.

Indeed, let us first see that π is an injection. Assume that $y = \pi(x,t) = \pi(w,s)$ for some $(x,t),(w,s) \in C_\varepsilon$. Then we have $y - p(t) = \beta(t)(x)$ and $y - p(s) = \beta(s)(w)$, so that $x = \beta^{-1}(t)(y-p(t))$ and $w = \beta^{-1}(s)(y-p(s))$, and, in order to conclude that $(x,t) = (w,s)$, it is enough to see that $t = s$. Note that $\beta(t)(x) - \beta(s)(w) = p(s) - p(t)$ and therefore, by (5) of lemma 2.2,

$$\begin{aligned}&\frac{1}{K}\min\{1, |t-s|\} \leq \|p(s) - p(t)\| = \|\beta(t)(x) - \beta(s)(w)\|\\ &\leq \|\beta(t)(x)\| + \|\beta(s)(w)\| \leq K(\|x\| + \|w\|) \leq 2K\varepsilon \leq \frac{1}{3K^4},\end{aligned}$$

so that $|t-s| \leq 2K^2\varepsilon \leq \frac{1}{3K^3}$. Now, since p and β are both K-Lipschitz, for every $r \in [t,s]$ we have that

$$\begin{aligned}&\|y - p(r)\| \leq \|y - p(t)\| + \|p(t) - p(r)\| = \|\beta(t)(x)\| + \|p(t) - p(r)\|\\ &\leq K\|x\| + K|t - r| \leq K\varepsilon + 2K^3\varepsilon \leq 3K^3\varepsilon.\end{aligned}$$

By combining this with the above estimation for $F_y'(r)$ we get

$$F_y'(r) \leq K\|y - p(r)\| - \frac{1}{K} \leq 3K^4\varepsilon - \frac{1}{K} \leq -\frac{1}{2K} \tag{7}$$

for every $r \in [t,s]$. Now suppose that $t \neq s$. Then, since $x = \beta^{-1}(t)(y - p(t))$ and $w = \beta^{-1}(s)(y-p(s))$ are both in H we have that $0 = z^*(x) = z^*(w) = F_y(t) = F_y(s)$, so that, by the classic Rolle's theorem, there should exist some $r \in (t,s)$ with $F_y'(r) = 0$. But this contradicts (7). Therefore $t = s$ and π is an injection.

If, for a given $y \in \pi(C_\varepsilon)$, we denote by $t(y)$ the unique $t = t(y)$ such that $y = \pi(\beta^{-1}(t)(y - p(t)), t)$ then it is clear that the inverse $\pi^{-1} : T_\varepsilon \longrightarrow C_\varepsilon$ is defined by

$$\pi^{-1}(y) = (\beta^{-1}(t(y))(y - p(t(y))), t(y)). \tag{8}$$

For each y the number $t(y)$ is uniquely determined by the equation

$$G(y,t) := F_y(t) = 0,$$

and the argument above shows that

$$\frac{\partial G}{\partial t}(y,t) = F_y'(t) \leq -\frac{1}{2K} \tag{9}$$

for every $y \in T_\varepsilon$ and t in a neighbourhood of $t(y)$. Then, according to the implicit function theorem we get that the function $y \mapsto t(y)$ is C^∞ smooth. Furthermore, we have that

$$t'(y) = \frac{-\frac{\partial G}{\partial y}(y, t(y))}{\frac{\partial G}{\partial t}(y, t(y))} = \frac{-z^* \circ \beta^{-1}(t(y))}{F_y'(t(y))},$$

and therefore, according to the above estimations,

$$\|t'(y)\| \leq \|z^* \circ \beta^{-1}(t(y))\| \frac{1}{|F'_y(t(y))|} \leq 2K^2,$$

which shows that $y \mapsto t(y)$ has a bounded derivative as well. Then it is clear that π^{-1} is C^∞ and has a bounded derivative (all the functions involved in (8) have been proved to have bounded derivatives). This concludes the proof of theorem 2.1.

We will finish this section with a simple alternative proof of the failure of Rolle's theorem in the non-Lipschitz case.

Remark 2.3. If we drop the Lipschitz condition from the statement of theorem 1.1, a much simpler proof based on the same idea is available. Let us make a sketch of this proof.

