HAMILTONIAN GROMOV-WITTEN INVARIANTS

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ABSTRACT. In this paper we introduce invariants of semi-free Hamiltonian actions of S^1 on compact symplectic manifolds (which satisfy some technical conditions related to positivity) using the space of solutions to certain gauge theoretical equations. These equations generalize at the same time the vortex equations and the holomorphicity equation used in Gromov–Witten theory. In the definition of the invariants we combine ideas coming from gauge theory and the ideas underlying the construction of Gromov–Witten invariants.

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1. INTRODUCTION

Gromov–Witten invariants are among the most important and useful tools in symplectic topology. They can be used for example to prove the existence of two symplectic structures on a compact smooth manifold which are not deformation equivalent (see [Ru1]). Other applications appear in the study of symplectic fibrations (see [McD2]), the topology of the group of Hamiltonian symplectomorphisms (see [Se]), and Weinstein's conjecture [LiuTi]. Finally, Gromov–Witten invariants play a prominent role in the celebrated mirror conjecture (see for example [Gi]).

It is a natural question whether Gromov–Witten invariants or some related construction can be used to study Hamiltonian actions of compact Lie groups on symplectic manifolds. The purpose of this paper is to make some steps towards an affirmative answer to this question. For any symplectic manifold with a Hamiltonian action of S^1 we introduce a set of equations whose moduli space of solutions, in a way very much similar to Gromov–Witten theory, allow to define invariants of the symplectic manifold together with the Hamiltonian action. These equations generalise on the one hand the

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holomorphicity equation used in Gromov–Witten theory and on the other hand the gauge theoretical vortex equations or their analogues as considered in full generality by Banfield [Ba]. Although we restrict ourselves to Hamiltonian actions of S^1 , the definition of the invariants can be given (at least heuristically) for any action of a compact group. We call these new invariants Hamiltonian Gromov–Witten invariants.

Let (F^{2n}, ω) be a compact symplectic manifold supporting a Hamiltonian action of S^1 with moment map $\mu : F \to \text{Lie}(S^1)^* = (\mathbf{i}\mathbb{R})^*$. Let $ES^1 \to BS^1$ be the universal principal S^1 -bundle. Recall that the Borel construction of F is

$$F_{S^1} = ES^1 \times_{S^1} F,$$

and that the equivariant homology (resp. cohomology) of F is by definition $H_*^{S^1}(F;\mathbb{Z}) := H_*(F_{S^1};\mathbb{Z})$ (resp. $H_{S^1}^*(F;\mathbb{Z}) := H^*(F_{S^1};\mathbb{Z})$). The Hamiltonian Gromov–Witten invariants depend on the choice of an element in $H_2^{S^1}(F;\mathbb{Z})$, a collection of elements in $H_{S^1}^*(F;\mathbb{Z})$, and an element of $\text{Lie}(S^1)$. It is an interesting question to relate these invariants to the equivariant Gromov–Witten invariants [GiKm, Lu] (note, however, that in the definition of equivariant Gromov–Witten invariants one does not use any element of $\text{Lie}(S^1)$, and that the action of S^1 does not need to be Hamiltonian for these invariants to be defined).

To define the Hamiltonian Gromov–Witten invariants we follow the ideas and techniques used in the definition of Gromov–Witten invariants in [McDS1]. We are forced to assume some technical conditions on the manifold F and on the action. However, we expect that using the techniques of virtual moduli cycles (as developed in the several papers which give a construction of Gromov–Witten invariants for general compact symplectic manifolds, see [FuOn, LiTi, Ru2, Si]) a definition of Hamiltonian Gromov–Witten invariants could be given in full generality.

This paper is essentially based on the Ph.D. Thesis of the author [Mu2], which was submitted in the Universidad Autónoma de Madrid in the spring of 1999. The author discovered the equations which are used to define the Hamiltonian Gromov– Witten invariants inspired by previous work on Hitchin–Kobayashi correspondence (see [Mu1]). In the summer of 1999 the author knew that K. Cieliebak, A.R. Gaio and D. Salamon had independently discovered the same equations and how to define the invariants (see [CiGaSa, Ga]). On the other hand, A. Bertram, G. Daskalopoulos and R. Wentworth studied in [BtDaWe] Gromov–Witten invariants of Grassmannians using ideas similar to the ones which we use.

This paper is organised as follows. In the rest of this section we introduce the equations and we give a heuristic definition of the invariants. In Section 2 we introduce Sobolev completions of our parameter space and we explain how to perturb the equations in order to get a smooth moduli space of solutions. We also prove a regularity result for Sobolev solutions of the equations. In Section 3 we define a compactification of the moduli space of solutions to the equation. This compactification generalises Gromov compactification of the moduli space of pseudo-holomorphic curves. In Section 4 we study the moduli of rational curves for a generic S^1 -invariant complex structure on F. Finally, in Section 5 we recall the basic definition of the theory of pseudo-cycles and we use them to give a rigorous definition of the invariants under the conditions specified in Subsection 1.8.

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1.1. The equations. Take a S^1 -invariant complex structure $I \in \text{End}(TF)$ such that $g(\cdot, \cdot) = \omega(\cdot, I \cdot)$ is a Riemannian metric on F. Such complex structures always exists, and they form a contractible space (see Lemma 5.49 in [McDS2]); this implies that the Chern classes of the complex bundle (TF, I) only depend on (the deformation class of) ω .

Let Σ be a compact connected Riemann surface, with a fixed Riemannian metric. Let $E \to \Sigma$ be a principal S^1 -bundle, and let $\pi : \mathcal{F} = E \times_{S^1} F \to \Sigma$ be the associated fibration with fibre F. Since the moment map is by definition S^1 -invariant, it extends to give a map $\mu : \mathcal{F} \to (\mathbf{i}\mathbb{R})^*$. Let \mathscr{A} be the space of connections on E, let $\mathscr{G} = \operatorname{Map}(\Sigma, S^1)$ be the gauge group of E, and let $\mathscr{S} = \Gamma(\mathcal{F})$ be the space of smooth sections of \mathcal{F} .

