# On cocycles with values in the group SU(2)

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#### Abstract

In this paper we introduce the notion of degree for  $C^1$ -cocycles over irrational rotations on the circle with values in the group SU(2). It is shown that if a  $C^1$ cocycle  $\varphi : \mathbb{T} \to SU(2)$  over an irrational rotation by  $\alpha$  has nonzero degree, then the skew product

 $\mathbb{T} \times SU(2) \ni (x,g) \mapsto (x+\alpha, g\,\varphi(x)) \in \mathbb{T} \times SU(2)$ 

is not ergodic and the group of essential values of  $\varphi$  is equal to the maximal Abelian subgroup of SU(2). Moreover, if  $\varphi$  is of class  $C^2$  (with some additional assumptions) the Lebesgue component in the spectrum of the skew product has countable multiplicity. Possible values of degree are discussed, too.

### 1 Introduction

Assume that  $T: (X, \mathcal{B}, \lambda) \to (X, \mathcal{B}, \lambda)$  is an ergodic measure-preserving automorphism of standard Borel space. Let G be a compact Lie group,  $\mu$  its Haar measure. For a given measurable function  $\varphi: X \to G$  we study spectral properties of the measure-preserving automorphism of  $X \times G$  (called skew product) defined by

$$T_{\varphi}: (X \times G, \lambda \otimes \mu) \to (X \times G, \lambda \otimes \mu), \ T_{\varphi}(x,g) = (Tx, g \, \varphi(x)).$$

A measurable function  $\varphi: X \to G$  determines a measurable cocycle over the automorphism T given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)\varphi(Tx)\dots\varphi(T^{n-1}x) & \text{for} \quad n > 0\\ e & \text{for} \quad n = 0\\ (\varphi(T^{-1}x)\varphi(T^{-2}x)\dots\varphi(T^{-n}x))^{-1} & \text{for} \quad n < 0, \end{cases}$$

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which we will identify with the function  $\varphi$ . Then  $T_{\varphi}^{n}(x,g) = (Tx, g \varphi^{(n)}(x))$  for any integer n. Two cocycles  $\varphi, \psi : X \to G$  are *cohomologous* if there exists a measurable map  $p : X \to G$  such that

$$\varphi(x) = p(x)^{-1} \,\psi(x) \, p(Tx).$$

In this case, p will be called a transfer function. If  $\varphi$  and  $\psi$  are cohomologous, then the map  $(x, g) \mapsto (x, p(x)g)$  establishes a metrical isomorphism of  $T_{\varphi}$  and  $T_{\psi}$ .

By  $\mathbb{T}$  we will mean the circle group  $\{z \in \mathbb{C}; |z| = 1\}$  which most often will be treated as the group  $\mathbb{R}/\mathbb{Z}; \lambda$  will denote Lebesgue measure on  $\mathbb{T}$ . We will identify functions on  $\mathbb{T}$ with periodic of period 1 functions on  $\mathbb{R}$ . Assume that  $\alpha \in \mathbb{T}$  is irrational. We will treat the case where T is the ergodic rotation on  $\mathbb{T}$  given by  $Tx = x + \alpha$ .

In the case where G is the circle and  $\varphi$  is a smooth cocycle, spectral properties of  $T_{\varphi}$  depend on the topological degree  $d(\varphi)$  of  $\varphi$ . For example, in [5] A. Iwanik, M. Lemańczyk, D. Rudolph have proved that if  $\varphi$  is a  $C^2$ -cocycle with  $d(\varphi) \neq 0$ , then  $T_{\varphi}$  is ergodic and it has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable. On the other hand, in [3] P. Gabriel, M. Lemańczyk, P. Liardet have proved that if  $\varphi$  is absolutely continuous with  $d(\varphi) = 0$ , then  $T_{\varphi}$  has singular spectrum.

The aim of this paper is to find a spectral equivalent of topological degree in case G = SU(2).

### 2 Degree of cocycle

In this section we introduce the notion of degree in case G = SU(2). For a given matrix  $A = [a_{ij}]_{i,j=1,...,d} \in M_d(\mathbb{C})$  define  $||A|| = \sqrt{\frac{1}{d} \sum_{i,j=1}^d |a_{ij}|^2}$ . Observe that if A is an element of the Lie algebra  $\mathfrak{su}(2)$ , i.e.

$$A = \left[ \begin{array}{cc} ia & b + ic \\ -b + ic & -ia \end{array} \right],$$

where  $a, b, c \in \mathbb{R}$ , then  $||A|| = \sqrt{\det A}$ . Moreover, if B is an element of the group SU(2), i.e.

$$A = \left[ \begin{array}{cc} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{array} \right]$$

where  $z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1$ , then  $\mathrm{Ad}_B A = BAB^{-1} \in \mathfrak{su}(2)$  and  $\|\mathrm{Ad}_B A\| = \|A\|$ .

Consider the scalar product of  $\mathfrak{su}(2)$  given by

$$\langle A, B \rangle = -\frac{1}{8} \operatorname{tr}(\operatorname{ad} A \circ \operatorname{ad} B).$$

Then  $||A|| = \sqrt{\langle A, A \rangle}$ . By  $L^2(X, \mathfrak{su}(2))$  we mean the space of all functions  $f: X \to \mathfrak{su}(2)$  such that

$$||f||_{L^2} = \sqrt{\int_X ||f(x)||^2 dx} < \infty.$$

For two  $f_1, f_2 \in L^2(X, \mathfrak{su}(2))$  set

$$\langle f_1, f_2 \rangle_{L^2} = \int_X \langle f_1(x), f_2(x) \rangle dx$$

The space  $L^2(X, \mathfrak{su}(2))$  endowed with the above scalar product is a Hilbert space.

By  $L^1(X, \mathfrak{su}(2))$  we mean the space of all functions  $f: X \to \mathfrak{su}(2)$  such that

$$||f||_{L^1} = \int_X ||f(x)|| dx < \infty.$$

The space  $L^1(X, \mathfrak{su}(2))$  endowed with the norm  $\|\|_{L^1}$  is a Banach space.

For a given measurable cocycle  $\varphi : \mathbb{T} \to SU(2)$  consider the unitary operator

(1) 
$$U: L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2)), \quad Uf(x) = \operatorname{Ad}_{\varphi(x)} f(Tx).$$

Then  $U^n f(x) = \operatorname{Ad}_{\varphi^{(n)}(x)} f(T^n x)$  for any integer n.

**Lemma 2.1** There exists an operator  $P: L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2))$  such that

$$\frac{1}{n}\sum_{j=0}^{n-1}U^jf\to Pf\quad in\quad L^2(\mathbb{T},\mathfrak{su}(2))$$

for any  $f \in L^2(\mathbb{T}, \mathfrak{su}(2))$  and  $U \circ P = P$ . Moreover, ||Pf|| is constant  $\lambda$ -a.e..

**Proof.** First claim of the lemma follows from the von Neuman ergodic theorem. Since  $U \circ P = P$ , we have  $\operatorname{Ad}_{\varphi(x)} Pf(Tx) = Pf(x)$ , for  $\lambda$ -a.e.  $x \in \mathbb{T}$ . It follows that  $\|Pf(Tx)\| = \|Pf(x)\|$ , for  $\lambda$ -a.e.  $x \in \mathbb{T}$ . Hence  $\|Pf(x)\| = c$ , for  $\lambda$ -a.e.  $x \in \mathbb{T}$ , by the ergodicity of T.

**Lemma 2.2** For every  $f \in L^2(\mathbb{T}, \mathfrak{su}(2))$  the sequence  $\frac{1}{n} \sum_{j=0}^{n-1} U^j f$  converges  $\lambda$ -almost everywhere.

**Proof.** Let  $\tilde{f} \in L^2(\mathbb{T} \times SU(2), \mathfrak{su}(2))$  be given by  $\tilde{f}(x, g) = \operatorname{Ad}_g f(x)$ . Then  $\tilde{f}(T^n_{\varphi}(x, g)) = \operatorname{Ad}_g(U^n f(x))$ 

for any integer n. By the Birkhoff ergodic theorem, the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1}\tilde{f}(T_{\varphi}^{n}(x,g)) = \operatorname{Ad}_{g}(\frac{1}{n}\sum_{j=0}^{n-1}U^{j}f(x))$$

converges for  $\lambda \otimes \mu$ -a.e.  $(x,g) \in \mathbb{T} \times SU(2)$ . Hence there exists  $g \in SU(2)$  such that  $\operatorname{Ad}_g(\frac{1}{n}\sum_{j=0}^{n-1}U^jf(x))$  converges for  $\lambda$ -a.e.  $x \in \mathbb{T}$ , and the proof is complete.

Recall that, if a function  $\varphi : \mathbb{T} \to SU(2)$  is of class  $C^1$ , then  $D\varphi(x)\varphi(x)^{-1} \in \mathfrak{su}(2)$  for every  $x \in \mathbb{T}$ .

**Lemma 2.3** For every  $C^1$ -cocycle  $\varphi : \mathbb{T} \to SU(2)$ , there exists  $\psi \in L^2(\mathbb{T}, \mathfrak{su}(2))$  such that

$$\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi \quad in \ L^2(\mathbb{T},\mathfrak{su}(2)) \ and \ \lambda-almost \ everywhere.$$

Moreover,  $\|\psi\|$  is a constant function  $\lambda$ -a.e. and  $\varphi(x)\psi(Tx)\varphi(x)^{-1} = \psi(x)$  for  $\lambda$ -a.e.  $x \in \mathbb{T}$ .

**Proof.** Since

$$D\varphi^{(n)}(x) = \sum_{j=0}^{n-1} \varphi(x) \dots \varphi(T^{j-1}x) D\varphi(T^j x) \varphi(T^{j+1}x) \dots \varphi(T^{n-1}x),$$

we have

$$D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} = \sum_{j=0}^{n-1} \varphi(x) \dots \varphi(T^{j-1}x) D\varphi(T^{j}x) \varphi(T^{j}x)^{-1} \varphi(T^{j-1}x)^{-1} \dots \varphi(x)^{-1}$$
  
$$= \sum_{j=0}^{n-1} \varphi^{(j)}(x) D\varphi(T^{j}x) \varphi(T^{j}x)^{-1} (\varphi^{(j)}(x))^{-1}$$
  
$$= \sum_{j=0}^{n-1} U^{j} (D\varphi \varphi^{-1})(x),$$

where U is the unitary operator given by (1). Applying Lemmas 2.1 and 2.2, we get  $\psi = P(D\varphi \varphi^{-1})$ , and the proof is complete.

**Definition 1** The number  $\|\psi\|$  will be called the *degree* of the cocycle  $\varphi$  and denoted by  $d(\varphi)$ .

Lemma 2.3 shows that

$$\frac{1}{n} \| D\varphi^{(n)}(\varphi^{(n)})^{-1} \|_{L^1} \to d(\varphi).$$

On the other hand,  $\|D\varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1}$  is the length of the curve  $\varphi^{(n)}$ . Geometrically speaking, the degree of  $\varphi$  is the limit of length $(\varphi^{(n)})/n$ .

A measurable cocycle  $\delta : \mathbb{T} \to SU(2)$  is called *diagonal* if there exists a measurable function  $\gamma : \mathbb{T} \to \mathbb{T}$  such that

$$\delta(x) = \left[\begin{array}{cc} \gamma(x) & 0\\ 0 & \overline{\gamma(x)} \end{array}\right].$$

**Theorem 2.4** Suppose that  $\varphi : \mathbb{T} \to SU(2)$  is a  $C^1$ -cocycle with  $d(\varphi) \neq 0$ . Then  $\varphi$  is cohomologous to a diagonal cocycle.

**Proof.** For every nonzero  $A \in \mathfrak{su}(2)$  there exists  $B_A \in SU(2)$  such that

$$B_A A (B_A)^{-1} = \begin{bmatrix} i \|A\| & 0 \\ 0 & -i \|A\| \end{bmatrix}.$$

Indeed, if  $A = \begin{bmatrix} ia & b+ic \\ -b+ic & -ia \end{bmatrix}$ , then we can take  $B_A = \begin{cases} \begin{bmatrix} -i\sqrt{\frac{\|A\|+a}{2\|A\|}} \frac{b+ic}{|b+ic|} & -\sqrt{\frac{\|A\|-a}{2\|A\|}} \\ \sqrt{\frac{\|A\|-a}{2\|A\|}} & i\sqrt{\frac{\|A\|+a}{2\|A\|}} \frac{b-ic}{|b+ic|} \end{bmatrix} & \text{if } |a| \neq \|A\| \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } a = -\|A\| \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } a = \|A\|. \end{cases}$ 

Set  $p(x) = B_{\psi(x)}$ . Then  $p: \mathbb{T} \to SU(2)$  is a measurable function and

$$\psi(x) = p(x)^{-1} \begin{bmatrix} i \, d(\varphi) & 0 \\ 0 & -i \, d(\varphi) \end{bmatrix} p(x).$$

Since  $\varphi(x)\psi(Tx)\varphi(x)^{-1}=\psi(x)$ , we have

$$\varphi(x)p(Tx)^{-1} \begin{bmatrix} i \, d(\varphi) & 0\\ 0 & -i \, d(\varphi) \end{bmatrix} p(Tx)\varphi(x)^{-1} = p(x)^{-1} \begin{bmatrix} i \, d(\varphi) & 0\\ 0 & -i \, d(\varphi) \end{bmatrix} p(x).$$

Hence

$$p(x)\varphi(x)p(Tx)^{-1}\left[\begin{array}{cc}i\,d(\varphi)&0\\0&-i\,d(\varphi)\end{array}\right] = \left[\begin{array}{cc}i\,d(\varphi)&0\\0&-i\,d(\varphi)\end{array}\right]p(x)\varphi(x)p(Tx)^{-1}.$$

Since  $d(\varphi) \neq 0$ , we see that the cocycle  $\delta : \mathbb{T} \to SU(2)$  defined by  $\delta(x) = p(x)\varphi(x)p(Tx)^{-1}$  is diagonal.