Consider the decomposition $X = H \times \mathbb{R}$ and pick a non-negative C^p smooth bump function φ on H whose support is contained on the ball $B_H(0, 1/16)$. First, we construct a C^∞ smooth path $q : [0, \infty) \longrightarrow B_H$, where B_H stands for the unit ball of the hyperplane H, with the property that q has no accumulation points at the infinity, that is, $\lim_{n\to\infty} q(t_n)$ does not exist for any (t_n) going to ∞. This can easily be done by having q lost in the infinitely many dimensions of H. For instance, take a biorthogonal sequence $\{x_n, x_n^*\} \subseteq H \times H^*$ so that $\|x_n\| = 1$ and $\|x_n^*\| \leq 4$, and consider a C^∞ function $\theta : \mathbb{R} \longrightarrow [0, 1]$ so that supp$\theta \subseteq [-1, 1]$, $\theta(0) = 1$, $\theta'(t) < 0$ for $t \in (0, 1)$, and $\theta(t - 1) = 1 - \theta(t)$ for $t \in [0, 1]$. The path q may be defined as

$$q(t) = \sum_{n=1}^{\infty} \theta(t - n + 1) x_n$$

for $t \geq 0$. Now we reparametrize q and define $p : [0, 1) \longrightarrow B_H$ by

$$p(t) = q\big(\frac{t}{1-t}\big).$$

Let $\alpha : \mathbb{R} \longrightarrow [0, 1]$ be a C^∞ smooth function so that $\alpha(t) = 0$ for all $t \leq 0$, and $\alpha'(t) > 0$ for all $t > 0$. Then the function $g : X = H \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(x,t) = \begin{cases} \varphi(x - p(t))\alpha(t) & \text{if } \quad t \in [0,1); \\ 0 \quad \text{otherwise} \end{cases}$$

is a C^p smooth bump function which does not satisfy Rolle's theorem. Indeed, it is easy to see that

$$g'(x,t)(p'(t), 1) = \varphi(x - p(t))\alpha'(t) > 0,$$

and in particular $g'(x, t) \neq 0$, for all (x, t) in the interior of the support of g.

3. Killing singularities. The failure of Brouwer's fixed point theorem in infinite dimensions.

Do not be afraid, this section does not contain any totalitarian propaganda. Here we will present two applications of theorem 1.5, both of which have in common the following principle: if you have a mapping with a single singular point or an isolated set of singularities that bother you, you can just kill them by composing your map with some deleting diffeomorphisms. In this way you obtain a new map which is as close as you want to the old one but does not have the adverse properties created by the singular points you eliminate.

For instance, if you want a smooth bump function g which does not satisfy Rolle's theorem and whose support is a smooth starlike body A, by composing the Minkowski functional of this body with a real bump function you get a function h whose support is A and whose derivative vanishes only at the origin and outside A; then, by composing h with a diffeomorphism f which extracts a small set containing the origin and which restricts to the identity outside A, you get a map g with the required properties.

On the other hand, suppose you want a smooth retraction r from a bounded starlike body A of a Banach space X onto its boundary ∂A. This is impossible if X is finite-dimensional, but otherwise you can use the following trick: it is trivial that there is a smooth retraction h from $A \setminus \{0\}$ onto ∂A; then take a diffeomorphism f which removes from X a small subset containing the origin and restricts to the identity outside A. The composition $r = h \circ f$ gives the required retraction.

Let us formalize these ideas and comment on the results that they provide.

The support of the bumps that violate Rolle's theorem.

The bump function constructed in the proof of theorem 1.1 has a weird support, namely a twisted tube. Some readers might judge this fact rather unpleasant and wonder whether it is possible to construct a bump function which does not satisfy Rolle's theorem and whose support is a nicer set, such as a ball or a starlike body. To comfort those readers let us first recall that in infinite dimensions there is no topological difference between a tube (whether it is twisted or not) and a ball or a starlike body, as theorem 2.4 in [4] shows. Furthermore, as we said above, theorem 1.5 allows us to show that for a given C^p smooth bounded starlike body A in an infinite-dimensional Banach space X, it is always possible to construct a C^p smooth bump function on X which does not satisfy Rolle's theorem and whose support is precisely the body A.