Any connection $A \in \mathscr{A}$ induces a projection $\alpha_A : T\mathcal{F} \to T\mathcal{F}_v = \operatorname{Ker} d\pi$. Using this map we define the covariant derivative with respect to A of a section $\phi \in \mathscr{S}$ to be

$$d_A \phi = \alpha_A \circ d\phi \in \Omega^1(\Sigma; \phi^* T \mathcal{F}_v).$$

On the other hand, since I is S^1 -invariant it can be extended to give a complex structure on $T\mathcal{F}_v$. Hence, we can split $d_A\phi$ as the sum of its holomorphic part $\partial_A\phi \in \Omega^{1,0}(\Sigma; \phi^*T\mathcal{F}_v)$ plus its antiholomorphic part $\overline{\partial}_A\phi \in \Omega^{0,1}(\Sigma; \phi^*T\mathcal{F}_v)$.

Let $\Lambda : \Omega^2(\Sigma) \to \Omega^0(\Sigma)$ denote the contraction with the volume form on Σ . We will henceforth identify $\operatorname{Lie}(S^1)^* = (\mathbf{i}\mathbb{R})^*$ with $\operatorname{Lie}(S^1) = (\mathbf{i}\mathbb{R})$ by assigning to $a \in \mathbf{i}\mathbb{R}$ the element $a^* : \mathbf{i}\mathbb{R} \to \mathbb{R}$ which maps $b \in \mathbf{i}\mathbb{R}$ to $a^*(b) = \langle a, b \rangle = -ab \in \mathbb{R}$. Let $c \in \mathbf{i}\mathbb{R}$. We will consider the following equations on $(A, \phi) \in \mathscr{A} \times \mathscr{S}$

$$\begin{cases} \overline{\partial}_A \phi = 0\\ \Lambda F_A + \mu(\phi) = c \end{cases}$$
(1)

where $F_A \in \Omega^2(\Sigma; \mathbf{i}\mathbb{R})$ denotes the curvature of A. Taking on $\mathscr{A} \times \mathscr{S}$ the diagonal action of the gauge group (acting by pullback both in \mathscr{A} and \mathscr{S}), the set of solutions to the equations is \mathscr{G} -invariant.

Definition 1.1. We call a tuple (Σ, E, A, ϕ, c) for which (1) is satisfied a twisted holomorphic curve, THC for short.

Example. When $F = \mathbb{C}$ and $S^1 \subset \mathbb{C}$ acts on it by multiplication, the equations (1) coincide with the abelian vortex equations (see for example [GP]). (Note, however, that in this paper we assume that F is compact.)

1.2. Relation with holomorphic curves in \mathcal{F} . Any connection $A \in \mathscr{A}$ gives rise to a splitting $T\mathcal{F} \simeq T\mathcal{F}_v \oplus \pi^*T\Sigma$ (which is given by taking as complementary to $T\mathcal{F}_v \subset T\mathcal{F}$ the kernel Ker α_A). Using this splitting we may combine the complex structure on $T\mathcal{F}_v$ (which was induced by the complex structure on F) with the complex structure of $T\Sigma$ to get a complex structre $I(A) \in \text{End}(T\mathcal{F})$. It is then straightforward to prove the following.

Lemma 1.2. A section $\phi \in \mathscr{S}$ is I(A) holomorphic as a map from Σ to \mathcal{F} if and only if $\overline{\partial}_A \phi = 0$.

1.3. Principal bundles and maps to Borel construction. In this subsection we will describe a construction which relates principal bundles on a topological space and sections of an associated bundle to maps into the Borel construction of the fibre of the associated bundle. We will use this construction twice in this paper: first, to fix the topology of the bundle $E \to \Sigma$ and of the section $\phi \in \Gamma(E \times_{S^1} F)$; and, later, to build an *equivariant evaluation map* which will play the role of the evaluation map in Gromov–Witten theory.

Let M be a CW-complex, let K be a compact connected Lie group, and let V be a CW-complex with a left action of K. Let $EK \to BK$ be the universal principal K-bundle¹. We denote by $V_K := EK \times_K V$ the Borel construction of V.

Lemma 1.3. Let $\mathscr{G}_P = \Gamma(P \times_{\operatorname{Ad}} K)$ be the gauge group of P. There is a canonical bijection between the set of homotopy classes of maps $[M, V_K]$ and the set of homotopy classes of pairs consisting of a principal K-bundle $P \to M$ and a gauge equivalence class $\mathscr{G}_P \sigma \subset \Gamma(P \times_K V)$ of sections of the associated bundle $P \times_K V$.

Proof. Take a pair (P, σ) , where $P \to M$ is a principal K-bundle and $\sigma \in \Gamma(P \times_K V)$. Let C be the set of K-equivariant continuous maps $P \to EK$. Note that any element of C gives a lift of the classifying map $M \to BK$ of P. Denote by EK^{op} the space EK with the action $\rho_L : K \times EK \to EK$ of K on the left given by

$$\rho_L(k,x) = \rho_R(x,k^{-1}),$$

where $\rho_R : EK \times K \to EK$ is the usual right action. There is a canonical bijection

$$C \simeq \Gamma(P \times_K EK^{\mathrm{op}}),$$

where Γ denotes the space of continuous sections. Since EK^{op} is contractible, we deduce that C is nonempty and contractible. Fix one such map $C \ni c : P \to EK$. If $g \in \mathscr{G}_P$, then $c \circ g$ is also K-equivariant, and it is homotopic (as K-equivariant maps) to c:

$$c \circ g \sim c.$$
 (2)

The section σ gives a K-equivariant map $\psi: P \to V$ (i.e. $\psi(pk) = k^{-1}\psi(p)$). Hence the map $(c, \psi): P \to EK \times V$ descends to give a map $\sigma_P: M \to V_K$. If $g \in \mathscr{G}_K$ then the section $g^*\sigma$ induces the K-equivariant map $\psi \circ g: P \to V$. Now, by (2)

$$(c, \psi \circ g) \sim (c \circ g^{-1}, \psi \circ g),$$

and hence $(g^*\sigma)_P \simeq \sigma_P$. Finally, it is clear that the homotopy class $[\sigma_P]$ only depends on the homotopy class of the section σ .