For a given  $C^1$ -cocycle  $\varphi : \mathbb{T} \to SU(2)$  with nonzero degree let  $\gamma = \gamma(\varphi) : \mathbb{T} \to \mathbb{T}$  be a measurable cocycle such that the cocycles  $\varphi$  and  $\begin{bmatrix} \gamma & 0\\ 0 & (\gamma)^{-1} \end{bmatrix}$  are cohomologous. It is easy to check that the choice of  $\gamma$  is unique up to a measurable cohomology with values in the circle and inverse.

Theorem 2.4 shows that if  $d(\varphi) \neq 0$ , then the skew product  $T_{\varphi}$  is metrically isomorphic to a skew product of an irrational rotation on the circle and a diagonal cocycle. It follows that  $T_{\varphi}$  is not ergodic. However, in the next sections we show that if  $d(\varphi) \neq 0$ , then  $\varphi$  is not cohomologous to a constant cocycle. Moreover, the skew product  $T_{\gamma} : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable. We prove also that (with some additional assumptions on  $\varphi$ ) the Lebesgue component in the spectrum of  $T_{\gamma}$  has countable multiplicity. It follows that if  $d(\varphi) \neq 0$ , then:

- all ergodic components of  $T_{\varphi}$  are metrically isomorphic to  $T_{\gamma}$ ,
- the spectrum of  $T_{\varphi}$  consists of two parts: discrete and mixing,
- (with some additional assumptions on  $\varphi$ ) the Lebesgue component in the spectrum of  $T_{\varphi}$  has countable multiplicity.

In case  $G = \mathbb{T}$  the topological degree of each  $C^1$ -cocycle is an integer number. An important question is: what can one say on values of degree in case G = SU(2)? If a cocycle  $\varphi$  is cohomologous to a diagonal cocycle via a smooth transfer function, then  $d(\varphi) \in 2\pi \mathbb{N}_0 = 2\pi (\mathbb{N} \cup \{0\})$ .

We call a function  $f : \mathbb{T} \to SU(2)$  absolutely continuous if  $f_{ij} : \mathbb{T} \to \mathbb{C}$  is absolutely continuous for i, j = 1, 2. Suppose that  $\varphi$  is cohomologous to a diagonal cocycle via an absolutely continuous transfer function. Then  $\varphi$  can be represented as  $\varphi(x) = p(x)^{-1}\delta(x)p(Tx)$ , where  $\delta, p : \mathbb{T} \to SU(2)$  are absolutely continuous and  $\delta$  is diagonal. Since  $\varphi^{(n)}(x) = p(x)^{-1}\delta^{(n)}(x)p(T^nx)$ , we have

$$\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} = \frac{1}{n}(-p(x)^{-1}Dp(x) + \varphi^{(n)}(x)p(T^nx)^{-1}Dp(T^nx)(\varphi^{(n)}(x))^{-1} + p(x)^{-1}D\delta^{(n)}(x)(\delta^{(n)}(x))^{-1}p(x)).$$

On the other hand,  $\delta(x) = \begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix}$ , where  $\gamma : \mathbb{T} \to \mathbb{T}$  is an absolutely continuous cocycle of the form  $\gamma(x) = \exp 2\pi i (\widetilde{\gamma}(x) + kx)$ , where k is the topological degree of  $\gamma$  and  $\widetilde{\gamma} : \mathbb{T} \to \mathbb{R}$  is an absolutely continuous function. Then

$$\frac{1}{n}D\gamma^{(n)}(x)(\gamma^{(n)}(x))^{-1} = 2\pi i (\frac{1}{n}\sum_{j=0}^{n-1}D\widetilde{\gamma}(T^jx) + k) \to 2\pi i k$$

in  $L^1(\mathbb{T},\mathbb{R})$ , by the Birkhoff ergodic theorem. It follows that

$$\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1} \to p(x)^{-1} \begin{bmatrix} 2\pi ik & 0\\ 0 & -2\pi ik \end{bmatrix} p(x)$$

in  $L^1(\mathbb{T}, \mathfrak{su}(2))$ . Hence  $d(\varphi) = 2\pi |d(\gamma)| \in 2\pi \mathbb{N}_0$ .

In Section 7 it is shown that if  $\alpha$  is the golden ration, then the degree of every  $C^2$ -cocycle belongs to  $2\pi\mathbb{N}_0$ , too.

# **3** Notation and facts from spectral theory

Let U be a unitary operator on a separable Hilbert space  $\mathcal{H}$ . By the cyclic space generated by  $f \in \mathcal{H}$  we mean the space  $\mathbb{Z}(f) = \operatorname{span}\{U^n f; n \in \mathbb{Z}\}$ . By the spectral measure  $\sigma_f$  of f we mean a Borel measure on  $\mathbb{T}$  determined by the equalities

$$\hat{\sigma}_f(n) = \int_{\mathbb{T}} e^{2\pi i n x} d\sigma_f(x) = (U^n f, f)$$

for  $n \in \mathbb{Z}$ . Recall that there exists a sequence  $\{f_n\}_{n \in \mathbb{N}}$  in  $\mathcal{H}$  such that

(2) 
$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n) \text{ and } \sigma_{f_1} \gg \sigma_{f_2} \dots$$

Moreover, for any sequence  $\{f'_n\}_{n\in\mathbb{N}}$  in  $\mathcal{H}$  satisfying (2) we have  $\sigma_{f_1} \equiv \sigma_{f'_1}, \sigma_{f_2} \equiv \sigma_{f'_2}, \ldots$ . The above decompositions of  $\mathcal{H}$  are called spectral decompositions of U.

The spectral type of  $\sigma_{f_1}$  (the equivalence class of measures) will be called the maximal spectral type of U. We say that U has Lebesgue (continuous singular, discrete) spectrum if  $\sigma_{f_1}$  is equivalent to Lebesgue (continuous singular, discrete) measure on the circle. An operator U is called mixing if

$$\hat{\sigma}_f(n) = (U^n f, f) \to 0$$

for any  $f \in \mathcal{H}$ . We say that the Lebesgue component in the spectrum of U has countable multiplicity if  $\lambda \ll \sigma_{f_n}$  for every natural n or equivalently if there exists a sequence  $\{g_n\}_{n \in \mathbb{N}}$ in  $\mathcal{H}$  such that the cyclic spaces  $\mathbb{Z}(g_n)$  are pairwise orthogonal and  $\sigma_{g_n} \equiv \lambda$  for every natural n.

For a skew product  $T_{\varphi}$  consider its Koopman operator

$$U_{T_{\varphi}}: L^{2}(\mathbb{T} \times G, \lambda \otimes \mu) \to L^{2}(\mathbb{T} \times G, \lambda \otimes \mu), \quad UT_{\varphi}f(x,g) = f(Tx, g \varphi(x)).$$

Denote by  $\widehat{G}$  the set of all equivalence classes of unitary irreducible representations of the group G. For any unitary irreducible representation  $\Pi : G \to \mathcal{U}(\mathcal{H}_{\Pi})$  by  $\{\Pi_{ij}\}_{i,j=1}^{d_{\Pi}}$  we mean the matrix elements of  $\Pi$ , where  $d_{\Pi} = \dim \mathcal{H}_{\Pi}$ . Let us decompose

$$L^{2}(\mathbb{T} \times G) = \bigoplus_{\Pi \in \widehat{G}} \bigoplus_{i=1}^{d_{\pi}} \mathcal{H}_{i}^{\Pi},$$

where

$$\mathcal{H}_i^{\Pi} = \{ \sum_{j=1}^{d_{\pi}} \Pi_{ij}(g) f_j(x); f_j \in L^2(\mathbb{T}, \lambda), j = 1, \dots, d_{\Pi} \} \simeq \overbrace{L^2(\mathbb{T}, \lambda) \oplus \ldots \oplus L^2(\mathbb{T}, \lambda)}^{d_{\Pi}}.$$

Observe that  $\mathcal{H}_i^{\Pi}$  is a closed  $U_{T_{\omega}}$ -invariant subspace of  $L^2(\mathbb{T} \times G)$  and

$$U_{T_{\varphi}}^{n}(\sum_{j=1}^{d_{\pi}} \Pi_{ij}(g)f_{j}(x)) = \sum_{j,k=1}^{d_{\pi}} \Pi_{ik}(g)\Pi_{kj}(\varphi^{(n)}(x))f_{j}(T^{n}x).$$

Consider the unitary operator  $M_i^{\Pi} : \mathcal{H}_i^{\Pi} \to \mathcal{H}_i^{\Pi}$  given by

$$M_i^{\Pi}(\sum_{j=1}^{d_{\pi}} \Pi_{ij}(g) f_j(x)) = \sum_{j=1}^{d_{\pi}} e^{2\pi i x} \Pi_{ij}(g) f_j(x).$$

(3) 
$$U_{T_{\varphi}}^{n}M_{i}^{\Pi}f = e^{2\pi i n\alpha}M_{i}^{\Pi}U_{T_{\varphi}}^{n}f$$

for any  $f \in \mathcal{H}_i^{\Pi}$ . It follows that

Then

$$\int_{\mathbb{T}} e^{2\pi i n x} d\sigma_{M_i^{\Pi} f}(x) = (U_{T_{\varphi}}^n M_i^{\Pi} f, M_i^{\Pi} f) = e^{2\pi i n \alpha} (U_{T_{\varphi}}^n f, f) = \int_{\mathbb{T}} e^{2\pi i n x} d(T^* \sigma_f)(x)$$

for any  $f \in \mathcal{H}_i^{\Pi}$ . Hence  $\sigma_{M_i^{\Pi} f} = T^* \sigma_f$ .

**Lemma 3.1** For every  $\Pi \in \widehat{G}$  and  $i = 1, \ldots, d_{\pi}$  if the operator  $U_{T_{\varphi}} : \mathcal{H}_{i}^{\Pi} \to \mathcal{H}_{i}^{\Pi}$  has absolutely continuous spectrum, then it has Lebesgue spectrum of uniform multiplicity.

**Proof.** Let  $\mathcal{H}_i^{\Pi} = \bigoplus_{n=1}^{\infty} \mathbb{Z}(f_n)$  be a spectral decomposition. Then

$$\mathcal{H}_i^{\Pi} = (M_i^{\Pi})^m \mathcal{H}_i^{\Pi} = \bigoplus_{n=1}^{\infty} \mathbb{Z}((M_i^{\Pi})^m f_n)$$

is a spectral decomposition for any integer m. Therefore  $\sigma_{f_n} \equiv \sigma_{(M_i^{\Pi})^m f_n} \ll \lambda$  for every natural n and integer m. Suppose that there exists a Borel set  $A \subset \mathbb{T}$  such that  $\sigma_{f_n}(A) = 0$ and  $\lambda(A) > 0$ . Then

$$\sigma_{f_n}(\bigcup_{m\in\mathbb{Z}}T^mA) = 0 \text{ and } \lambda(\bigcup_{m\in\mathbb{Z}}T^mA) = 1,$$

by the ergodicity of T. It follows that  $\sigma_{f_n} \equiv \lambda$  or  $\sigma_{f_n} = 0$  for every natural n.

Lemma 3.2 If

$$\sum_{n \in \mathbb{Z}} |\int_{\mathbb{T}} \prod_{j \in \mathbb{Z}} |\varphi^{(n)}(x)| dx|^2 < \infty$$

for  $j = 1, \ldots, d_{\Pi}$ , then  $U_{T_{\varphi}}$  has Lebesgue spectrum of uniform multiplicity on  $\mathcal{H}_{i}^{\Pi}$  for  $i = 1, \ldots, d_{\Pi}$ .