Theorem 3.1. *Let X be an infinite-dimensional Banach space with a C^p smooth bounded starlike body A. Then there exists a C^p smooth bump function g on X whose support is precisely the body A, and with the property that $g'(x) \neq 0$ for all x in the interior of A (that is, g does not satisfy Rolle's theorem).*

Proof. Let q_A be the Minkowski functional of A. We may assume that $B_X \subseteq A$. By theorem 1.5 there is a closed subset D of A and a C^p diffeomorphism $f : X \longrightarrow X \setminus D$ which is the identity outside A. It can be assumed that the origin belongs to D. Then the function $h : X \longrightarrow \mathbb{R}$ defined by

$$h(x) = q_A(f(x))$$

is C^p smooth on X, restricts to the gauge q_A outside A, and has the remarkable property that $h'(x) \neq 0$ for all $x \in X$ (indeed, $h'(x) = q_A'(f(x)) \circ f'(x)$ is non-zero everywhere because $q_A'(y) \neq 0$ whenever $y \neq 0$, $0 \notin f(X)$, and $f'(x)$ is a linear isomorphism at each point x).

Now, take a C^∞ real function $\theta : \mathbb{R} \longrightarrow [0, 1]$ such that $\theta(t) > 0$ for $t \in (-1, 1)$, $\theta = 0$ outside $[-1, 1]$, $\theta(t) = \theta(-t)$, $\theta(0) = 1$, and $\theta'(t) < 0$ for all $t \in (0, 1)$. Then, if we define $g : X \longrightarrow \mathbb{R}$ by

$$g(x) = \theta(h(x)),$$

it is immediately checked that g is a C^p smooth bump on X which does not satisfy Rolle's theorem and whose support is precisely the body A. □

The failure of Brouwer's fixed point theorem in infinite dimensions.

The celebrated Brouwer's fixed point theorem tells us that every continuous self-map of the unit ball of a finite-dimensional normed space admits a fixed point. This is the same as saying that there is no continuous retraction from the unit ball onto the unit sphere, or that the unit sphere is not contractible (the identity map on the sphere is not homotopic to a constant map). The *Scottish book* (see [31]) contains the following question, asked in 1935 by S. Ulam. Can one transform continuously the solid sphere of a Hilbert space into its boundary such the transformation should be the identity on the boundary of the ball? In other words, is the unit sphere of a Hilbert space a retract of its unit ball? The first answer to this question is commonly attributed to S. Kakutani [26], who solved the problem by exhibiting several examples of continuous self-mappings of the unit ball of the Hilbert space without fixed points. Thus, none of the above forms of Brouwer's fixed point theorem remains valid in infinite dimensions.

A very nice solution to the retraction problem, and one which has the advantage of holding in arbitrary infinite-dimensional Banach spaces, was given by the pioneering results of Klee's on topological negligibility of points and compacta [27, 28]: for every infinite-dimensional Banach space X there always exists a homeomorphism $h : X \longrightarrow X \setminus \{0\}$ so that h restricts to the identity outside the unit ball B_X. The required retraction of B_X onto the unit sphere S_X is then given by $R(x) = h(x)/\|h(x)\|$ for $x \in B_X$. By taking into account the subsequent progress on topological negligibility of subsets made by C. Bessaga, T. Dobrowolski and the first-named author among others (see [11, 13, 18, 19, 1, 5]), this mapping h may even be assumed C^p smooth provided that the sphere S_X is C^p smooth. Thus, there are very regular retractions that provide an answer to Ulam's question.

In [30] B. Nowak showed that for several infinite-dimensional Banach spaces Brouwer's theorem fails even for Lipschitz mappings (that is, under the strongest uniform-continuity condition), and in [9] Y. Benyamini and Y. Sternfeld generalized Nowak's result for all infinite-dimensional normed spaces, establishing that for every infinite-dimensional space $(X, \|\cdot\|)$ there exists a Lipschitz retraction from the unit ball B_X onto the sphere S_X, and that S_X is Lipschitz contractible.

More recently, M. Cepedello and the first-named author showed that these results hold for the smooth category as well (see [2]). In fact they proved that for every infinite-dimensional Banach space with a C^p (Lipschitz) bounded starlike body A (where $p = 0, 1, 2, \ldots, \infty$), there is a C^p Lipschitz retraction of A onto its boundary ∂A, the boundary ∂A is C^p (Lipschitz) contractible, and there is a C^p smooth (Lipschitz) mapping $f : A \longrightarrow A$ such that f has no (approximate) fixed points.