Conversely, if $f : M \to V_K$ is any continuous map, we let P_f be the pullback $f^*\pi_K^*EK$, where $\pi_K : V_K \to BK$ is the projection. Since there is a canonical isomorphism $P_f \times_K V \simeq f^*\pi_K^*V_K$, the map f induces a section $\sigma_f \in \Gamma(P_f \times_K V)$. This construction is the inverse of the preceeding one. In particular, we have

$$P \simeq \sigma_f^* \pi_K^* E K. \tag{3}$$

 $^{^{1}}$ We assume that the action of the structure group on any principal bundle is on the right.

Remark. Another way to look at this lemma is the following. Given a K principal bundle $P \to M$ there is, up to homotopy, a unique K-equivariant map $P \to EK$. Combining it with the identity map from V to V and quotienting by K we get a map $P \times_K V \to V_K$ which induces a canonical map in cohomology $c_P^* : H_K^*(V;\mathbb{Z}) \to$ $H^*(P \times_K V;\mathbb{Z})$. Furthermore, the image of c_P^* lies inside the fixed point set of the action of \mathscr{G}_P on $H^*(P \times_K V;\mathbb{Z})$. Then, if $\sigma \in \Gamma(P \times_K V)$, we have

$$\sigma_P^* = \sigma^* c_P^*$$

An important consequence of this equality is the following. If



is a Cartesian diagram of K principal bundles, and if we denote $s_V : P \times_K V \to P' \times_K V$ the map induced by s, then for any section $\sigma \in \Gamma(P \times_K V)$ we have

$$\sigma_P^* = (s_V \sigma)^* c_{P'}.\tag{4}$$

1.4. The moduli space. Similarly to what one does in Gromov–Witten theory, we will study the set of solutions to equations (1) for all pairs (E, ϕ) satisfying certain topological constraints. In Gromov–Witten theory the constraints are specified by the choice of an element of H_2 of the target space. In our case we will need to chose an element of $H_2^{S_1}(F;\mathbb{Z})$. We will apply the construction of Lemma 1.3 to S^1 principal bundles $E \to \Sigma$. Observe that if $\sigma \in \Gamma(E \times_{S^1} F)$, we may recover the degree of E using σ_E , since by (3)

$$\deg E = \langle c_1(E), [\Sigma] \rangle = \langle c_1(ES^1), \pi_{S^1*} \sigma_E \rangle$$

where c_1 denotes the first Chern class and $\pi_{S^1}: F_{S^1} \to BS^1$ is the projection.

So fix a homology class $\beta \in H_2^{S^1}(F;\mathbb{Z})$. Let $E \to \Sigma$ be the principal S^1 -bundle of degree $\langle c_1(ES^1), \pi_{S^1*}\beta \rangle$. Define as in Section 1.1 \mathscr{A} to be the set of connections on E and \mathscr{S} the set of smooth sections of $\mathcal{F} = E \times_{S^1} F$. Let

$$\widetilde{\mathcal{M}}(\beta, c) = \widetilde{\mathcal{M}}_{I}(\beta, c) = \{ (A, \phi) \in \mathscr{A} \times \mathscr{S} \mid \phi_{E_{*}}[\Sigma] = \beta \text{ and } (A, \phi) \text{ satisfies } (1) \},$$

where $\phi_E : \Sigma \to F_{S^1}$ is the map constructed in Lemma 1.3. By gauge equivariance of the equations and of the condition $\phi_{E_*}[\Sigma] = \beta$ (by Lemma 1.3, if $\phi_{E_*}[\Sigma] = \beta$ and $g \in \mathscr{G}$ then $(g^*\phi_E)_*[\Sigma] = \beta$), it makes sense to define

$$\mathcal{M}(\beta, c) = \mathcal{M}_I(\beta, c) = \widetilde{\mathcal{M}}_I(\beta, c) / \mathscr{G}.$$

We will only use the notations $\widetilde{\mathcal{M}}_{I}(\beta, c)$ and $\mathcal{M}_{I}(\beta, c)$ when the complex structure I is not clear from the context.

1.5. The choice of c. Assume that the action of S^1 on F is semi-free (i.e., S^1 acts freely on the complementary $F \setminus F^f$ of the fixed point sets). Let $F_1, \ldots, F_r \subset F$ be the connected components of the fixed point set F^f . It follows from the properties of the moment map that for any $1 \leq j \leq r$ the restriction $\mu|_{F_j}$ takes a constant value, say $c_j \in \mathbf{i}\mathbb{R}$. Let

$$\mathscr{C} = \{ c_j - 2\pi \mathbf{i} \deg(E) / \operatorname{Vol}(\Sigma) \mid 1 \le j \le r \}.$$

Lemma 1.4. If $c \in \mathbf{i}\mathbb{R} \setminus \mathscr{C}$, then the action of \mathscr{G} on $\widetilde{\mathcal{M}}(\beta, c)$ is free.

Proof. Take any $(A, \phi) \in \mathscr{A} \times \mathscr{S}$, and assume that there is a nontrivial gauge transformation which fixes (A, ϕ) . Since the stabiliser of any connection is the set of constant gauge transformations, we deduce that (A, ϕ) is fixed by a nontrivial constant gauge. Because the action of S^1 on F is semi-free, this implies that $\phi(\Sigma) \subset E \times_{S^1} F^f \subset \mathcal{F}$, and since Σ is connected, we deduce that

$$\phi(\Sigma) \subset E \times_{S^1} F_j \tag{5}$$

for some $1 \leq j \leq r$. Now assume that (A, ϕ) satisfies

$$\Lambda F_A + \mu(\phi) = c$$

for some $c \in \mathbf{i}\mathbb{R}$. Integrating this equality over Σ , dividing by $\operatorname{Vol}(\Sigma)$ and using Chern-Weil theory to write $\frac{\mathbf{i}}{2\pi} \int_{\Sigma} F_A = \deg(E)$, we deduce from (5) that

$$c = c_j - 2\pi \mathbf{i} \deg(E) / \operatorname{Vol}(\Sigma) \in \mathscr{C}.$$

1.6. The universal bundle. Let

$$\widetilde{\mathbb{E}} = \pi_{\Sigma}^* E = \mathscr{A} \times \mathscr{S} \times E \to \mathscr{A} \times \mathscr{S} \times \Sigma,$$

where $\pi_{\Sigma} : \mathscr{A} \times \mathscr{S} \times \Sigma \to \Sigma$ is the projection. Consider the section of the associated bundle

$$\widetilde{\Phi} : \mathscr{A} \times \mathscr{S} \times \Sigma \quad \to \quad \widetilde{\mathbb{E}} \times_{S^1} F = \mathscr{A} \times \mathscr{S} \times \mathcal{F} (A, \phi, x) \quad \mapsto \quad (A, \phi, \phi(x)).$$

Take on $\mathscr{A} \times \mathscr{S} \times \Sigma$ the diagonal action of \mathscr{G} , acting trivially on Σ . There is a canonical lift of this action to $\widetilde{\mathbb{E}}$ (resp. to $\widetilde{\mathbb{E}} \times_{S^1} F$) which is the diagonal action on $\mathscr{A} \times \mathscr{S} \times E$ (resp. $\mathscr{A} \times \mathscr{S} \times \mathscr{F}$), and the section $\widetilde{\Phi}$ is \mathscr{G} -equivariant.