**Proof.** Fix  $1 \leq i \leq d_{\Pi}$ . Note that

$$\langle U_{T_{\varphi}}^{n}\Pi_{ij},\Pi_{ij}\rangle = \sum_{k=1}^{d_{\Pi}} \int_{\mathbb{T}} \int_{G} \langle \Pi_{ik}(g)\Pi_{kj}(\varphi^{(n)}(x)),\Pi_{ij}(g)\rangle dgdx = \frac{1}{d_{\Pi}} \int_{\mathbb{T}} \Pi_{jj}(\varphi^{(n)}(x))dx$$

Since

$$\sum_{n\in\mathbb{Z}}|\langle U_{T_{\varphi}}^{n}\Pi_{ij},\Pi_{ij}\rangle|^{2}<\infty,$$

we have  $\sigma_{\Pi_{ij}} \ll \lambda$  for  $j = 1, \ldots, d_{\Pi}$ . From (3) we get  $\sigma_{(M_i^{\Pi})^m \Pi_{ij}} \ll \lambda$  for any integer m. Since  $\{f \in \mathcal{H}_i^{\Pi}; \sigma_f \ll \lambda\}$  is a closed linear subspace of  $L^2(\mathbb{T} \times G)$  and the set  $\{(M_i^{\Pi})^m \Pi_{ij}; j = 1, \ldots, d_{\Pi}, m \in \mathbb{Z}\}$  generates the space  $\mathcal{H}_i^{\Pi}$ , it follows that  $U_{T_{\varphi}}$  has absolutely continuous spectrum on  $\mathcal{H}_i^{\Pi}$ . By Lemma 3.1,  $U_{T_{\varphi}}$  has Lebesgue spectrum of uniform multiplicity on  $\mathcal{H}_i^{\Pi}$ .

**Corollary 3.3** For every  $\Pi \in \widehat{G}$ , if

$$\sum_{n\in\mathbb{Z}} \|\int_{\mathbb{T}} \Pi(\varphi^{(n)}(x)) dx\|^2 < \infty,$$

then  $U_{T_{\varphi}}$  has Lebesgue spectrum of uniform multiplicity on  $\bigoplus_{i=1}^{d_{\Pi}} \mathcal{H}_{i}^{\Pi}$ .

Similarly one can prove the following result.

**Theorem 3.4** For every  $\Pi \in \widehat{G}$ , if

$$\lim_{n\to\infty}\int_{\mathbb{T}}\Pi(\varphi^{(n)}(x))dx=0,$$

then  $U_{T_{\varphi}}$  is mixing on  $\bigoplus_{i=1}^{d_{\Pi}} \mathcal{H}_{i}^{\Pi}$ .

# 4 Representations of SU(2)

In this section some basic information about the theory of representations of the group SU(2) are presented. By  $\mathcal{P}_k$  we mean the linear space of all homogeneous polynomials of degree  $k \in \mathbb{N}_0$  in two variables u and v. Denote by  $\Pi_k$  the representation of the group SU(2) in  $\mathcal{P}_k$  given by

$$\left[\Pi_k\left(\left[\begin{array}{cc}z_1&z_2\\-\overline{z_2}&\overline{z_1}\end{array}\right]\right)f\right](u,v)=f(z_1u-\overline{z_2}v,z_2u+\overline{z_1}v).$$

Of course, all  $\Pi_k$  are unitary (under an appropriate inner product on  $\mathcal{P}_k$ ) and the family  $\{\Pi_0, \Pi_1, \Pi_2, \ldots\}$  is a complete family of continuous unitary irreducible representations of SU(2). In the Lie algebra  $\mathfrak{su}(2)$ , we choose the following basis:

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Let  $V_k$  be a k+1-dimension linear space. For every natural k there exists a basis  $v_0, \ldots, v_k$  of  $V_k$  such that the corresponding representation  $\Pi_k^*$  of  $\mathfrak{su}(2)$  in  $V_k$  has the following form:

$$\Pi_{k}^{*}(e)v_{i} = i(k-i+1)v_{i-1}$$
$$\Pi_{k}^{*}(f)v_{i} = v_{i+1}$$
$$\Pi_{k}^{*}(h)v_{i} = (k-2i)v_{i}$$

for i = 0, ..., k. Then (4)  $||A|| \le ||\Pi_k^*(A)|| \le k^2 ||A||$ 

for any  $A \in \mathfrak{su}(2)$ .

Lemma 4.1

$$\det \Pi_{2k-1}^*(A) = ((2k-1)!!)^2 (\det A)^k$$

for any  $A \in \mathfrak{su}(2)$  and  $k \in \mathbb{N}$ .

**Proof.** For every  $A \in \mathfrak{su}(2)$  there exists  $B \in SU(2)$  and  $d \in \mathbb{R}$  such that  $A = \operatorname{Ad}_B \begin{bmatrix} id & 0 \\ 0 & -id \end{bmatrix}$ . Then

$$\Pi_{2k-1}^*(A) = \Pi_{2k-1}^*(\operatorname{Ad}_B\left[\begin{array}{cc} id & 0\\ 0 & -id \end{array}\right]) = \operatorname{Ad}_{\Pi_{2k-1}(B)}\Pi_{2k-1}^*\left(\left[\begin{array}{cc} id & 0\\ 0 & -id \end{array}\right]\right).$$

It follows that

$$\det \Pi_{2k-1}^*(A) = \det \Pi_{2k-1}^* \left( \begin{bmatrix} id & 0\\ 0 & -id \end{bmatrix} \right) = ((2k-1)!!)^2 d^{2k} = ((2k-1)!!)^2 (\det A)^k. \blacksquare$$

**Lemma 4.2** For any nonzero  $A \in \mathfrak{su}(2)$  the matrix  $\prod_{2k-1}^{*}(A)$  is invertible. Moreover, for every natural k there exists a real constant  $K_k > 0$  such that

$$\|\Pi_{2k-1}^*(A)^{-1}\| \le K_k \|A\|^{-1}$$

for every nonzero  $A \in \mathfrak{su}(2)$ .

**Proof.** First claim of the lemma follows from Lemma 4.1. Set  $C = \prod_{2k-1}^{*}(A)$ . Then

$$|[C]_{ij}| \le (2k)^{4k}(2k-1)! ||A||^{2k-1}$$

for  $i, j = 1, \ldots, 2k$ . It follows that

$$|(C^{-1})_{ij}| = \frac{|[C]_{ij}|}{\det \Pi^*_{2k-1}(A)} \le \frac{(2k)^{4k}(2k-1)! ||A||^{2k-1}}{((2k-1)!!)^2 ||A||^{2k}} = \frac{(2k)^{4k}(2k-1)!}{((2k-1)!!)^2} ||A||^{-1}.$$

Hence

$$\|C^{-1}\| \le \frac{(2k)^{4k+1}(2k-1)!}{((2k-1)!!)^2} \|A\|^{-1}. \blacksquare$$

# 5 Ergodicity and mixing of $T_{\gamma}$

**Lemma 5.1** Suppose that  $\{f_n\}_{n\in\mathbb{N}}$  is a sequence in  $L^2(\mathbb{T},\mathbb{C})$  such that  $\int_0^x f_n(y)dy \to 0$  for any  $x \in \mathbb{T}$ . Let  $g: \mathbb{T} \to \mathbb{C}$  be a bounded measurable function. Then

$$\lim_{n \to \infty} \int_{\mathbb{T}} f_n(y) g(T^n y) dy = 0 \quad and \quad \lim_{n \to \infty} \int_0^x f_n(y) g(y) dy = 0$$

for any  $x \in \mathbb{T}$ .

**Proof.** By assumption, the sequence  $\{f_n\}_{n\in\mathbb{N}}$  tends to zero in the weak topology in  $L^2(\mathbb{T},\mathbb{C})$ , which implies immediately the second claim of the lemma. Since  $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to zero, for every integer m we have

$$\lim_{n \to \infty} \int_{\mathbb{T}} f_n(T^{-n}y) \exp 2\pi i m y \, dy = \lim_{n \to \infty} \int_{\mathbb{T}} f_n(y) \exp 2\pi i m (y + n\alpha) \, dy = 0.$$

It follows that the sequence  $\{f_n \circ T^{-n}\}_{n \in \mathbb{N}}$  converges weakly to zero. Therefore

$$\int_{\mathbb{T}} f_n(y)g(T^n y)dy = \int_{\mathbb{T}} f_n(T^{-n}y)g(y)dy = 0. \blacksquare$$

This gives immediately the following conclusion.

**Corollary 5.2** Suppose that  $\{f_n\}_{n\in\mathbb{N}}$  is a sequence in  $L^2(\mathbb{T}, M_k(\mathbb{C}))$  (k is a natural number) such that  $\int_0^x f_n(y)dy \to 0$  for any  $x \in \mathbb{T}$ . Let  $g : \mathbb{T} \to M_k(\mathbb{C})$  be a bounded measurable function. Then

$$\lim_{n \to \infty} \int_{\mathbb{T}} f_n(y) g(T^n y) dy = 0 \quad and \quad \lim_{n \to \infty} \int_0^x f_n(y) g(y) dy = 0$$

for any  $x \in \mathbb{T}$ .

**Theorem 5.3** Let  $\varphi : \mathbb{T} \to SU(2)$  be a  $C^1$ -cocycle with nonzero degree. Then the skew product  $T_{\gamma(\varphi)} : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$  is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first variable.

**Proof.** By Theorem 3.4, it suffices to show that

$$\lim_{n \to \infty} \int_{\mathbb{T}} (\gamma^{(n)}(x))^k dx = 0$$

for every nonzero integer k. Fix  $k \in \mathbb{N}$ . Denote by  $\psi : \mathbb{T} \to \mathfrak{su}(2)$  the limit (in  $L^2(\mathbb{T}, \mathfrak{su}(2))$ ) of the sequence  $\{\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}d\lambda\}_{n\in\mathbb{N}}$ . Let  $p:\mathbb{T}\to SU(2)$  be a measurable function such that

$$\begin{bmatrix} \gamma(x) & 0\\ 0 & \overline{\gamma(x)} \end{bmatrix} = p(x)\varphi(x)p(Tx)^{-1} \text{ and } \operatorname{Ad}_{p(x)}(\psi(x)) = \begin{bmatrix} id & 0\\ 0 & -id \end{bmatrix},$$

where d is the degree of  $\varphi$  (see the proof of Theorem 2.4). Then

(5) 
$$\begin{bmatrix} (\gamma^{(n)})^k & & & \\ & (\gamma^{(n)})^{k-2} & & 0 \\ & & \ddots & & \\ & 0 & & (\gamma^{(n)})^{-k+2} \\ & & & & (\gamma^{(n)})^{-k} \end{bmatrix} = \Pi_k(p)\Pi_k(\varphi^{(n)})\Pi_k(p \circ T^n)^{-1}$$

for any natural n and

(6) 
$$\operatorname{Ad}_{\Pi_{k}(p(x))}\Pi_{k}^{*}(\psi(x)) = \Pi_{k}^{*}(\operatorname{Ad}_{p(x)}\psi(x))$$
  

$$= \Pi_{k}^{*}(\begin{bmatrix} id & 0\\ 0 & -id \end{bmatrix})$$

$$= \begin{bmatrix} kid & & & \\ (k-2)id & & 0 & \\ & & \ddots & & \\ 0 & & (-k+2)id & \\ & & & -kid \end{bmatrix}.$$

Recall that for any differentiable function  $\xi : \mathbb{T} \to SU(2)$  and for any representation  $\Pi$  of SU(2) we have

$$D(\Pi\xi(x))(\Pi\xi(x))^{-1} = \Pi^*(D\xi(x)\xi(x)^{-1}).$$

Therefore

$$\int_0^x \frac{1}{n} \Pi_k^* (D\varphi^{(n)}(y)(\varphi^{(n)}(y))^{-1}) \Pi_k(\varphi^{(n)}(y)) dy = \int_0^x \frac{1}{n} D(\Pi_k \varphi^{(n)}(y)) dy$$
$$= \frac{1}{n} (\Pi_k(\varphi^{(n)}(x)) - \Pi_k(\varphi^{(n)}(0)))$$

tends to zero for any  $x \in \mathbb{T}$ . Since

$$\frac{1}{n} \Pi_k^* (D\varphi^{(n)}(\varphi^{(n)})^{-1}) \to \Pi_k^* \psi$$

in  $L^2(\mathbb{T}, M_{k+1}(\mathbb{C}))$ , it follows that

$$\int_0^x \Pi_k^*(\psi(y)) \Pi_k(\varphi^{(n)}(y)) dy \to 0$$

for any  $x \in \mathbb{T}$ . By Corollary 5.2,

$$\int_{\mathbb{T}} \Pi_k(p(y)) \Pi_k^*(\psi(y)) \Pi_k(\varphi^{(n)}(y)) \Pi_k(p(T^n y))^{-1} dy \to 0$$

On the other hand,

$$\Pi_k(p(y))\Pi_k^*(\psi(y))\Pi_k(\varphi^{(n)}(y))\Pi_k(p(T^n y))^{-1} = \begin{bmatrix} ikd(\gamma^{(n)}(y))^k & 0 \\ & \ddots & \\ 0 & -ikd(\gamma^{(n)}(y))^{-k} \end{bmatrix},$$

by (5) and (6). Therefore

$$\lim_{n \to \infty} \int_{\mathbb{T}} (\gamma^{(n)}(y))^m dy = 0$$

for any nonzero  $m \in \{-k, -k+2, \dots, k-2, k\}$ , which completes the proof.