The proof of these results in the general case is somewhat involved, but if we drop the Lipschitz condition then the fact that Brouwer's theorem is false in infinite dimensions even for smooth self-mappings of balls or starlike bodies is a trivial consequence of theorem 1.5.

Corollary 3.2 (Azagra–Cepedello)**.** *Let X be an infinite-dimensional Banach space and let A be a C^p smooth bounded starlike body. Then:*

(1) *The boundary ∂A is C^p contractible.*

(2) *There is a C^p smooth retraction from A onto ∂A.*
(3) *There exists a C^p smooth mapping $\varphi : A \longrightarrow A$ without fixed points.*

Proof. Let $f : X \longrightarrow X \setminus D$ be the diffeomorphism from theorem 1.5. We may assume that the origin belongs to the deleted set D and that $B_X \subseteq A$, so that f restricts to the identity outside A. Then the formula

$$R(x) = \frac{f(x)}{q_A(f(x))},$$

where q_A is the Minkowski functional of A, defines a C^p smooth retraction from A onto the boundary ∂A. This proves (2).

Once we have such a retraction it is easy to prove parts (1) and (3): the formula $\varphi(x) = -R(x)$ defines a C^p smooth self-mapping of A without fixed points. On the other hand, if we pick a non-decreasing C^∞ function $\zeta : \mathbb{R} \longrightarrow \mathbb{R}$ so that $\zeta(t) = 0$ for $t \leq \frac{1}{4}$ and $\zeta(t) = 1$ for $t \geq \frac{3}{4}$, then the formula

$$H(t, x) = R((1 - \zeta(t))x),$$

for $t \in [0, 1]$, $x \in \partial A$, defines a C^p homotopy joining the identity to a constant on ∂A, that is, H contracts the pseudosphere ∂A to a point. □

4. How small can the set of gradients of a bump be?

If b is a smooth bump function on a Banach space X it is natural to ask how large or how small the cone generated by the set of gradients $b'(X)$ can be. In general, as a consequence of Ekeland's variational principle, one has that the cone $\mathcal{C}(b) = \{\lambda b'(x) : \lambda \geq 0, x \in X\}$ is norm-dense in the dual space X^* (see [16], pag. 58, proposition 5.2).

In [4] a study was initiated on the topological properties of the set of derivatives of smooth functions. Among other results it was proved that an infinite-dimensional separable Banach space has a C^1 smooth bump function (resp. is Asplund) if and only if there exists another C^1 smooth bump function b on X with the property that $b'(X) = X^*$. This answers the question as to how large can the cone $\mathcal{C}(b)$ be. But what is the smallest possible size of $\mathcal{C}(b)$?

To begin with, by using theorem 1.1, one can easily construct smooth bump functions whose sets of gradients lack not only the point zero, but any pre-set finite-dimensional linear subspace of the dual.

Corollary 4.1. *Let X be an infinite-dimensional Banach space and W a finite-dimensional subspace of X^*. The following statements are equivalent.*

(1) *X has a C^p smooth (Lipschitz) bump function.*
(2) *X has a C^p smooth (Lipschitz) bump function f satisfying that $\{\lambda f'(x) : x \in X, \lambda \in \mathbb{R}\} \cap W = \{0\}$. Moreover, $\{f'(x) : f(x) \neq 0\} \cap W = \emptyset$.*

Proof. We can write $X = Y \oplus Z$, where $Y = \cap_{w^* \in W} \mathrm{Ker} w^*$ and $\dim Z = \dim W$ is finite. Pick a C^p smooth (Lipschitz) bump function $\varphi : Y \longrightarrow \mathbb{R}$ such that φ does not satisfy Rolle's theorem, and let θ be a C^∞ smooth Lipschitz bump function on Z so that $\theta'(z) = 0$ whenever $\theta(z) = 0$. Then the function $f : X = Y \oplus Z \longrightarrow \mathbb{R}$ defined by $f(y, z) = \varphi(y)\theta(z)$ is a C^p smooth (Lipschitz) bump which satisfies $\{f'(x) : f(x) \neq 0\} \cap W = \emptyset$. Indeed, we have

$$f'(y, z) = \big(\theta(z)\varphi'(y), \varphi(y)\theta'(z)\big) \in X^* = Y^* \oplus Z^* = Y^* \oplus W$$

and, since $\theta(z)\varphi'(y) \neq 0$ whenever $\varphi(y)\theta'(z) \neq 0$, it is clear that $f'(y,z) \notin W$ unless $f'(y,z) = 0$. □