Suppose that $c \in \mathbf{i}\mathbb{R} \setminus \mathscr{C}$. By Lemma 1.4 the action of \mathscr{G} on $\widetilde{\mathcal{M}}(\beta, c) \times \Sigma$ is free. This implies that the restriction of $\widetilde{\mathbb{E}}$ to $\widetilde{\mathcal{M}}(\beta, c) \times \Sigma$ descends to give a bundle

$$\mathbb{E} \to \mathcal{M}(\beta, c) \times \Sigma.$$

We call \mathbb{E} the universal bundle. By equivariance the section $\widetilde{\Phi}$ also descends to give a section Φ of the bundle $\mathbb{E} \times_{S^1} F \to \mathcal{M}(\beta, c) \times \Sigma$. Arguing like in Lemma 1.3 we obtain out of \mathbb{E} and Φ a canonical (homotopy class of) map

$$\operatorname{ev}_{\beta,c}: \mathcal{M}(\beta,c) \times \Sigma \to F_{S^1},$$

which we call the equivariant evaluation map.

1.7. The invariants. We now give a heuristic definition of the invariants. Suppose that $c \in \mathbf{i}\mathbb{R} \setminus \mathscr{C}$, so that by the preceeding section we have an equivariant evaluation map $\operatorname{ev}_{\beta,c} : \mathcal{M}(\beta,c) \times \Sigma \to F_{S^1}$. Suppose that there is a *canonical* fundamental class

$$[\mathcal{M}(\beta, c)] \in H_*(\mathcal{M}(\beta, c); \mathbb{Z}).$$

Let $\alpha_1, \ldots, \alpha_p \in H^*_{S^1}(F; \mathbb{Z})$ be equivariant cohomology classes and let $\gamma \in H^*(\mathscr{A}/\mathscr{G}; \mathbb{Z})$. Let $\nu : \mathcal{M} \to \mathscr{A}/\mathscr{G}$ be the map induced by the projection $\mathscr{A} \times \mathscr{S} \to \mathscr{A}$. We define the Hamiltonian Gromov–Witten invariant $\Psi_{\Sigma,\beta,c}$ evaluated at $\alpha_1, \ldots, \alpha_p$ and γ to be

$$\Psi_{\Sigma,\beta,c}(\alpha_1,\ldots,\alpha_p;\gamma) = \langle \operatorname{ev}_{\beta,c}^* \alpha_1 / [\Sigma] \cup \cdots \cup \operatorname{ev}_{\beta,c}^* \alpha_p / [\Sigma] \cup \nu^* \gamma, [\mathcal{M}(\beta,c)] \rangle \in \mathbb{Z}.$$
(6)

Of course, the existence of a canonical fundamental class is far from obvious, since in general $\mathcal{M}(\beta, c)$ is neither compact nor a smooth manifold. However, perturbing slightly the equations (1) we obtain a smooth set of gauge orbits of solutions, which can be compactified in a natural way. And although we will not construct any fundamental homology class of the compactification, in some situations we will manage to give a rigorous definition of the invariants sketched above using the technique of pseudocycles.

1.8. Assumptions on F and the action of S^1 . In order to give a rigorous definition of the invariants we will assume that F and the action of S^1 satisfy several technical conditions, which we now list.

- 1. The action of S^1 on F is semi-free;
- 2. the manifold F is monotone; this means that there is a real number $\lambda > 0$ such that $[\omega] = \lambda c_1(TF)$;
- 3. the connected components F_1, \ldots, F_r of the fixed point set F^f are also monotone, i.e., there exists positive real numbers $\lambda_1, \ldots, \lambda_r > 0$ such that for any $1 \le k \le r$

$$[\omega_k] = \lambda_k c_1(TF_k),$$

where $\omega_k \in \Omega^2(F_k)$ is the restriction of ω ; furthermore, for any $1 \le k \le r$ we also have

$$\lambda_k \le \lambda;$$

4. we have for any $1 \le k \le r$ a bound

$$\operatorname{codim}_{\mathbb{R}} F_k \le 6. \tag{7}$$

For example, any monotone symplectic manifold of dimension ≤ 6 with a Hamiltonian action with isolated fixed points satisfies the above conditions.

We will use the following consequences of the above assumptions. Let $s: \mathbb{CP}^1 \to F$ be a *I*-holomorphic map. Then

$$0 < \langle c_1(s^*TF), [\mathbb{CP}^1] \rangle.$$
(8)

Furthermore, if $s(\mathbb{CP}^1) \subset F_k$, then

$$\langle c_1(s^*TF_k), [\mathbb{CP}^1] \rangle \le \langle c_1(s^*TF), [\mathbb{CP}^1] \rangle.$$
 (9)

Both (8) and (9) follow from our assumptions, thanks to the formula

$$\int_{\mathbb{CP}^1} |ds|^2 = \int_{\mathbb{CP}^1} s^* \omega_s$$

which is valid because s is I-holomorphic.

Remarks. Some comments on the technical conditions are now in order.

1. The condition that the action is semi-free is used in two places. First to assure that there is universial bundle $\mathbb{E} \to \mathcal{M} \times \Sigma$, and then to prove that a generic perturbation

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of the equation gives a smooth moduli of solutions. If the action is not semi-free, one might encounter two different (but related) problems. First, the moduli of solutions \mathcal{M} could be forced to be an orbifold, and not a manifold, and there would not exist a universal bundle, but an orbibundle. Second, the kind of perturbations which we use will not suffice to get smoothness, and multivalued perturbations will have to be used instead (see [Sa]). Finally, we recall that, although the condition of being semi-free is rather restrictive, it has also a very natural geometrical meaning: namely, it is equivalent to the condition of all Marsden–Weinstein quotients at regular values of μ being smooth.