### 6 Spectral analysis of cocycles with nonzero degree

In this section it is shown that for every cocycle  $\varphi : \mathbb{T} \to SU(2)$  if  $d(\varphi) \neq 0$  and it satisfies some additional assumptions, then the Lebesgue component in the spectrum of  $T_{\varphi}$  has countable multiplicity.

Now we introduce a notation, which is necessary to prove the main theory. Let  $f, g : \mathbb{T} \to M_k(\mathbb{C})$  be functions of bounded variation (i.e.  $f_{ij}, g_{ij} : \mathbb{T} \to \mathbb{C}$  have bounded variation for  $i, j = 1, \ldots, k$ ) and let one of them be continuous. We will use the symbol  $\int_{\mathbb{T}} f \, dg$  to denote the  $k \times k$ -matrix given by

$$(\int_{\mathbb{T}} f \, dg)_{ij} = \sum_{l=1}^{k} \int_{\mathbb{T}} f_{il} \, dg_{lj}$$

for  $i, j = 1, \ldots, d$ . It is clear that if g is absolutely continuous, then

(7) 
$$\int_{\mathbb{T}} f \, dg = \int_{\mathbb{T}} f(x) \, Dg(x) dx.$$

Moreover, applying integration by parts, we have

(8) 
$$\int_{\mathbb{T}} f \, dg = -\left(\int_{\mathbb{T}} g^T \, df^T\right)^T$$

**Theorem 6.1** Let  $\varphi : \mathbb{T} \to SU(2)$  be a  $C^2$ -cocycle with  $d(\varphi) \neq 0$ . Suppose that the sequence  $\{\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$  is uniformly convergent and  $\{D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$  is bounded in  $L^1(\mathbb{T},\mathfrak{su}(2))$ . Then the Lebesgue component in the spectrum of  $T_{\varphi}$  has countable multiplicity. Moreover, the Lebesgue component in the spectrum of  $T_{\gamma(\varphi)}$  has countable multiplicity, too.

**Proof.** First observe that it suffices to show that for every natural k there exists a real constant  $C_k > 0$  such that

(9) 
$$\|\int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x))dx\| \le \frac{C_k}{n}$$

for large enough natural n. Indeed, let  $p: \mathbb{T} \to SU(2)$  be a measurable function such that

$$p(x)\varphi(x)p(Tx)^{-1} = \delta(x) = \begin{bmatrix} \gamma(x) & 0\\ 0 & \overline{\gamma(x)} \end{bmatrix}.$$

Consider the unitary operator  $V : \mathcal{H}_1^{\Pi_{2k-1}} \to \mathcal{H}_1^{\Pi_{2k-1}}$  given by

$$V(\sum_{i=1}^{d_{\Pi_{2k-1}}} \Pi_{1i}(g)f_i(x)) = \sum_{i,j=1}^{d_{\Pi_{2k-1}}} \Pi_{1j}(g)\Pi_{ji}(p(x)^{-1})f_i(x)).$$

Then

$$V^{-1}U_{T_{\varphi}}V(\sum_{i=1}^{d_{\Pi_{2k-1}}}\Pi_{1i}(g)f_{i}(x)) = \sum_{i,j,l,m=1}^{d_{\Pi_{2k-1}}}\Pi_{1m}(g)\Pi_{ml}(p(x))\Pi_{lj}(\varphi(x))\Pi_{ji}(p(Tx)^{-1})f_{i}(Tx)$$
$$= \sum_{i=1}^{d_{\Pi_{2k-1}}}\Pi_{1i}(g)\Pi_{ii}(\delta(x))f_{i}(Tx).$$

From (9),  $U_{T_{\varphi}} : \mathcal{H}_{1}^{\Pi_{2k-1}} \to \mathcal{H}_{1}^{\Pi_{2k-1}}$  has Lebesgue spectrum of uniform multiplicity, by Corollary 3.3. Hence  $V^{-1}U_{T_{\varphi}}V$  has Lebesgue spectrum of uniform multiplicity and it is the product of the operators  $U_{j} : L^{2}(\mathbb{T}, \mathbb{C}) \to L^{2}(\mathbb{T}, \mathbb{C})$  given by  $U_{j}f(x) = (\gamma(x))^{2k-2j+1}f(Tx)$ for  $j = 1, \ldots, 2k$ . Therefore  $U_{j}$  has absolutely continuous spectrum for  $j = 1, \ldots, 2k$ . By Lemma 3.1,  $U_{j}$  has Lebesgue spectrum for all  $j = 1, \ldots, 2k$  and  $k \in \mathbb{N}$ . It follows that the Lebesgue component in the spectrum of  $T_{\gamma(\varphi)}$  has countable multiplicity.

By assumption,

$$\left\|\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1}\right\| \to d(\varphi)$$

uniformly. Therefore

(10) 
$$\|\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\| \ge d(\varphi)/2$$

for large enough natural n. For all  $A, B \in M_k(\mathbb{C})$  we have  $||AB|| \leq \sqrt{k} ||A|| ||B||$ . Applying these facts, (7) and (8) we get

$$\begin{split} \| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x)) dx \| &= \| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x)) (D\Pi_{2k-1}(\varphi^{(n)}(x)))^{-1} d\Pi_{2k-1}(\varphi^{(n)}(x))) \| \\ &= \| \int_{\mathbb{T}} (\Pi_{2k-1}^{*} (D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1} d\Pi_{2k-1}(\varphi^{(n)}(x)) \| \\ &= \| \int_{\mathbb{T}} (\Pi_{2k-1}(\varphi^{(n)}(x)))^{T} d((\Pi_{2k-1}^{*} (D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^{T} \| \\ &= \| \int_{\mathbb{T}} [(\Pi_{2k-1}(\varphi^{(n)}(x)))^{T} ((\Pi_{2k-1}^{*} (D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1})^{T} \\ (\Pi_{2k-1}^{*} (D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1}] dx \| \\ &\leq 2k \int_{\mathbb{T}} [\| (\Pi_{2k-1}^{*} (D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1} \|^{2} \\ \| \Pi_{2k-1}^{*} D(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}) \| ] dx. \end{split}$$

By Lemma 4.2, we have

$$\|(\Pi_{2k-1}^*(D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}))^{-1}\| \le K_k \|D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}\|^{-1}.$$

From this, (4) and (10) we obtain

$$\begin{split} \| \int_{\mathbb{T}} \Pi_{2k-1}(\varphi^{(n)}(x)) dx \| \\ &\leq \frac{K_k^2 (2k)^3}{n} \int_{\mathbb{T}} [\| \frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}) \|^{-2} \| D(\frac{1}{n} D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1}) \|] dx \\ &\leq \frac{1}{n} (\frac{8K_k k^2}{d(\varphi)})^2 \| D(\frac{1}{n} D\varphi^{(n)}(\varphi^{(n)})^{-1}) \|_{L^1} \end{split}$$

for large enough natural n. By assumption, there exists a real constant M > 0 such that  $\|D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\|_{L^1} \leq M$ . Then

$$\left\|\int_{\mathbb{T}}\Pi_{2k-1}(\varphi^{(n)}(x))dx\right\| \leq \frac{C_k}{n}$$

for large enough natural n, where  $C_k = (\frac{8K_kk^2}{d(\varphi)})^2 M$ .

In this section we also present a class of cocycles satisfying the assumptions of Theorem 6.1. We will need the following lemma.

**Lemma 6.2** Let  $\{f_n : \mathbb{T} \to \mathbb{C}^d; n \in \mathbb{N}\}$  be a sequence of absolutely continuous functions. Assume that the sequence  $\{f_n\}_{n\in\mathbb{N}}$  converges in  $L^1(\mathbb{T}, \mathbb{R}^d)$  to a function f and it is bounded for the sup norm. Suppose that there is a sequence  $\{h_n\}_{n\in\mathbb{N}}$  convergent in  $L^2_+(\mathbb{T}, \mathbb{R})$ and a sequence  $\{k_n\}_{n\in\mathbb{N}}$  bounded in  $L^2_+(\mathbb{T}, \mathbb{R})$  such that

$$||Df_n(x)|| \le h_n(x)k_n(x) \text{ for } \lambda - a.e. \ x \in \mathbb{T}$$

and for any natural n. Then  $\{f_n\}_{n\in\mathbb{N}}$  converges to f uniformly.

**Proof.** Denote by  $h \in L^2_+(\mathbb{T}, \mathbb{R})$  the limit of the sequence  $\{h_n\}_{n \in \mathbb{N}}$ . Let M > 0 be a real number such that  $||k_n||_{L^2} \leq M$  for all natural n. First observe that the sequence  $\{f_n\}_{n \in \mathbb{N}}$  is equicontinuous. Fix  $\varepsilon > 0$ . Take  $n_0 \in \mathbb{N}$  such that  $||h_n - h||_{L^2} < \varepsilon/2M$  for any  $n \geq n_0$ . Then for all  $x, y \in \mathbb{T}$  and  $n \geq n_0$  we have

$$\begin{aligned} \|f_n(x) - f_n(y)\| &= \|\int_x^y Df_n(t)dt\| \le \int_x^y \|Df_n(t)\|dt\\ &\le \int_x^y h_n(t)k_n(t)dt \le \|k_n\|_{L^2} \sqrt{\int_x^y h_n^2(t)dt}\\ &\le M(\sqrt{\int_x^y h^2(t)dt} + \|h_n - h\|_{L^2}) \le M(\sqrt{\int_x^y h^2(t)dt} + \frac{\varepsilon}{2M}) \end{aligned}$$

Choose  $\delta_1 > 0$  such that  $|x - y| < \delta_1$  implies  $\int_x^y h^2(t) dt < (\varepsilon/2M)^2$ . Hence if  $|x - y| < \delta_1$ , then  $||f_n(x) - f_n(y)|| < \varepsilon$  for any  $n \ge n_0$ . Next choose  $0 < \delta \le \delta_1$  such that  $|x - y| < \varepsilon$ 

 $\delta$  implies  $||f_n(x) - f_n(y)|| < \varepsilon$  for any  $n \le n_0$ . It follows that if  $|x - y| < \delta$ , then  $||f_n(x) - f_n(y)|| < \varepsilon$  for every natural n.

By the Arzela–Ascoli theorem, for any subsequence of  $\{f_n\}_{n\in\mathbb{N}}$  there exists a subsequence convergent to f uniformly. Consequently, the sequence  $\{f_n\}_{n\in\mathbb{N}}$  converges to f uniformly.

This gives the following corollary.

**Corollary 6.3** Let  $\{f_n : \mathbb{T} \to \mathbb{C}^d; n \in \mathbb{N}\}$  be a sequence of absolutely continuous functions. Assume that the sequence  $\{f_n\}_{n\in\mathbb{N}}$  converges in  $L^1(\mathbb{T}, \mathbb{R}^d)$  to a function f and it is bounded for the sup norm. Suppose that there is a sequence  $\{h_n\}_{n\in\mathbb{N}}$  convergent in  $L^2_+(\mathbb{T}, \mathbb{R})$ , a sequence  $\{k_n\}_{n\in\mathbb{N}}$  bounded in  $L^2_+(\mathbb{T}, \mathbb{R})$  and a sequence  $\{l_n\}_{n\in\mathbb{N}}$  convergent in  $L^1_+(\mathbb{T}, \mathbb{R})$  such that

$$||Df_n(x)|| \leq l_n(x) + h_n(x)k_n(x)$$
 for  $\lambda$ -a.e.  $x \in \mathbb{T}$ 

and for any natural n. Then  $\{f_n\}_{n\in\mathbb{N}}$  converges to f uniformly.

We will denote by  $BV^2(\mathbb{T}, SU(2))$  the set of all functions  $f : \mathbb{T} \to SU(2)$  of bounded variation such that  $Df(f)^{-1} \in L^2(\mathbb{T}, \mathfrak{su}(2))$ .