We also have the following

Corollary 4.2. *Let X be an infinite-dimensional Banach space such that $X = X_1 \oplus X_2$, where X_1 and X_2 are both infinite-dimensional. The following statements are equivalent.*

(1) *X has a C^p smooth (Lipschitz) bump function.*
(2) *X has a C^p smooth (Lipschitz) bump function f satisfying that $\{\lambda f'(x) : x \in X, \lambda \in \mathbb{R}\} \cap (X_1^* \cup X_2^*) = \{0\}$. Moreover, $\{f'(x) : f(x) \neq 0\} \cap (X_1^* \cup X_2^*) = \emptyset$.*

Proof. The proof is similar to that of the preceding corollary. Pick φ_1 and φ_2 smooth (Lipschitz) bump functions on X_1 and X_2, respectively, so that φ_1 and φ_2 do not satisfy Rolle's theorem. Then the function $f : X = X_1 \oplus X_2 \longrightarrow \mathbb{R}$ defined by $f(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$ is a smooth (Lipschitz) bump which satisfies $\{f'(x) : f(x) \neq 0\} \cap (X_1^* \cup X_2^*) = \emptyset$. □

If we restrict the scope of our search to classic Banach spaces, much stronger results are available. On the one hand, if $X = c_0$ the size of $\mathcal{C}(b)$ can be really small. Indeed, as a consequence of P. Hájek's work [24] on smooth functions on c_0 we know that if b is C^1 smooth with a locally uniformly continuous derivative (note that there are bump functions with this property in c_0), then $b'(X)$ is contained in a countable union of compact sets in X^* (and in particular has empty interior). On the other hand, if X is non-reflexive and has a separable dual, there are bumps b on X so that $\mathcal{C}(b)$ has empty interior, as it was shown in [4].

In the reflexive case, however, the problem is far from being settled. To begin with, the cone $\mathcal{C}(b)$ cannot be very small, since it is going to be a residual subset of the dual X^*. Indeed, as a straightforward consequence of Stegall's variational principle, for every Banach space X having the Radon-Nikodym Property (RNP) it is easy to see that $\mathcal{C}(b)$ is a residual set in X^*. Therefore, for infinite-dimensional Banach spaces X enjoying RNP one can hardly expect a better answer to the above question than the following one: there are smooth bumps b on X such that the cone $\mathcal{C}(b)$ has empty interior in X^*. In the case of the Hilbert space $X = \ell_2$ we next show that this indeed happens.

Theorem 4.3. *In the Hilbert space ℓ_2 the following holds:*

(1) *The usual norm $||\cdot||_2$ can be uniformly approximated by C^1 Lipschitz functions ψ (with Lipschitz derivative) so that the cones $\mathcal{C}(\psi)$ generated by the sets of derivatives of ψ have empty interior, and $\psi'(x) \neq 0$ for all $x \in \ell_2$.*
(2) *There is a C^1 Lipschitz bump function b (with Lipschitz derivative) on ℓ_2 satisfying that the cone $\mathcal{C}(b)$ generated by its set of derivatives $b'(\ell_2)$ has empty interior, and $b'(x) \neq 0$ for every x in the interior of its support.*

Proof. To save notation, let us just write $||\cdot||$ when referring to the usual norm in ℓ_2. We will make use of the following restatement of a striking result due to S. A. Shkarin (see [32]).

Theorem 4.4 (Shkarin)**.** *There is a C^∞ diffeomorphism φ from ℓ_2 onto $\ell_2 \setminus \{0\}$ such that all the derivatives $\varphi^{(n)}$ are uniformly continuous on ℓ_2, and $\varphi(x) = x$ for $||x|| \geq 1$.*

Let us consider, for $0<\varepsilon<1$, the difeomorphism $\varphi_\varepsilon:\ell_2\longrightarrow\ell_2\setminus\{0\}$, $\varphi_\varepsilon(x)=\varepsilon\varphi(x/\varepsilon)$, and the function $U=U_\varepsilon:\ell_2\longrightarrow\mathbb{R}$ defined by $U(x)=\varepsilon^2+||\varphi_\varepsilon(x)||^2$. Then U satisfies the following properties:

(i) U is C^∞ smooth.
(ii) $||x||^2\le U(x)\le 2\varepsilon^2+||x||^2$ and $\varepsilon^2\le U(x)$, for every $x\in\ell_2$.
(iii) $U(x)=\varepsilon^2+||x||^2$, for every $x\in\ell_2$, $||x||\ge\varepsilon$.
(iv) $U'(x)\neq 0$ for every $x\in\ell_2$.
(v) U is Lipschitz in bounded sets and U' is Lipschitz.