2. Conditions (8), (9) and (7) are of technical nature, and using more sophisticated techniques such as as virtual moduli cycles one might presumably get rid of them.

3. It seems natural that one could define invariants of Hamiltonian actions of arbitrary compact Lie groups using the same ideas given as in this paper. However, giving a rigorous definition of them (at least using the techniques which we deal with here) seems to involve a considerable amount of extra work. In [Mu2], parts of the programme developped in this paper are also worked out for arbitrary compact Lie groups (for example, the compactification of the moduli of solutions). On the other hand, note that if we replace S^1 by tre trivial group we recover the usual Gromov–Witten invariants.

1.9. Kaehler situation. The second equation in (1) was studied in [Mu1] under the assumption that the complex structure I is integrable (i.e., when it gives a Kaehler structure to F). In this case the action of S^1 extends to a holomorphic action of \mathbb{C}^* . This allows to extend the action of \mathscr{G} on \mathscr{S} to an action of $\mathscr{G}^{\mathbb{C}} = \operatorname{Map}(\Sigma, \mathbb{C}^*)$. On the other hand, there is a natural extension of the action of \mathscr{G} on \mathscr{A} to an action of $\mathscr{G}^{\mathbb{C}}$. The theorem proved in [Mu1] describes which orbits of the action of $\mathscr{G}^{\mathbb{C}}$ on $\mathscr{A} \times \mathscr{S}$ contain solutions to the second equation in (1). When the Kaehler manifold (F, ω, I) is projective, the description might be given using concepts coming from GIT. This description was used in [Mu2] to compute some nonzero Hamiltonian Gromov–Witten invariants of S^2 with the action of S^1 given by rotations.

1.10. Symplectic interpretation. As most gauge theoretical equations, the second equation in (1) admits an infinite dimensional symplectic interpretation, which we now briefly explain (see [Mu1] for more details). The space $\mathscr{A} \times \mathscr{S}$ admits a natural symplectic structure (which is the sum of the symplectic structure in \mathscr{A} defined by M. Atiyah and R. Bott in [AtBo] and a symplectic structure on \mathscr{S} obtained using ω). Then the action of \mathscr{G} on $\mathscr{A} \times \mathscr{S}$ is Hamiltonian, and

$$\nu(A,\phi) := \Lambda F_A + \mu(\phi) \in \Omega^0(\mathbf{i}\mathbb{R}^*) \subset \Omega^0(\mathbf{i}\mathbb{R})^*$$

is a moment map for this action. On the other hand, the set

$$\mathcal{D}_{\beta} = \{ (A, \phi) \mid \overline{\partial}_A \phi = 0, \ \phi_{E_*}[\Sigma] = \beta \}$$

is \mathscr{G} -invariant. So $\mathcal{M}(\beta, c)$ is the symplectic quotient of \mathcal{D}_{β} at the central element $c \in \Omega^0(\mathbf{i}\mathbb{R})^*$ (by a slight abuse of notation here c denotes the constant section with value c). Hence, the smooth locus of $\mathcal{M}(\beta, c)$ has a natural symplectic structure. On the other hand, using the action of \mathscr{G} on the restriction $\widetilde{E}|_{\mathcal{D}_{\beta}\times\Sigma}$, we get a S^1 -principal bundle on the Borel construction

$$\mathbb{E}_{\mathscr{G}} \to E\mathscr{G} \times_{\mathscr{G}} (\mathcal{D}_{\beta} \times \Sigma)$$

and, proceeding exactly as in the construction of Φ in Section 1.6, we can construct a section $\Phi_{\mathscr{G}} \in \Gamma(\mathbb{E}_{\mathscr{G}} \times_{S^1} F)$. Using this data we get, by means of Lemma 1.3, a map

$$\operatorname{ev}_{\mathscr{G}} : E\mathscr{G} \times_{\mathscr{G}} (\mathcal{D}_{\beta} \times \Sigma) \to F_{S^1}.$$

Now, it is straightforward to prove that

$$\operatorname{ev}_{\beta,c}^* = \kappa_c \circ \operatorname{ev}_{\mathscr{G}}^*$$

where $\kappa_c : H^*_{\mathscr{G}}(\mathcal{D}_{\beta} \times \Sigma) \to H^*(\mathcal{M}(\beta, c) \times \Sigma)$ is the Kirwan map.

2. Smoothness of the moduli space

2.1. Sobolev completions. To be able to deal with Banach manifold techniques (such as Sard–Smale theorem) we will work with the completion of the space $\mathscr{A} \times \mathscr{S}$ with respect to suitable Sobolev norms. Take any p > 2. Let $A_0 \in \mathscr{A}$ be any smooth connection, and define $\mathscr{A}_{L_1^p} = A_0 + L_1^p(T^*\Sigma \otimes i\mathbb{R})$ (if $E \to \Sigma$ is a vector bundle with a metric $|\cdot|$ and a connection ∇ , we write $L_1^p(E)$ for the completion of $\Omega^0(\Sigma; E)$ with respect to the norm $\|\sigma\|_{L_1^p} = \|\sigma\|_{L^p} + \|\nabla\sigma\|_{L^p}$; and to define $L_1^p(T^*\Sigma \otimes i\mathbb{R})$ we use any connection on $T^*\Sigma$). It is easy to check that $\mathscr{A}_{L_1^p}$ is independent of A_0 . Take any smooth embedding $j: \mathcal{F} \to \mathbb{R}^N$; any $\phi \in \mathscr{S}$ can be seen as a map $\phi: \Sigma \to \mathcal{F}$ and we set, for any $\phi, \phi' \in \mathscr{S}$,

$$d_p(\phi, \phi') = \|j(\phi) - j(\phi')\|_{L^p_1} = \sum_{1 \le k \le N} \|e_k(j(\phi)) - e_k(j(\phi'))\|_{L^p_1},$$

where $e_k : \mathbb{R}^N \to \mathbb{R}$ is the k-th coordinate. Let $\mathscr{S}_{L_1^p}$ be the completion of \mathscr{S} with respect to the metric d_p . Since $\dim_{\mathbb{R}} \Sigma = 2$, the Sobolev theorems tell us that there is an inclusion $L_1^p(\Sigma) \subset C^0(\Sigma)$. Hence, the elements of $\mathscr{S}_{L_1^p}$ are continuous sections. Furthermore, $\mathscr{S}_{L_1^p}$ is independent of the embedding j. Finally, we complete $\mathscr{G} =$ $\operatorname{Map}(\Sigma, S^1)$ using L_2^p norm. Then $\mathscr{G}_{L_2^p}$ is a Banach Lie group, which acts smoothly on $\mathscr{A}_{L_1^p}$ and $\mathscr{S}_{L_1^p}$.