**Lemma 6.4** Let  $\varphi : \mathbb{T} \to SU(2)$  be a  $C^2$ -cocycle. Suppose that  $\varphi$  is cohomologous to a diagonal cocycle with a transfer function in  $BV^2(\mathbb{T}, SU(2))$ . Then the sequence  $\{\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$  is uniformly convergent and  $\{D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$  is bounded in  $L^1(\mathbb{T},\mathfrak{su}(2))$ 

**Proof.** By Corollary 6.3, it suffices to show that there exist a sequence  $\{h_n\}_{n\in\mathbb{N}}$  convergent in  $L^2_+(\mathbb{T},\mathbb{R})$ , a sequence  $\{k_n\}_{n\in\mathbb{N}}$  bounded in  $L^2_+(\mathbb{T},\mathbb{R})$  and a sequence  $\{l_n\}_{n\in\mathbb{N}}$  convergent in  $L^1_+(\mathbb{T},\mathbb{R})$  such that

$$\|D(\frac{1}{n}D\varphi^{(n)}(x)(\varphi^{(n)}(x))^{-1})\| \le l_n(x) + h_n(x)k_n(x) \text{ for } \lambda \text{-a.e. } x \in \mathbb{T}.$$

By assumption, there exist  $\delta, p \in BV^2(\mathbb{T}, SU(2))$  such that  $\varphi(x) = p(x)^{-1}\delta(x)p(Tx)$ , where  $\delta$  is a diagonal cocycle. Then

$$D\varphi(x)\varphi(x)^{-1} = -p(x)^{-1}Dp(x) + p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x) + \varphi(x)p(Tx)^{-1}Dp(Tx)\varphi(x)^{-1}$$

for  $\lambda$ -a.e.  $x \in \mathbb{T}$ . Set

$$\widetilde{\varphi}(x) = D\varphi(x)\varphi(x)^{-1}, \quad \widetilde{p}(x) = p(x)^{-1}Dp(x) \text{ and } \widetilde{\delta}(x) = p(x)^{-1}D\delta(x)\delta(x)^{-1}p(x).$$

Then  $\widetilde{\varphi}(x) = -\widetilde{p}(x) + U\widetilde{p}(x) + \widetilde{\delta}(x)$ , where  $\widetilde{p}, \widetilde{\delta} \in L^2(\mathbb{T}, \mathfrak{su}(2))$ . Since

$$\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1} = \frac{1}{n}\sum_{k=0}^{n-1}\varphi^{(k)}\,\widetilde{\varphi}\circ T^k\,(\varphi^{(k)})^{-1},$$

we have

$$D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} (\operatorname{Ad}_{\varphi^{(j)}}(\widetilde{\varphi}\circ T^{j})\operatorname{Ad}_{\varphi^{(k)}}(\widetilde{\varphi}\circ T^{k}) - \operatorname{Ad}_{\varphi^{(k)}}(\widetilde{\varphi}\circ T^{k})\operatorname{Ad}_{\varphi^{(j)}}(\widetilde{\varphi}\circ T^{j}))$$

$$+ \frac{1}{n}\sum_{k=0}^{n-1}\operatorname{Ad}_{\varphi^{(k)}}(D\widetilde{\varphi}\circ T^{k})$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} [U^{j}\widetilde{\varphi}, U^{k}\widetilde{\varphi}] + \frac{1}{n}\sum_{k=0}^{n-1}U^{k}(D\widetilde{\varphi}).$$

However,

$$\begin{split} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^{j} \widetilde{\varphi}, U^{k} \widetilde{\varphi}] &= \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^{j+1} \widetilde{p} - U^{j} \widetilde{p} + U^{j} \widetilde{\delta}, U^{k} \widetilde{\varphi}] \\ &= \sum_{k=0}^{n-1} [U^{k} \widetilde{p} - \widetilde{p}, U^{k} \widetilde{\varphi}] + \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^{j} \widetilde{\delta}, U^{k+1} \widetilde{p} - U^{k} \widetilde{p} + U^{k} \widetilde{\delta}] \\ &= \sum_{k=0}^{n-1} [U^{k} \widetilde{p} - \widetilde{p}, U^{k} \widetilde{\varphi}] + \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} [U^{j} \widetilde{\delta}, U^{k} \widetilde{\delta}] + \sum_{j=0}^{n-2} [U^{j} \widetilde{\delta}, U^{n} \widetilde{p} - U^{j+1} \widetilde{p}]. \end{split}$$

Since

$$U^{j}\widetilde{\delta}(x) = \operatorname{Ad}_{\varphi^{(n)}(x)p(T^{n}x)^{-1}}(D\delta(T^{n}x)\delta(T^{n}x)^{-1})$$
  
$$= \operatorname{Ad}_{p(x)^{-1}\delta^{(n)}(x)}(D\delta(T^{n}x)\delta(T^{n}x)^{-1})$$
  
$$= \operatorname{Ad}_{p(x)^{-1}}(D\delta(T^{n}x)\delta(T^{n}x)^{-1}),$$

we have  $[U^j \widetilde{\delta}, U^k \widetilde{\delta}] = 0$  for any integers j, k. Observe that  $||[A, B]|| \le 2||A|| ||B||$  for any  $A, B \in \mathfrak{su}(2)$ . It follows that

$$\begin{split} \|D(\frac{1}{n}D\varphi^{(n)}(\varphi^{(n)})^{-1})\| \\ &\leq \frac{2}{n}\sum_{k=0}^{n-1}(\|D\widetilde{\varphi}\circ T^k\| + \|\widetilde{p}\circ T^k\| \|\widetilde{\varphi}\circ T^k\| + \|\widetilde{p}\| \|\widetilde{\varphi}\circ T^k\| + \|\widetilde{\delta}\circ T^k\| \|\widetilde{p}\circ T^{k+1}\|) \\ &+ \|\widetilde{p}\circ T^n\|\frac{2}{n}\sum_{k=0}^{n-1}\|\widetilde{\delta}\circ T^k\|. \end{split}$$

 $\operatorname{Set}$ 

$$h_n = \frac{2}{n} \sum_{k=0}^{n-1} \|\widetilde{\delta} \circ T^k\|$$

$$k_n = \|\widetilde{p} \circ T^n\|$$
  

$$l_n = \frac{2}{n} \sum_{k=0}^{n-1} (\|D\widetilde{\varphi} \circ T^k\| + \|\widetilde{p} \circ T^k\| \|\widetilde{\varphi} \circ T^k\| + \|\widetilde{p}\| \|\widetilde{\varphi} \circ T^k\| + \|\widetilde{\delta} \circ T^k\| \|\widetilde{p} \circ T^{k+1}\|).$$

By the Birkhoff ergodic theorem, the sequence  $\{h_n\}_{n\in\mathbb{N}}$  converges in  $L^2_+(\mathbb{T},\mathbb{R})$  and the sequence  $\{l_n\}_{n\in\mathbb{N}}$  converges in  $L^1_+(\mathbb{T},\mathbb{R})$ . This completes the proof.

Theorem 6.1 and Lemma 6.4 lead to the following conclusion.

**Corollary 6.5** Let  $\varphi : \mathbb{T} \to SU(2)$  be a  $C^2$ -cocycle with  $d(\varphi) \neq 0$ . Suppose that  $\varphi$  is cohomologous to a diagonal cocycle with a transfer function in  $BV^2(\mathbb{T}, SU(2))$ . Then the Lebesgue component in the spectrum of  $T_{\varphi}$  has countable multiplicity. Moreover, the Lebesgue component in the spectrum of  $T_{\gamma(\varphi)}$  has countable multiplicity, too.

The following result will be useful in the next section of the paper.

**Proposition 6.6** For every  $C^2$ -cocycle  $\varphi : \mathbb{T} \to SU(2)$ , the sequence

$$\frac{1}{n^2}D(D\varphi^{(n)}(\varphi^{(n)})^{-1})$$

converges to zero in  $L^1(\mathbb{T}, \mathfrak{su}(2))$ .

The following lemmas are some simple generalizations of some classical results. Their proofs are left to the reader.

**Lemma 6.7** Let  $\{a_n\}_{n\in\mathbb{N}}$  be an increasing sequence of natural numbers and let  $\{f_n\}_{n\in\mathbb{N}}$  be a sequence in the Banach space  $L^2(\mathbb{T}, M_2(\mathbb{C}))$ . Then

$$\frac{f_{n+1} - f_n}{a_{n+1} - a_n} \to g \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})) \implies \frac{f_n}{a_n} \to g \text{ in } L^2(\mathbb{T}, M_2(\mathbb{C})).$$

**Lemma 6.8** Let  $\{g_k^n; n \in \mathbb{N}, 0 \leq k < n\}$  be a triangular matrix of elements from  $L^2(\mathbb{T}, M_2(\mathbb{C}))$  such that  $||g_k^n|| = O(1/n)$  and

$$g_0^n + g_1^n + \ldots + g_{n-1}^n \to g \quad in \quad L^2(\mathbb{T}, M_2(\mathbb{C})).$$

Then  $f_n \to f$  in  $L^2(\mathbb{T}, M_2(\mathbb{C}))$  implies

$$\sum_{k=0}^{n-1} g_k^n f_k \to g f \quad and \quad \sum_{k=0}^{n-1} f_k g_k^n \to f g \quad in \quad L^1(\mathbb{T}, M_2(\mathbb{C})).$$

**Proof of Proposition 6.6.** First recall that

$$\frac{1}{n^2}D(D\varphi^{(n)}(\varphi^{(n)})^{-1}) = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^j\widetilde{\varphi}, U^k\widetilde{\varphi}] + \frac{1}{n^2}\sum_{k=0}^{n-1}U^k(D\widetilde{\varphi}),$$

where  $\widetilde{\varphi} = D\varphi(\varphi)^{-1}$  and

$$\frac{1}{n}\sum_{k=0}^{n-1}U^k\widetilde{\varphi}\to\psi\quad\text{in }L^2(\mathbb{T},\mathfrak{su}(2)).$$

Since  $\frac{1}{n^2} \sum_{k=0}^{n-1} U^k(D\widetilde{\varphi})$  uniformly converges to zero, it suffices to show that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j \widetilde{\varphi} U^k \widetilde{\varphi} = \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^k \widetilde{\varphi} U^j \widetilde{\varphi} = \frac{1}{2} \psi \psi \quad \text{in} \quad L^2(\mathbb{T}, M_2(\mathbb{C})).$$

Set  $f_n = \sum_{k=0}^{n-1} (n-k) U^k \widetilde{\varphi}$  and  $a_n = n^2$ . Then

$$\frac{f_{n+1} - f_n}{a_{n+1} - a_n} = \frac{\sum_{k=0}^n (n+1-k)U^k \widetilde{\varphi} - \sum_{k=0}^{n-1} (n-k)U^k \widetilde{\varphi}}{(n+1)^2 - n^2} = \frac{\sum_{k=0}^n U^k \widetilde{\varphi}}{2n+1} \to \frac{1}{2}\psi$$

in  $L^2(\mathbb{T}, M_2(\mathbb{C}))$ . Applying Lemma 6.7, we get

$$\frac{1}{n^2} \sum_{k=0}^{n-1} (n-k) U^k \widetilde{\varphi} \to \frac{1}{2} \psi \quad \text{in} \quad L^2(\mathbb{T}, M_2(\mathbb{C})).$$

Therefore

$$\frac{1}{n^2}\sum_{k=0}^{n-1}k\,U^k\widetilde{\varphi} = \frac{1}{n}\sum_{k=0}^{n-1}U^k\widetilde{\varphi} - \frac{1}{n^2}\sum_{k=0}^{n-1}(n-k)U^k\widetilde{\varphi} \to \psi - \frac{1}{2}\psi = \frac{1}{2}\psi$$

in  $L^2(\mathbb{T}, M_2(\mathbb{C}))$ . Applying Lemma 6.8 with  $g_k^n = \frac{k}{n^2} U^k \widetilde{\varphi}$  and  $f_k = \frac{1}{k} \sum_{j=0}^{k-1} U^j \widetilde{\varphi}$ , we conclude that

$$\sum_{k=0}^{n-1} g_k^n f_k = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^k \widetilde{\varphi} U^j \widetilde{\varphi} \to \frac{1}{2} \psi \psi$$

and

$$\sum_{k=0}^{n-1} f_k g_k^n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} U^j \widetilde{\varphi} U^k \widetilde{\varphi} \to \frac{1}{2} \psi \psi$$

in  $L^2(\mathbb{T}, M_2(\mathbb{C}))$ , which completes the proof.

### 7 Possible values of degree

One may ask what we know about the set of possible values of degree. For  $G = \mathbb{T}$  the degree of each smooth cocycle is an integer number. Probably, in the case of cocycles with values in SU(2) the set of possible values of degree is more complicated. However, in this section we show that if  $\alpha$  is the golden ratio, then the degree of each smooth cocycle belongs to  $2\pi\mathbb{N}_0$ . The idea of renormalization, which is used to prove this result is due to Rychlik [8].

Let  $\alpha$  be the golden ratio (i.e. the positive root of the equation  $\alpha^2 + \alpha = 1$ ). It will be advantageous for our notation to consider the interval  $[-\alpha^2, \alpha)$  to be the model of the circle. Then the map  $T : [-\alpha^2, \alpha) \rightarrow [-\alpha^2, \alpha)$  given by

$$T(x) = \begin{cases} x + \alpha & \text{for} \quad x \in [-\alpha^2, 0] \\ x - \alpha^2 & \text{for} \quad x \in [0, \alpha) \end{cases}$$

is the rotation by  $\alpha$ . Let  $X = [-\alpha^2, \alpha^3)$ . Then the first return time to X, which we call  $\tau$ , satisfies the following formula

$$\tau(x) = \begin{cases} 1 & \text{for} \quad x \in [0, \alpha^3) \\ 2 & \text{for} \quad x \in [-\alpha^2, 0) \end{cases}$$

and the first return map  $T_X : X \to X$  is equal to T up to a linear scaling. Indeed, if  $M : \mathbb{T} \to X$  is the map given by  $M(x) = -\alpha x$ , then  $T_X \circ M = M \circ T$ .