Now, we define the functions $U_n:\ell_2\longrightarrow\mathbb{R}$ by $U_n(x)=\frac{1}{2^{2n}}U(2^n x)$, whenever $x\in\ell^2$. We will identify ℓ_2 with the infinite sum $\sum_2\ell_2\equiv\ell_2\oplus_2\ell_2\oplus_2\ell_2\cdots$, where an element $x=(x_n)$ belongs to $\sum_2\ell_2$ if and only if every x_n is in ℓ_2 and $\sum_n||x_n||^2<\infty$, being $||x||^2=\sum_n||x_n||^2$. Then, we define the function $f:\sum_2\ell_2\longrightarrow\mathbb{R}$ by

$$f(x)=\sum_n U_n(x_n),\ \text{ where } x=(x_n)_n.$$

First, note that f is well-defined, since condition (ii) implies that, whenever $x=(x_n)\in\sum_2\ell_2$,

$$\begin{aligned}0<f(x)&=\sum_n\frac{1}{2^{2n}}U(2^n x_n)\le\sum_n\frac{1}{2^{2n}}(2\varepsilon^2+||2^n x_n||^2)\\&=\sum_n(\frac{2\varepsilon^2}{2^{2n}}+||x_n||^2)<\infty.\end{aligned}\tag{4.1}$$

On the one hand, note that, if U' has Lipschitz constant less than or equal to M then U_n' is also Lipschitz with constant less than or equal to M, since for x and y in ℓ_2 we have

$$||U_n'(x)-U_n'(y)||=\frac{1}{2^n}||U'(2^n x)-U'(2^n y)||\le\frac{1}{2^n}M2^n||x-y||.$$

This implies that, if $x=(x_n)\in\sum_2\ell_2$, the functionals $U_n'(x_n)\in\ell_2$ satisfy that $(U_n'(x_n))_n\in\sum_2\ell_2$. Indeed, we have $||U_n'(x_n)-U_n'(0)||\le M||x_n||$, and therefore $\sum_n||U_n'(x_n)-U_n'(0)||^2<\infty$. Also, $(U_n'(0))=(\frac{1}{2^n}U'(0))\in(\sum_2\ell_2)^*\equiv\sum_2\ell_2$, and then we get that $T(x)=(U_n'(x_n))$ also belongs to $\sum_2\ell_2$.

Let us now prove that f is C^1 smooth. For every $x=(x_n)$ and $h=(h_n)$ in $\sum_2\ell_2$, we can estimate

$$\begin{aligned}&|f(x+h)-f(x)-T(x)(h)|\le\sum_n|U_n(x_n+h_n)-U_n(x_n)-U_n'(x_n)(h_n)|\\&\le\sum_n|U_n'(x_n+t_nh_n)(h_n)-U_n'(x_n)(h_n)|\qquad(\text{for some }0\le t_n\le 1)\\&\le M\sum_n||h_n||^2=M||h||^2.\end{aligned}$$

Therefore f is Fréchet differentiable and $f'(x)=(\frac{1}{2^n}U'(2^n x_n))$. Moreover, f' is Lipschitz since $||f'(x)-f'(y)||^2=\sum_n||U_n'(x_n)-U_n'(y_n)||^2\le M^2\sum_n||x_n-y_n||^2=M^2||x-y||^2$. This implies, in particular, that f is Lipschitz on bounded sets.

Let us check that $f = f_\varepsilon$ uniformly approximates $||\cdot||^2$. Indeed, from condition (ii) on U and (4.1), we have that, for every $x = (x_n) \in \sum_2 \ell_2$,

$$\max\{\frac{1}{3}\varepsilon^2, ||x||^2\} \le f(x) \le \frac{2}{3}\varepsilon^2 + ||x||^2, \tag{4.2}$$

and then,

$$0 \le f(x) - ||x||^2 \le \frac{2}{3}\varepsilon^2. \tag{4.3}$$

In order to obtain functions which approximate the norm uniformly in ℓ_2 let us consider $\psi = \psi_\varepsilon = \sqrt{f_\varepsilon}$. According to inequalities (4.2) and (4.3) we have that

$$0 \le \psi - ||x|| \le \frac{2\varepsilon^2}{3(\psi + ||x||)} \le \frac{2}{\sqrt{3}}\varepsilon$$

for any $x \in \sum_2 \ell_2$.