Let $\mathscr{S}^* = \{ \phi \in \mathscr{S} \mid \phi(\Sigma) \not\subset E \times_{S^1} F^f \}$, and let $\mathscr{S}^*_{L^p_1}$ be the closure of \mathscr{S}^* in $\mathscr{S}_{L^p_1}$. One can give a natural Banach manifold structure to the set

$$\mathscr{B} = (\mathscr{A}_{L_1^p} \times \mathscr{S}_{L_1^p}^*) / \mathscr{G}_{L_2^p}.$$

There is also a natural Banach bundle structure on $\widetilde{\mathcal{W}} \to \mathscr{A}_{L_1^p} \times \mathscr{S}_{L_1^p}^*$, where the fibre over (A, ϕ) is

$$\widetilde{\mathcal{W}}_{(A,\phi)} = L^p(T^{0,1}\Sigma \otimes \phi^*T\mathcal{F}_v) \oplus L^p(\mathbf{i}\mathbb{R}).$$

The action of $\mathscr{G}_{L_2^p}$ lifts to a smooth action on $\widetilde{\mathcal{W}}$ and there is a quotient Banach bundle $\mathcal{W} \to \mathscr{B}$. Finally, equations (1) provide a smooth section $\Psi : \mathscr{B} \to \mathcal{W}$.

In the sequel we will omit the Sobolev subscripts in \mathscr{A} , \mathscr{S} and \mathscr{G} , and Sobolev completions will be implicitly assumed unless otherwise stated.

2.2. Virtual dimension of the moduli. The section Ψ is Fredholm, that is, its covariant derivative at any $[A, \phi] \in \Psi^{-1}(0) \subset \mathscr{B}$ (here the brackets denote gauge equivalence class) is a Fredholm operator $D_{[A,\phi]}\Psi : T_{[A,\phi]}\mathscr{B} \to \mathcal{W}_{[A,\phi]}$. Its index (we will be talking on real indices unless otherwise specified) is equal to minus the index of the deformation complex

$$C^0_{A,\phi} \xrightarrow{d_0} C^1_{A,\phi} \xrightarrow{d_1} C^2_{A,\phi}, \tag{10}$$

where $C_{A,\phi}^0 = L_2^p(\mathbf{i}\mathbb{R}) = \text{Lie}\,\mathscr{G}, C_{A,\phi}^1 = L_1^p(\phi^*T\mathcal{F}_v) \oplus L_1^p(T^*\Sigma \otimes \mathbf{i}\mathbb{R}) = T_{(A,\phi)}\mathscr{A} \times \mathscr{S}$ and $C_{A,\phi}^2 = L^p(T^{0,1}\Sigma \otimes \phi^*T\mathcal{F}_v) \oplus L^p(\mathbf{i}\mathbb{R}); d_0$ is given by the infinitesial action of \mathscr{G} on $\mathscr{A} \times \mathscr{S}$ and d_1 is the linearisation of equations (1). Modulo compact operators, the complex (10) splits as the sum of the de Rham complex

$$L_2^p(\mathbf{i}\mathbb{R}) \xrightarrow{d} L_1^p(T^*\Sigma \otimes \mathbf{i}\mathbb{R}) \xrightarrow{d'} L^p(\Lambda^2 T^*\Sigma \otimes \mathbf{i}\mathbb{R}) \simeq L^p(\mathbf{i}\mathbb{R})$$

(the isomorphism being given by Λ), whose index is

$$\operatorname{Ind} d' + d^* = 2 - 2g,$$

plus a (shifted one unity) Dolbeault complex

$$L_1^p(\phi^*T\mathcal{F}_v) \xrightarrow{\partial_\phi} L^p(T^{0,1}\Sigma \otimes \phi^*T\mathcal{F}_v),$$

whose index is, by Riemann–Roch, equal to

Ind
$$\overline{\partial}_{\phi} = 2\langle c_1(\phi^*T\mathcal{F}_v), [\Sigma] \rangle + 2n(1-g)$$

= $2\langle \phi^*c_1(T\mathcal{F}_v), [\Sigma] \rangle + 2n(1-g).$

Consequently, we have Ind $D_{[A,\phi]}\Psi = 2\langle \phi^* c_1(T\mathcal{F}_v), [\Sigma] \rangle + 2(n-1)(1-g)$. Observe that this is a function of $\beta = \phi_{E_*}[\Sigma]$. More precisely, we may write

Ind
$$D_{[A,\phi]}\Psi = 2\langle c_1^{S^1}(TF), \beta \rangle + 2(n-1)(1-g),$$
 (11)

where $c_1^{S^1}$ denotes the first equivariant Chern class. As usual, we call this number the real virtual dimension of our moduli space.