By  $W^1$  we mean the space of all cocycle  $\varphi : \mathbb{T} \to SU(2)$  such that the functions  $\varphi : [-\alpha^2, 0) \to SU(2), \varphi : [0, \alpha) \to SU(2)$  are both of class  $C^1$  and

$$\lim_{x \to 0^{-}} D\varphi(x)\varphi(x)^{-1} \quad \text{and} \quad \lim_{x \to \alpha^{-}} D\varphi(x)\varphi(x)^{-1}$$

exist. The topology of  $W^1$  is induced from  $C^1((-\alpha^2, 0) \cup (0, \alpha))$ . Consider the renormalization operator  $\Phi: W^1 \to W^1$  defined by

$$\Phi \varphi(x) = \varphi^{(\tau(Mx))}(Mx).$$

Then

$$\Phi^{n}\varphi(x) = \begin{cases} \varphi^{(q_{n+1})}(M^{n}x) & \text{for} \quad x \in [-\alpha^{2}, 0) \\ \varphi^{(q_{n+2})}(M^{n}x) & \text{for} \quad x \in [0, \alpha) \end{cases}$$

for any natural n, where  $\{q_n\}_{n\in\mathbb{N}}$  is the Fibonacci sequence. By  $W_0^1$  we mean the set of all cocycles  $\varphi \in W^1$  such that  $\varphi^{(2)}$  is continuous at 0. The set  $W_0^1$  is a closed subset of  $W^1$  and

(11) 
$$\Phi(W_0^1) \subset W_0^1$$

(see [8]). It is easy to check that  $||D(\Phi\varphi)(\Phi\varphi)^{-1}||_{L^1} \leq ||D\varphi(\varphi)^{-1}||_{L^1}$  for any  $\varphi \in W^1$ . The following result is due to M. Rychlik [8].

**Proposition 7.1** If  $||D(\Phi^k \varphi)(\Phi^k \varphi)^{-1}||_{L^1} = ||D\varphi(\varphi)^{-1}||_{L^1}$  for all natural k, then

$$D\varphi(x)(\varphi(x))^{-1} = \alpha \operatorname{Ad}_{\varphi(x)}[D\varphi(Tx)(\varphi(Tx))^{-1}]$$

for every  $x \in [-\alpha^2, 0)$ .

**Lemma 7.2** Let  $\varphi : \mathbb{T} \to SU(2)$  be a  $C^2$ -cocycle. Assume that

$$\frac{1}{n}D\varphi^{(n)}(0)(\varphi^{(n)}(0))^{-1} \to H \in \mathfrak{su}(2)$$

and there is an increasing sequence  $\{n_k\}_{k\in\mathbb{N}}$  of even numbers such that

$$\lim_{k \to \infty} \alpha^{n_k} \int_0^{\alpha^{n_k}} |D(D\varphi^{(q_{n_k+i})}(x)(\varphi^{(q_{n_k+i})}(x))^{-1})| dx = 0$$

for i = 1, 2. Then  $||H|| \in 2\pi \mathbb{N}_0$ .

**Proof.** First note that

$$D\Phi^{n}\varphi(x)(\Phi^{n}\varphi(x))^{-1} = \begin{cases} \alpha^{n}D\varphi^{(q_{n+1})}(M^{n}x)(\varphi^{(q_{n+1})}(M^{n}x))^{-1} & \text{for} \quad x \in [-\alpha^{2}, 0] \\ \alpha^{n}D\varphi^{(q_{n+2})}(M^{n}x)(\varphi^{(q_{n+2})}(M^{n}x))^{-1} & \text{for} \quad x \in [0, \alpha) \end{cases}$$

for any even n. Since

$$\begin{aligned} |\frac{1}{q_{n+i}} D\varphi^{(q_{n+i})} (M^n x) (\varphi^{(q_{n+i})} (M^n x))^{-1} &- \frac{1}{q_{n+i}} D\varphi^{(q_{n+i})} (0) (\varphi^{(q_{n+i})} (0))^{-1} | \\ &\leq \frac{1}{q_{n+i}} \int_0^{\alpha^n x} |D(D\varphi^{(q_{n+i})} (\varphi^{(q_{n+i})})^{-1})| d\lambda \leq \frac{1}{q_{n+i}\alpha^n} \alpha^n \int_0^{\alpha^n} |D(D\varphi^{(q_{n+i})} (\varphi^{(q_{n+i})})^{-1})| d\lambda \end{aligned}$$

for all even n, i = 1, 2 and

$$\lim_{n \to \infty} \alpha^n q_{n+1} = 1/(1 + \alpha^2), \quad \lim_{n \to \infty} \alpha^n q_{n+2} = 1/(\alpha + \alpha^3),$$

it follows that

$$\lim_{k \to \infty} D\Phi^{n_k} \varphi(x) (\Phi^{n_k} \varphi(x))^{-1} = \lim_{k \to \infty} \alpha^{n_k} q_{n_k+1} \frac{1}{q_{n_k+1}} D\varphi^{(q_{n_k+1})}(0) (\varphi^{(q_{n_k+1})}(0))^{-1}$$
$$= \frac{1}{1+\alpha^2} H$$

uniformly on  $[-\alpha^2,0)$  and

$$\lim_{k \to \infty} D\Phi^{n_k} \varphi(x) (\Phi^{n_k} \varphi(x))^{-1} = \lim_{k \to \infty} \alpha^{n_k} q_{n_k+2} \frac{1}{q_{n_k+2}} D\varphi^{(q_{n_k+2})}(0) (\varphi^{(q_{n_k+2})}(0))^{-1} = \frac{1}{\alpha + \alpha^3} H$$

uniformly on  $[0, \alpha)$ . Therefore we can assume that there exists  $v \in W^1$  such that

$$\Phi^{n_k} \varphi \to v \quad \text{and} \quad D\Phi^{n_k} \varphi (\Phi^{n_k} \varphi)^{-1} \to Dv \ v^{-1}$$

uniformly. Then

$$Dv(x)(v(x))^{-1} = \begin{cases} \alpha A & \text{for} \quad x \in [-\alpha^2, 0] \\ A & \text{for} \quad x \in [0, \alpha), \end{cases}$$

where  $A = 1/(\alpha + \alpha^3) H \in \mathfrak{su}(2)$ . Therefore

$$v(x) = \begin{cases} e^{\alpha x A} B & \text{for} \quad x \in [-\alpha^2, 0) \\ e^{x A} C & \text{for} \quad x \in [0, \alpha), \end{cases}$$

where  $B = v_{-}(0)$  and  $C = v_{+}(0)$ . Since the set  $W_0^1 \subset W^1$  is closed and  $\Phi$ -invariant,  $v \in W_0^1$ . It follows that

(12) 
$$Ce^{-\alpha^3 A}B = Be^{\alpha A}C.$$

Since v is a limit point of the sequence  $\{\Phi^n \varphi\}_{n \in \mathbb{N}}$ , we have  $\|D\Phi^k v(\Phi^k v)^{-1}\|_{L^1} = \|Dvv^{-1}\|_{L^1}$ for any natural k. By Proposition 7.1,

$$\lim_{x \to 0^{-}} Dv(x)(v(x))^{-1} = \alpha \operatorname{Ad}_{v_{-}(0)} \lim_{x \to \alpha^{-}} Dv(x)(v(x))^{-1}.$$

Hence

$$\alpha A = \alpha \operatorname{Ad}_B(A)$$

and finally AB = BA. Therefore

$$\Phi v(x) = \begin{cases} e^{-\alpha xA}C & \text{for} \quad x \in [-\alpha^2, 0] \\ e^{-xA + \alpha A}BC & \text{for} \quad x \in [0, \alpha). \end{cases}$$

By Proposition 7.1,

$$\lim_{x \to 0^{-}} D\Phi v(x) (\Phi v(x))^{-1} = \alpha \operatorname{Ad}_{\Phi v_{-}(0)} \lim_{x \to \alpha^{-}} D\Phi v(x) (\Phi v(x))^{-1}.$$

Hence

$$-\alpha A = \alpha \operatorname{Ad}_C(-A)$$

and finally AC = CA. It follows that B and C commute, by (12). From (12), we obtain  $e^{(\alpha+\alpha^3)A} = \text{Id.}$  Therefore  $||H|| = ||(\alpha+\alpha^3)A|| \in 2\pi\mathbb{N}_0$ .

**Theorem 7.3** Suppose that  $\alpha$  is the golden ratio. Then for every  $C^2$ -cocycle  $\varphi : \mathbb{T} \to SU(2)$ , we have  $d(\varphi) \in 2\pi \mathbb{N}_0$ .

**Proof.** Fix  $n \in \mathbb{N}$  such that  $2\alpha^{2n}[1/2\alpha^{2n}] \geq 4/5$ . Set  $I_j = [2(j-1)\alpha^{2n}, 2j\alpha^{2n}]$  for  $j \in E = \{1, \ldots, [1/2\alpha^{2n}]\}$  and  $\varepsilon_n = \frac{1}{q_n^2} \int_{\mathbb{T}} |D(D\varphi^{(q_n)}(\varphi^{(q_n)})^{-1})| d\lambda$ . By Proposition 6.6,  $\varepsilon_n$  tends to zero. For i = 1, 2 define

$$E_{i} = \{ j \in E; \ \frac{1}{2\alpha^{2n}q_{2n+i}^{2}} \int_{I_{j}} |D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})| d\lambda \le 10\varepsilon_{2n+i} \}.$$

Then

$$\varepsilon_{2n+i} = \frac{1}{q_{2n+i}^2} \int_{\mathbb{T}} |D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})| d\lambda \\
\geq \frac{1}{q_{2n+i}^2} \sum_{j \in E \setminus E_i} \int_{I_j} |D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})| d\lambda \\
\geq 20\alpha^{2n} \varepsilon_{2n+i}([1/2\alpha^{2n}] - \#E_i).$$

Hence

$$\#E_i \ge [1/2\alpha^{2n}](1 - \frac{1}{10}\frac{1/2\alpha^{2n}}{[1/2\alpha^{2n}]}) \ge \frac{7}{8}[1/2\alpha^{2n}]$$

for i = 1, 2. Therefore

$$#(E_1 \cap E_2) \ge #E_1 + #E_2 - #E \ge \frac{3}{4}[1/2\alpha^{2n}].$$

Define

$$G_n = \bigcup_{j \in E_1 \cap E_2} [(2j-2)\alpha^{2n}, (2j-1)\alpha^{2n}].$$

Observe that  $y \in G_n$  implies

$$\frac{1}{2\alpha^{2n}q_{2n+i}^2} \int_y^{y+\alpha^{2n}} |D(D\varphi^{(q_{2n+i})}(\varphi^{(q_{2n+i})})^{-1})| d\lambda \le 10\varepsilon_{2n+i}$$

for i = 1, 2 and

$$\lambda(G_n) \ge \alpha^{2n} \# (E_1 \cap E_2) \ge \frac{3}{8} 2\alpha^{2n} [1/2\alpha^{2n}] \ge \frac{3}{10}.$$

Set  $G' = \bigcap \bigcup G_k$ . Then  $\lambda(G') \ge 3/10$ . Since  $\frac{1}{n} D \varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi$  almost everywhere,  $n \in \mathbb{N} k > n$ 

we see that the set

$$G = \{ x \in G'; \frac{1}{n} D\varphi^{(n)}(x) (\varphi^{(n)}(x))^{-1} \to \psi(x) \}$$

has positive measure.

For every  $y \in \mathbb{T}$  denote by  $\varphi_y : \mathbb{T} \to SU(2)$  the  $C^2$ -cocycle  $\varphi_y(x) = \varphi(x+y)$ . Suppose that  $y \in G$ . Then  $\frac{1}{n}D\varphi_y^{(n)}(0)(\varphi_y^{(n)}(0))^{-1} \to \psi(y)$  and there exists an increasing sequence  $\{n_k\}_{k\in\mathbb{N}}$  of natural numbers such that  $y \in G_{n_k}$  for any natural k. Hence

$$\alpha^{2n_k} \int_0^{\alpha^{2n_k}} |D(D\varphi_y^{(q_{2n_k+i})}(\varphi_y^{(q_{2n_k+i})})^{-1})| d\lambda \le 20(\alpha^{2n_k}q_{2n_k+i})^2 \varepsilon_{2n_k+i}$$

for i = 1, 2. Since the sequence  $\{\alpha^n q_{n+i}\}_{n \in \mathbb{N}}$  converges for i = 1, 2 and  $\varepsilon_n$  tends to zero, letting  $k \to \infty$  we have

$$\lim_{k \to \infty} \alpha^{2n_k} \int_0^{\alpha^{2n_k}} |D(D\varphi_y^{(q_{2n_k+i})}(\varphi_y^{(q_{2n_k+i})})^{-1})| d\lambda = 0$$

for i = 1, 2. By Lemma 7.2,  $\|\psi(y)\| \in 2\pi \mathbb{N}_0$  for every  $y \in G$ . Since  $d(\varphi) = \|\psi(y)\|$  for a.e.  $y \in \mathbb{T}$ , we conclude that  $d(\varphi) \in 2\pi \mathbb{N}_0$ .