Let us check that ψ' is bounded. By equation (4.2) we have, for any $x \in \sum_2 \ell_2$,

$$||\psi'(x)|| = \frac{||f'(x)||}{2\psi(x)} \le \frac{||f'(x) - f'(0)||}{2\psi(x)} + \frac{||f'(0)||}{2\psi(x)} \le \frac{M}{2} + \frac{\sqrt{3}}{2\varepsilon}||f'(0)||.$$

Consequently, ψ is Lipschitz with Lipschitz constant, say N. In a similar way, we obtain that ψ' is Lipschitz, since for any x, y in $\sum_2 \ell_2$,

$$\begin{aligned}
&||\psi'(x) - \psi'(y)|| = ||\frac{f'(x) - f'(y)}{2\psi(x)} + \frac{f'(y)}{2}(\frac{1}{\psi(x)} - \frac{1}{\psi(y)})|| \\
&\le \frac{1}{2}\frac{||f'(x) - f'(y)||}{\psi(x)} + \frac{||\psi(y) - \psi(x)||}{\psi(x)}\frac{||f'(y)||}{2\psi(y)} \\
&\le \frac{\sqrt{3}M}{2\varepsilon}||x - y|| + \frac{\sqrt{3}N^2}{\varepsilon}\|x - y\|.
\end{aligned}$$

Finally, note that the set $\{\lambda f'(x) = \lambda(U_n'(x_n)) : \ x = (x_n) \in \sum_2 \ell_2, \lambda > 0\}$ is contained in $\{z = (z_n) \in \sum_2 \ell_2 : \ z_n \neq 0 \ \text{for every}\ n \in \mathbb{N}\}$, which has empty interior in $\sum_2 \ell_2$. This concludes the proof of (1).

In order to prove (2), we consider a C^∞ function $\theta : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $\theta'(t) < 0$ for $t \in (0,1)$, and $\operatorname{supp}\theta = (0,1]$. Then, we can define a required bump function as the composition $b(x) = \theta(f(x))$. Indeed, on the one hand, $f(0) \le \frac{2}{3}\varepsilon^2 < 1$ and therefore $b(0) > 0$. On the other hand, $f(x) \ge ||x||^2 \ge 1$, whenever $||x|| \ge 1$, and hence $b(x) = 0$ for $||x|| \ge 1$. The bump function b is clearly Lipschitz with Lipschitz derivative since θ, θ' and f' are Lipschitz and f is Lipschitz on bounded sets. □

It is clear that the proof of the preceding theorem could only be adapted for superreflexive Banach spaces X which admit a decomposition as an infinite sum of subspaces isomorphic to X, and therefore the problem whether in a separable reflexive space there can be bumps whose sets of gradients have empty interior remains open in the general case, though the information that has already been gathered seems to point to a final positive solution.

Open Problem 4.5. Let X be a (separable reflexive) infinite-dimensional Banach space which admits a C^p smooth (Lipschitz) bump function. Is there another C^p smooth (Lipschitz) bump function f on X so that the cone $\mathcal{C}(f) = \{\lambda f'(x) : \lambda \ge 0, x \in X\}$ has empty interior in X^*?

The proof of theorem 4.3 naturally raises the following question.

Open Problem 4.6. Can every equivalent norm on ℓ_2 be uniformly approximated by C^1 Lipschitz functions satisfying that the cone generated by their derivatives has empty interior? Furthermore, is it possible to approximate squared equivalent norms in ℓ_2 uniformly on bounded sets by real-analytic functions (resp. polynomials) ψ such that the cones generated by the sets $\psi'(\ell_2)$ have empty interior?

Acknowledgements

This research was carried out during a postdoctoral stay of the authors in the Equipe d'Analyse de l'Université Pierre et Marie Curie, Paris 6. The authors are indebted to the Equipe d'Analyse and very especially to Gilles Godefroy for their kind hospitality and generous advice.

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