2.3. Perturbing the equations. Recall that $\pi : \mathcal{F} \to \Sigma$ denotes the projection. Let $\operatorname{Hom}^{0,1}(\pi^*T\Sigma, T\mathcal{F}_v)$ be the space of antiholomorphic smooth vector bundle maps from $\pi^*T\Sigma$ to $T\mathcal{F}_v$. This space has an action of S^1 induced by the action on \mathcal{F} . Let

$$\mathscr{P} = \mathrm{Hom}^{0,1}(\pi^*T\Sigma, T\mathcal{F}_v)^{S^1} \oplus \Omega^0(\Sigma; \mathbf{i}\mathbb{R})$$

(the superscript S^1 denotes the subspace of invariant elements). For any $\rho = (\rho_1, \rho_2) \in \mathscr{P}$, consider the following perturbed equations

$$\begin{cases} \overline{\partial}_A \phi = \rho_1 \\ \Lambda F_A + \mu(\phi) = c + \rho_2 \end{cases}$$
(12)

and define the perturbed set of solutions to be

$$\mathcal{M}_{\rho}(\beta, c) = \{ (A, \phi) \in \mathscr{A} \times \mathscr{S} \mid \phi_{E_*}[\Sigma] = \beta \text{ and } (A, \phi) \text{ satisfies (12)} \}.$$

Since the elements of \mathscr{P} are S^1 -invariant, the set $\widetilde{\mathcal{M}}_{\rho}(\beta, c)$ is gauge invariant, so we may define the perturbed moduli space to be

$$\mathcal{M}_{\rho}(\beta, c) = \mathcal{M}_{\rho}(\beta, c) / \mathscr{G}.$$

In order to avoid having solutions of the perturbed equations which are entirely contained in the fixed point set, we assume that $c \in \mathbf{i}\mathbb{R} \setminus \mathbb{C}$ and we define

$$\mathscr{P}_c = \{(\rho_1, \rho_2) \in \mathscr{P} \mid |\rho_2| < d(c, \mathscr{C})\},\$$

where $d(c, \mathscr{C})$ denotes the distance from c to the set \mathscr{C} . Then we may prove, exactly like in Lemma 1.4:

Lemma 2.1. If $\rho \in \mathscr{P}_c$, then the action of \mathscr{G} on $\widetilde{\mathcal{M}}_{\rho}(\beta, c)$ is free.

The following Theorem justifies the use of the perturbed equations.

Theorem 2.2. Assume that $c \in \mathbf{i}\mathbb{R} \setminus \mathscr{C}$. There is a subset $\mathscr{P}_c^{\operatorname{reg}} \subset \mathscr{P}_c$ of Baire of the second category (with respect to the C^{∞} topology on \mathscr{P}_c) such that for any $\rho \in \mathscr{P}_c^{\operatorname{reg}}$ the perturbed moduli space $\mathcal{M}_{\rho}(\beta, c)$ is a smooth manifold of dimension equal to (11) with a natural orientation. Furthermore, for any pair of perturbations $\rho, \rho' \in \mathscr{P}_c$ there is a path $P : [0, 1] \to \mathscr{P}_c$ with $P(0) = \rho$, $P(1) = \rho'$ and such that

$$\mathcal{M}_P(\beta, c) = \bigcup_{t \in [0,1]} \mathcal{M}_{P(t)}(\beta, c)$$

is a smooth oriented cobordism between $\mathcal{M}_{\rho}(\beta, c)$, and $\mathcal{M}_{\rho'}(\beta, c)$.

Proof. The techniques needed to prove this result are rather standard, so we will be very sketchy. More details are given in [Mu2], Theorem 3.4.4, and see also [FrUh, McDS1] for analogous results.

Take a big enough (to be specified later) integer l > 0, and consider the completion \mathscr{P}_{c}^{l} of \mathscr{P}_{c} with respect to the C^{l} norm. Let

 $\mathcal{M}_{\mathscr{P}}(\beta, c) = \{(\rho, A, \phi) \in \mathscr{P}_{c}^{l} \times \mathscr{A} \times \mathscr{S} \mid \phi_{E_{*}}[\Sigma] = \beta \text{ and } (A, \phi, rho) \text{ satisfies } (12)\}/\mathscr{G}.$ One first proves that this is a Banach manifold. Indeed, consider the following section of the pullback bundle $\pi_{\mathscr{R}}^{*}\mathcal{W} \to \mathscr{P}_{c}^{l} \times \mathscr{B}$ (where $\pi_{\mathscr{R}}$ denotes the projection):

$$\Psi_{\mathscr{P}}((\rho_1,\rho_2),[A,\phi]) = (\overline{\partial}_A \phi - \rho_1,\Lambda F_A + \mu(\phi) - c - \rho_2)$$

(here $[A, \phi]$ denotes de gauge equivalence class of $(A, \phi) \in \mathscr{A} \times \mathscr{S}$). Then $\mathcal{M}_{\mathscr{P}}(\beta, c) = \Psi_{\mathscr{P}}^{-1}(0)$. So to prove that $\widetilde{\mathcal{M}}_{\mathscr{P}}(\beta, c)$ is a Banach manifold it suffices to prove that if $z = ((\rho_1, \rho_2), [A, \phi]) \in \Psi_{\mathscr{P}}^{-1}(0)$ then the differential $D\Psi_{\mathscr{P}}(z) : T(\mathscr{P}_c^l \times \mathscr{B}) \to (\pi_{\mathscr{B}}^* \mathcal{W})_z$ is onto. By ellipticity, we know that the image of $D\Psi_{\mathscr{P}}(z)$ is closed. So if $D\Psi_{\mathscr{P}}(z)$ were not onto, there would be some nonzero $\eta \in L^q(T^{0,1}\Sigma \otimes \phi^*T\mathcal{F}_v) \oplus L^q(\mathbf{i}\mathbb{R})$, where $p^{-1} + q^{-1} = 1$, and such that $\langle \eta, D\Psi_{\mathscr{P}}(z)\xi \rangle = 0$ for any $\xi \in T(\mathscr{P}_c^l \times \mathscr{B})$. Now, it is easy to prove, using that $\phi \in \mathscr{S}^*$ (here we use Lemma 1.4), that one can find some $\xi \in T\mathscr{P}_c^l \subset T(\mathscr{P}_c^l \times \mathscr{B})$ which does not satisfy that equality. This proves that $\widetilde{\mathcal{M}}_{\mathscr{P}}(\beta, c)$ is a Banach manifold.

Consider the projection $\pi_{\mathscr{P}} : \mathcal{M}_{\mathscr{P}}(\beta, c) \to \mathscr{P}^{l}$. This is a map whose differential is everywhere Fredholm, and its index $\operatorname{Ind}(D\pi_{\mathscr{P}})$ is equal, at any point, to the virtual dimension (11). Now, provided $l > 2 + \operatorname{Ind}(D\pi_{\mathscr{P}})$, we may apply Sard–Smale theorem to deduce that there is a set of Baire of the second category $\mathscr{P}_{c}^{\operatorname{reg},l} \subset \mathscr{P}_{c}^{l}$ of regular values of $\pi_{\mathscr{P}}$. And $\pi_{\mathscr{P}}^{-1}(\rho) = \mathcal{M}_{\rho}(\beta, c)$ by definition. Finally, a trick of Taubes (see p. 36 in [McDS1]) allows to deduce from this result that there is also a subset $\mathscr{P}_{c}^{\operatorname{reg}} \subset \mathscr{P}_{c}$ of Baire of the second category (with respect to the C^{∞} topology on \mathscr{P}_{c}) of regular values of $\pi_{\mathscr{P}}$. The result on cobordisms is proved in the same way.