### 8 2-dimensional case

In this section we will be concerned with properties of smooth cocycles over ergodic rotations on the 2-dimensional torus with values in SU(2). By  $\mathbb{T}^2$  we will mean the group  $\mathbb{R}^2/\mathbb{Z}^2$ . We will identify functions on  $\mathbb{T}^2$  with periodic of period 1 in each coordinates functions on  $\mathbb{R}^2$ . Suppose that  $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$  is an ergodic rotation on  $\mathbb{T}^2$ . Let  $\varphi : \mathbb{T}^2 \to SU(2)$  be a  $C^1$ -cocycle over the rotation T. Analysis similar to that in Section 2 shows that there exists  $\psi_i \in L^2(\mathbb{T}^2, \mathfrak{su}(2)), i = 1, 2$  such that

$$\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1} \to \psi_i \quad \text{in} \quad L^2(\mathbb{T}^2,\mathfrak{su}(2)).$$

Moreover,  $\|\psi_i\|$  is a  $\lambda \otimes \lambda$ -a.e. constant function and  $\varphi(\bar{x})\psi_i(T\bar{x})\varphi(\bar{x})^{-1} = \psi_i(\bar{x})$  for  $\lambda \otimes \lambda$ -a.e.  $\bar{x} \in \mathbb{T} \times \mathbb{T}$  for i = 1, 2.

**Definition 2** The pair

$$(\|\psi_1\|, \|\psi_2\|) = \lim_{n \to \infty} \frac{1}{n} (\|\frac{\partial}{\partial x_1} \varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1}, \|\frac{\partial}{\partial x_2} \varphi^{(n)}(\varphi^{(n)})^{-1}\|_{L^1})$$

will be called the *degree* of the cocycle  $\varphi : \mathbb{T}^2 \to SU(2)$  and denoted by  $d(\varphi)$ .

Similarly, one can prove the following

**Theorem 8.1** If  $d(\varphi) \neq 0$ , then  $\varphi$  is cohomologous to a diagonal cocycle  $\mathbb{T}^2 \ni \bar{x} \mapsto \begin{bmatrix} \gamma(\bar{x}) & 0\\ 0 & \overline{\gamma(\bar{x})} \end{bmatrix} \in SU(2)$ , where  $\gamma : \mathbb{T}^2 \to \mathbb{T}$  is measurable. Moreover, the skew product  $T_{\gamma} : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2 \times \mathbb{T}$  is ergodic and it is mixing on the orthocomplement of the space of functions depending only on the first two variables.

Analysis similar to that in the proof of Theorem 6.1 gives

**Theorem 8.2** Let  $\varphi : \mathbb{T}^2 \to SU(2)$  be a  $C^2$ -cocycle with  $d(\varphi) \neq 0$ . Suppose that the sequence  $\{\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$  is uniformly convergent and  $\{\frac{\partial}{\partial x_i}(\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$  is bounded in  $L^2(\mathbb{T}^2, \mathfrak{su}(2))$  for i = 1, 2. Then the Lebesgue component in the spectrum of  $T_{\varphi}$  has countable multiplicity.

By  $BV^{\mathcal{R}}(\mathbb{T}^2, SU(2))$  we mean the set of all measurable functions  $f: \mathbb{T}^2 \to SU(2)$  such that

- the functions  $f(x, \cdot), f(\cdot, x) : \mathbb{T} \to SU(2)$  are of bounded variation for any  $x \in \mathbb{T}$ ;
- the functions  $\frac{\partial}{\partial x_1} f(f)^{-1}$ ,  $\frac{\partial}{\partial x_2} f(f)^{-1} : \mathbb{T}^2 \to \mathfrak{su}(2)$  are Riemann integrable for i = 1, 2.

Then we immediately get the following

**Lemma 8.3** Let  $\varphi : \mathbb{T}^2 \to SU(2)$  be a  $C^2$ -cocycle. Suppose that  $\varphi$  is cohomologous to a diagonal cocycle with a transfer function in  $BV^{\mathcal{R}}(\mathbb{T}^2, SU(2))$ . Then the sequence  $\{\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1}\}_{n\in\mathbb{N}}$  is uniformly convergent and  $\{\frac{\partial}{\partial x_i}(\frac{1}{n}\frac{\partial}{\partial x_i}\varphi^{(n)}(\varphi^{(n)})^{-1})\}_{n\in\mathbb{N}}$  is uniformly bounded for i = 1, 2.

It is easy to check that if  $\varphi$  is cohomologous to a diagonal cocycle via a  $C^1$  transfer function, then  $d(\varphi) \in 2\pi(\mathbb{N}_0 \times \mathbb{N}_0)$ . However, in the next section we show that for every ergodic rotation  $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$  there exists a smooth cocycle whose degree is equal to  $2\pi(|\beta|, |\alpha|)$ .

# 9 Cocycles over flows

Let  $\omega$  be an irrational number. By  $S: \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}$  we mean the ergodic flow defined by

(13)  $S_t(x_1, x_2) = (x_1 + t\omega, x_2 + t).$ 

Let  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$  be a smooth cocycle over S, i.e.

$$\varphi_{t+s}(\bar{x}) = \varphi_t(\bar{x})\varphi_s(S_t\bar{x})$$

for all  $t, s \in \mathbb{R}, \bar{x} \in \mathbb{T}^2$  or equivalently,  $\varphi$  is the fundamental matrix solution for a linear differential system

$$\frac{d}{dt}y(t) = y(t)A(S_t\bar{x})$$

where  $A: \mathbb{T}^2 \to \mathfrak{su}(2)$ , i.e.  $\varphi$  satisfies

$$\begin{cases} \frac{d}{dt}\varphi_t(\bar{x}) &= \varphi_t(\bar{x})A(S_t\bar{x}) \\ \varphi_0(\bar{x}) &= \text{Id.} \end{cases}$$

Then

$$\frac{\partial}{\partial x_i}\varphi_{t+s}(\bar{x})\varphi_{t+s}(\bar{x})^{-1} = \frac{\partial}{\partial x_i}\varphi_t(\bar{x})\varphi_t(\bar{x})^{-1} + \operatorname{Ad}_{\varphi_t(\bar{x})}\frac{\partial}{\partial x_i}\varphi_s(S_t\bar{x})\varphi_s(S_t\bar{x})^{-1}.$$

Hence

$$\left\|\frac{\partial}{\partial x_i}\varphi_{t+s}(\varphi_{t+s})^{-1}\right\|_{L^1} \le \left\|\frac{\partial}{\partial x_i}\varphi_t(\varphi_t)^{-1}\right\|_{L^1} + \left\|\frac{\partial}{\partial x_i}\varphi_s(\varphi_s)^{-1}\right\|_{L^1}$$

It follows that the limit

$$\lim_{t \to \infty} \frac{1}{|t|} \|\frac{\partial}{\partial x_i} \varphi_t(\varphi_t)^{-1}\|_{L^1}$$

exists for i = 1, 2.

**Definition 3** The pair

$$\lim_{t \to \infty} \frac{1}{|t|} \left( \left\| \frac{\partial}{\partial x_1} \varphi_t(\varphi_t)^{-1} \right\|_{L^1}, \left\| \frac{\partial}{\partial x_2} \varphi_t(\varphi_t)^{-1} \right\|_{L^1} \right)$$

will be called the *degree* of the cocycle  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$  and denoted by  $d(\varphi)$ .

For a given cocycle  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$  over the flow S, by  $\widehat{\varphi} : \mathbb{T} \to SU(2)$  we will mean the cocycle over the rotation  $Tx = x + \omega$  defined by  $\widehat{\varphi}(x) = \varphi_1(x, 0)$ . Then  $\widehat{\varphi}^{(n)}(x) = \varphi_n(x, 0)$ .

**Lemma 9.1**  $d(\varphi) = (1, |\omega|)d(\widehat{\varphi}).$ 

**Proof.** First observe that

 $\varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_n(x_1, x_2) = \varphi_{n+x_2}(x_1 - x_2\omega, 0) = \varphi_n(x_1 - x_2\omega, 0)\varphi_{x_2}(x_1 - x_2\omega + n\omega, 0).$ 

Hence

$$\varphi_n(x_1, x_2) = \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1} \widehat{\varphi}^{(n)}(x_1 - x_2\omega) \varphi_{x_2}(x_1 - x_2\omega + n\omega, 0)$$

for all  $x_1, x_2 \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Fix  $(x_1, x_2) \in [0, 1] \times [0, 1]$ . Then

$$\frac{\partial}{\partial x_{1}}\varphi_{n}(x_{1},x_{2})\varphi_{n}(x_{1},x_{2})^{-1} = -\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)^{-1}\frac{\partial}{\partial x_{1}}\varphi_{x_{2}}(x_{1}-x_{2}\omega,0) 
+ \operatorname{Ad}_{\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)^{-1}}(D\widehat{\varphi}^{(n)}(x_{1}-x_{2}\omega)\widehat{\varphi}^{(n)}(x_{1}-x_{2}\omega)^{-1}) 
+ \operatorname{Ad}_{\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)^{-1}\widehat{\varphi}^{(n)}(x_{1}-x_{2}\omega)}(\frac{\partial}{\partial x_{1}}\varphi_{x_{2}}(x_{1}-x_{2}\omega+n\omega,0)\varphi_{x_{2}}(x_{1}-x_{2}\omega+n\omega,0)^{-1}).$$

It follows that

$$\begin{aligned} &|||\frac{\partial}{\partial x_{1}}\varphi_{n}(\varphi_{n})^{-1}||_{L^{1}} - ||D\widehat{\varphi}^{(n)}(\widehat{\varphi}^{(n)})^{-1}||_{L^{1}}| \\ &= |||\frac{\partial}{\partial x_{1}}\varphi_{n}(\varphi_{n})^{-1}||_{L^{1}} - \int_{0}^{1}\int_{0}^{1}||D\widehat{\varphi}^{(n)}(x_{1} - x_{2}\omega)\widehat{\varphi}^{(n)}(x_{1} - x_{2}\omega)^{-1}||dx_{1}dx_{2}| \\ &\leq 2\int_{0}^{1}\int_{0}^{1}||\frac{\partial}{\partial x_{1}}\varphi_{x_{2}}(x_{1} - x_{2}\omega, 0)\varphi_{x_{2}}(x_{1} - x_{2}\omega, 0)^{-1}||dx_{1}dx_{2}|. \end{aligned}$$

Therefore

$$\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_1} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} = \lim_{n \to \infty} \frac{1}{n} \left\| D\widehat{\varphi}^{(n)}(\widehat{\varphi}^{(n)})^{-1} \right\|_{L^1} = d(\widehat{\varphi}).$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial x_2} \varphi_n(x_1, x_2) \varphi_n(x_1, x_2)^{-1} &= -\varphi_{x_2} (x_1 - x_2 \omega, 0)^{-1} \frac{\partial}{\partial t} \varphi_{x_2} (x_1 - x_2 \omega, 0) \\ &+ \omega \, \varphi_{x_2} (x_1 - x_2 \omega, 0)^{-1} \frac{\partial}{\partial x_1} \varphi_{x_2} (x_1 - x_2 \omega, 0) \\ &- \omega \, \mathrm{Ad}_{\varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1}} (D \widehat{\varphi}^{(n)} (x_1 - x_2 \omega) \widehat{\varphi}^{(n)} (x_1 - x_2 \omega)^{-1}) \\ &+ \, \mathrm{Ad}_{\varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1} \widehat{\varphi}^{(n)} (x_1 - x_2 \omega)} (\frac{\partial}{\partial t} \varphi_{x_2} (x_1 - x_2 \omega + n \omega, 0) \varphi_{x_2} (x_1 - x_2 \omega + n \omega, 0)^{-1}) \\ &- \omega \, \mathrm{Ad}_{\varphi_{x_2}(x_1 - x_2 \omega, 0)^{-1} \widehat{\varphi}^{(n)} (x_1 - x_2 \omega)} (\frac{\partial}{\partial x_1} \varphi_{x_2} (x_1 - x_2 \omega + n \omega, 0) \varphi_{x_2} (x_1 - x_2 \omega + n \omega, 0)^{-1}). \end{aligned}$$

It follows that

$$\begin{aligned} \|\frac{\partial}{\partial x_{2}}\varphi_{n}(\varphi_{n})^{-1}\|_{L^{1}} &- |\omega| \|D\widehat{\varphi}^{(n)}(\widehat{\varphi}^{(n)})^{-1}\|_{L^{1}} |\\ &\leq 2\int_{0}^{1}\int_{0}^{1} \|\frac{\partial}{\partial t}\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)^{-1}\|dx_{1}dx_{2} \\ &+ 2|\omega|\int_{0}^{1}\int_{0}^{1} \|\frac{\partial}{\partial x_{1}}\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)\varphi_{x_{2}}(x_{1}-x_{2}\omega,0)^{-1}\|dx_{1}dx_{2} \end{aligned}$$

Therefore

$$\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_2} \varphi_n(\varphi_n)^{-1} \right\|_{L^1} = |\omega| \lim_{n \to \infty} \frac{1}{n} \left\| D\widehat{\varphi}^{(n)}(\widehat{\varphi}^{(n)})^{-1} \right\|_{L^1} = |\omega| d(\widehat{\varphi}),$$

and the proof is complete.  $\blacksquare$ 

**Lemma 9.2** For any  $C^2$ -cocycle  $\psi : \mathbb{T} \to SU(2)$  over the rotation T there exists a  $C^2$ -cocycle  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$  over the flow S such that  $\widehat{\varphi} = \psi$ .

**Proof.** Since the fundamental group of SU(2) is trivial, we can choose a  $C^2$ -homotopy  $\psi: [0,1] \times \mathbb{T} \to SU(2)$  such that

$$\psi(t,x) = \begin{cases} \text{Id for } t \in [0,1/4] \\ \psi(x) \text{ for } t \in [3/4,1]. \end{cases}$$

By  $\psi : \mathbb{R} \times \mathbb{T} \to SU(2)$  we mean the C<sup>2</sup>-function determined by

$$\psi(n+t,x) = \psi^{(n)}(x)\psi(t,x+n\omega)$$

for any  $t \in [0, 1]$  and  $n \in \mathbb{Z}$ . Then it is easy to check that

(14) 
$$\psi(n+t,x) = \psi^{(n)}(x)\psi(t,x+n\omega)$$

for any  $t \in \mathbb{R}$  and  $n \in \mathbb{Z}$ . Let  $\varphi : \mathbb{R} \times \mathbb{R}^2 \to SU(2)$  be defined by

$$\varphi_t(x_1, x_2) = \psi(x_2, x_1 - x_2\omega)^{-1} \psi(t + x_2, x_1 - x_2\omega).$$

It is easy to see that  $\varphi_t(x_1 + 1, x_2) = \varphi_t(x_1, x_2)$  and  $\varphi_t(x_1, x_2 + 1) = \varphi_t(x_1, x_2)$ , by (14). Then  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$  is a  $C^2$ -function and

Moreover,

$$\widehat{\varphi}(x) = \varphi_1(x,0) = \psi(0,x)^{-1}\psi(1,x) = \psi(x),$$

which completes the proof.  $\blacksquare$ 

Suppose that  $\alpha, \beta, 1$  are independent over  $\mathbb{Q}$ . Set  $\omega = \alpha/\beta$ .

**Theorem 9.3** For every ergodic rotation  $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$  and for every natural k there exists a  $C^2$ -cocycle over T whose degree is equal to  $2\pi k(|\beta|, |\alpha|)$ .

**Proof.** Let S denote the ergodic flow given by (13). Suppose that  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$ is a  $C^2$ -cocycle over S such that  $d(\widehat{\varphi}) = 2\pi k$ . Consider the cocycle  $\varphi_{\beta} : \mathbb{T} \to SU(2)$  over the rotation  $T = S_{\beta}$ . Then  $\varphi_{\beta}^{(n)} = \varphi_{\beta n}$  and

$$\lim_{n \to \infty} \frac{1}{n} \left\| \frac{\partial}{\partial x_i} \varphi_{\beta}^{(n)} (\varphi_{\beta}^{(n)})^{-1} \right\|_{L_1} = |\beta| \lim_{n \to \infty} \frac{1}{|\beta| n} \left\| \frac{\partial}{\partial x_i} \varphi_{\beta n} (\varphi_{\beta n})^{-1} \right\|_{L_1}.$$

It follows that

$$d(\varphi_{\beta}) = |\beta| d(\varphi) = |\beta| (1, |\omega|) d(\widehat{\varphi}) = (|\beta|, |\alpha|) d(\widehat{\varphi}),$$

which proves the theorem.  $\blacksquare$ 

Suppose that  $\beta \in (0,1)$ . Let  $\varphi : \mathbb{R} \times \mathbb{T}^2 \to SU(2)$  be a  $C^2$ -cocycle over S such that  $\widehat{\varphi}$  is a diagonal  $C^2$ -cocycle with nonzero degree. Set  $T = S_\beta$  and  $\psi = \varphi_\beta$ . Let  $p : \mathbb{T}^2 \to SU(2)$ be a  $BV^{\mathcal{R}}$ -function such that

$$p(x_1, x_2) = \varphi_{x_2}(x_1 - x_2\omega, 0)^{-1}$$

for  $(x_1, x_2) \in \mathbb{R} \times [0, 1)$ . Then

$$p(T(x_1, x_2)) = \begin{cases} \varphi_{x_2+\beta}(x_1 - x_2\omega, 0)^{-1} & \text{for } x_2 \in [0, 1-\beta) \\ \varphi_{x_2+\beta-1}(x_1 - (x_2-1)\omega, 0)^{-1} & \text{for } x_2 \in [1-\beta, 1). \end{cases}$$

Moreover,

$$\varphi_{x_2+\beta}(x_1 - x_2\omega, 0) = \varphi_{x_2}(x_1 - x_2\omega, 0)\varphi_{\beta}(x_1, x_2)$$

and

$$\begin{aligned} \varphi_{x_2+\beta-1}(x_1 - (x_2 - 1)\omega, 0) &= \varphi_{-1}(x_1 - (x_2 - 1)\omega, 0)\varphi_{x_2+\beta}(x_1 - x_2\omega, 0) \\ &= \varphi_1(x_1 - x_2\omega, 0)^{-1}\varphi_{x_2+\beta}(x_1 - x_2\omega, 0). \end{aligned}$$

It follows that  $p(\bar{x})\delta(\bar{x})p(T\bar{x})^{-1} = \psi(\bar{x})$ , where  $\delta : \mathbb{T}^2 \to SU(2)$  is the diagonal  $BV^{\mathcal{R}}_{-}$  cocycle given by

$$\delta(x_1, x_2) = \begin{cases} \text{Id for } x_2 \in [0, 1 - \beta) \\ \widehat{\varphi}(x_1 - x_2 \omega) \text{ for } x_2 \in [1 - \beta, 1). \end{cases}$$

**Lemma 9.4** Let  $\phi : \mathbb{T}^2 \to \mathbb{T}$  be a cocycle over the rotation  $T(x_1, x_2) = (x_1 + \alpha, x_2 + \beta)$ . Suppose that  $\phi | \mathbb{T} \times [0, \gamma), \phi | \mathbb{T} \times [\gamma, 1)$  are  $C^1$ -functions, where  $\gamma$  is irrational. If  $d(\phi(\cdot, 0)) \neq d(\phi(\cdot, \gamma))$ , then  $\phi$  is not a coboundary. **Proof.** Set  $I_1 = [0, \gamma), I_2 = [\gamma, 1), a_1 = d(\phi(\cdot, 0))$  and  $a_2 = d(\phi(\cdot, \gamma))$ . Then there exists a function  $\tilde{\phi} : \mathbb{T}^2 \to \mathbb{R}$  such that  $\tilde{\phi}|\mathbb{T} \times I_j$  is of class  $C^1$  for j = 1, 2 and  $\phi(x_1, x_2) = \exp 2\pi i (\tilde{\phi}(x_1, x_2) + a_j x_1)$  for any  $(x_1, x_2) \in \mathbb{T} \times I_j$ .

Clearly, it suffices to show that

$$\int_{\mathbb{T}^2} \phi^{(n)}(x_1, x_2) dx_1 dx_2 \to 0$$

Next note that

$$\phi^{(n)}(x_1, x_2) = \exp 2\pi i (\widetilde{\phi}^{(n)}(x_1, x_2) + (a_1 S_1^n(x_2) + a_2 S_2^n(x_2)) x_1 + c_n(x_2)),$$

where  $S_i^n(x) = \sum_{k=0}^{n-1} \mathbf{1}_{I_i}(x+k\beta)$  and  $c_n(x) = \sum_{k=0}^{n-1} k\alpha(a_1\mathbf{1}_{I_1}+a_2\mathbf{1}_{I_2})(x+k\beta)$ . Since the rotation by  $\beta$  is uniquely ergodic,

$$\frac{1}{n}(a_1S_1^n + a_2S_2^n) \to a_1\gamma + a_2(1-\gamma)$$

uniformly. Since  $a_1 \neq a_2$  and  $\gamma$  is irrational, there exists S > 0 and  $n_0 \in \mathbb{N}$  such that  $|a_1S_1^n(x) + a_2S_2^n(x)| \geq nS$  for all  $x \in \mathbb{T}$  and  $n \geq n_0$ . Applying integration by parts, we get

$$\begin{split} &|\int_{\mathbb{T}^{2}} \phi^{(n)}(x_{1}, x_{2}) dx_{1} dx_{2}| \\ &\leq \int_{0}^{1} |\int_{0}^{1} e^{2\pi i \widetilde{\phi}^{(n)}(x_{1}, x_{2}) + (a_{1}S_{1}^{n}(x_{2}) + a_{2}S_{2}^{n}(x_{2}))x_{1}} dx_{1}| dx_{2} \\ &= \int_{0}^{1} \frac{1}{2\pi |a_{1}S_{1}^{n}(x_{2}) + a_{2}S_{2}^{n}(x_{2})|} |\int_{0}^{1} e^{2\pi i \widetilde{\phi}^{(n)}(x_{1}, x_{2})} de^{2\pi i (a_{1}S_{1}^{n}(x_{2}) + a_{2}S_{2}^{n}(x_{2}))x_{1}} |dx_{2} \\ &= \int_{0}^{1} \frac{1}{2\pi |a_{1}S_{1}^{n}(x_{2}) + a_{2}S_{2}^{n}(x_{2})|} |\int_{0}^{1} e^{2\pi i (a_{1}S_{1}^{n}(x_{2}) + a_{2}S_{2}^{n}(x_{2}))x_{1}} de^{2\pi i \widetilde{\phi}^{(n)}(x_{1}, x_{2})} |dx_{2} \\ &\leq \int_{0}^{1} \frac{1}{nS} |\int_{0}^{1} e^{2\pi i \widetilde{\phi}^{(n)}(x_{1}, x_{2}) + (a_{1}S_{1}^{n}(x_{2}) + a_{2}S_{2}^{n}(x_{2}))x_{1}} \frac{\partial}{\partial x_{1}} \widetilde{\phi}^{(n)}(x_{1}, x_{2}) dx_{1} |dx_{2} \\ &\leq \frac{1}{nS} \int_{\mathbb{T}^{2}} |\frac{\partial}{\partial x_{1}} \widetilde{\phi}^{(n)}(x_{1}, x_{2})| dx_{1} dx_{2}. \end{split}$$

Since  $\frac{\partial}{\partial x_1} \widetilde{\phi} \in L^1(\mathbb{T}^2, \mathbb{C}),$ 

$$\frac{1}{n}\frac{\partial}{\partial x_1}\widetilde{\phi}^{(n)} \to \int_{\mathbb{T}^2} \frac{\partial}{\partial x_1}\widetilde{\phi}(x_1, x_2)dx_1dx_2 = 0$$

in  $L^1(\mathbb{T}^2, \mathbb{C})$ , by the Birkhoff ergodic theorem, and the proof in complete.

This leads to the following conclusion.

**Corollary 9.5** For every ergodic rotation T on  $\mathbb{T}^2$  there exists a  $C^2$ -cocycle  $\psi$  with nonzero degree such that the Lebesgue component in the spectrum of  $T_{\psi}$  has countable multiplicity and  $\psi$  is not cohomologous to any diagonal  $C^1$ -cocycle.

**Proof.** Let  $\check{\varphi} : \mathbb{T} \to \mathbb{T}$  be a  $C^2$ -function with nonzero topological degree. Let  $\varphi$  :  $\mathbb{R} \times \mathbb{T}^2 \to SU(2)$  be a  $C^2$ -cocycle over S such that  $\widehat{\varphi} = \begin{bmatrix} \check{\varphi} & 0 \\ 0 & (\check{\varphi})^{-1} \end{bmatrix}$ . Define  $\psi = \varphi_{\beta}$ . Then  $d(\psi) = 2\pi(|\beta|, |\alpha|)|d(\check{\varphi})| \neq 0$ . Moreover,  $\psi$  and the diagonal cocycle  $\delta : \mathbb{T}^2 \to SU(2)$  given by

$$\delta(x_1, x_2) = \begin{cases} \text{Id for } x_2 \in [0, 1 - \beta) \\ \widehat{\varphi}(x_1 - x_2\omega) & \text{for } x_2 \in [1 - \beta, 1) \end{cases}$$

are cohomologous with a transfer function in  $BV^{\mathcal{R}}(\mathbb{T}^2, SU(2))$ . Applying Theorem 8.2 and Lemma 8.3, we get the first part of our claim.

Next suppose that  $\psi$  is cohomologous to a diagonal  $C^1$ -cocycle. Then it is easy to see that the cocycle  $\eta : \mathbb{T}^2 \to \mathbb{T}$  given by

$$\eta(x_1, x_2) = \begin{cases} \text{Id for } x_2 \in [0, 1 - \beta) \\ \check{\varphi}(x_1 - x_2\omega) \text{ for } x_2 \in [1 - \beta, 1). \end{cases}$$

is cohomologous to a  $C^1$ -cocycle  $g : \mathbb{T}^2 \to \mathbb{T}$ . Applying Lemma 9.4 for  $\phi = \eta g^{-1}$  and  $\gamma = 1 - \beta$  we find that  $\eta g^{-1}$  is not a coboundary, which completes the proof.

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