Finally, the orientability follows from identifying the tangent space at $[A, \phi] \in \mathcal{M}_{\rho}(\beta, c)$ with the first cohomology group of the complex (10). By Hodge theory this group can be identified with the kernel of the elliptic operator $d_0^* + d_1$ and, when $\rho \in \mathscr{P}_c^{\text{reg}}$, this kernel carries a natural orientation because $d_0^* + d_1$ has the same symbol as a Cauchy–Riemann operator plus the Hodge operator $d + d^*$ acting on 1-forms. \Box

2.4. **Regularity.** The following theorem proves that the peturbed moduli spaces $\mathcal{M}_{\rho}(\beta, c)$ which we get by taking smooth perturbations $\rho \in \mathscr{P}$ are independent of the chosen Sobolev completion.

Theorem 2.3. Let $\rho \in \mathscr{P}$, and let $(A, \phi) \in \mathscr{A} \times \mathscr{S}$ be a solution to the perturbed equations (12). There is a gauge transformation $g \in \mathscr{G}$ such that g^*A and $g^*\phi$ are both smooth.

Proof. Take $g \in \mathscr{G}$ such that g^*A is in Coulomb gauge with respect to the smooth connection $A_0 \in \mathscr{A}$, i.e., such that $d^*(A_0 - g^*A) = 0$, and define $A_s = g^*A$ and $\phi_s = g^*\phi$. We will prove that A_s and ϕ_s are smooth. To begin with we know that the L_1^p norms of $A_0 - A_s$ and of ϕ_s are bounded. Let ω_{Σ} be the volume form of Σ . The second equation in (12) may be written

$$d(A_0 - A_s) = -F_{A_0} + \omega_{\Sigma}(c - \mu(\phi) + \rho_2).$$

Since μ is smooth and ϕ has bounded L_1^p norm, we deduce that the L_1^p norm of $\mu(\phi)$ is also bounded. So the above equation gives an L_1^p bound to $d(A_0 - A_s)$. This bound, combined with $d^*(A_0 - A_s) = 0$ implies, by the ellipticity of $d + d^*$, an L_2^p norm on $A_0 - A_s$. From this we obtain a bound on the L_2^p norm of the complex structure $I(A_s)$ constructed in Subsection 1.2. Now, standard results on regularity of (perturbed) holomorphicity equation for curves (see for example Theorem B.3.4 in [McDS1]), allow to deduce from this L_2^p bounds on the section ϕ (which is a $I(A_s)$ holomorphic map from Σ to \mathcal{F} , by Lemma 1.2). So we have passed from L_1^p bounds on A_s and ϕ_s to L_2^p bounds on both. This step can be repeated to obtain L_k^p bounds for any k. And this implies, by the Sobolev theorems, that both A_s and ϕ_s are smooth.

What this theorem proves is, strictly speaking, that $\mathcal{M}_{\rho}(\beta, c)$ is independent of p as a set. To prove also that, when $\rho \in \mathscr{P}_{c}^{\text{reg}}$, the structure of $\mathcal{M}_{\rho}(\beta, c)$ as a differentiable manifold is intrinsic, one may use standard elliptic theory applied to the Kuranishi models which describe locally the moduli spaces (see Section 3.3 in [Mu2]).

3. Compactification of the moduli space

3.1. The Yang–Mills–Higgs functional. Define the Yang–Mills–Higgs functional $\mathcal{YMH}_c: \mathscr{A} \times \mathscr{S} \to \mathbb{R}$ as

$$\mathcal{YMH}_c(A,\phi) = \|F_A\|_{L^2}^2 + \|d_A\phi\|_{L^2}^2 + \|\mu(\phi) - c\|_{L^2}^2$$

for any $(A, \phi) \in \mathscr{A} \times \mathscr{S}$. The following is Lemma 7.9 in [Mu1] (note that in [Mu1] we assume the manifold F to be Kaehler; however, the results on the functional \mathcal{YMH}_c remain valid when the complex structure on F is not integrable).

Lemma 3.1. For any $(A, \phi) \in \mathscr{A} \times \mathscr{S}$ we have

$$\mathcal{YMH}_c(A,\phi) = \|\Lambda F_A + \mu(\phi) - c\|_{L^2}^2 + 2\|\overline{\partial}_A \phi\|_{L^2}^2 + \int_{\Sigma} \langle F_A, c \rangle + \langle \phi^*[\omega_\mathcal{F}], \Sigma \rangle,$$

where $[\omega_{\mathcal{F}}] \in H^2(\mathcal{F}; \mathbb{R})$ is the cohomology class represented by the coupling form of the symplectic fibration $\mathcal{F} \to \Sigma$ and the connection A (this cohomology class does not depend on A, see [GuLeS]).

Corollary 3.2. Given $\beta \in H_2^{S^1}(F;\mathbb{Z})$, $c \in i\mathbb{R}$ and $\rho \in \mathscr{P}$, there is a constant $C(\beta, c, \rho) > 0$ such that for any $(A, \phi) \in \widetilde{\mathcal{M}}_{\rho}(\beta, c)$ we have $||F_A||_{L^2} < C(\beta, c, \rho)$ and $||d_A\phi||_{L^2} < C(\beta, c, \rho)$.

Proof. Indeed, the term $\int_{\Sigma} \langle F_A, c \rangle + \langle \phi^*[\omega_{\mathcal{F}}], \Sigma \rangle$ only depends on β and c.

In fact, the bound for F_A is obvious when the manifold F is compact, since then μ is bounded; although we will not need it, we mention that the result remains valid even when F is not compact.

3.2. Compactness. Let $\rho \in \mathscr{P}$ be any perturbation.

Definition 3.3. A cusp ρ -THC is the following set of data.

- 1. A compact connected singular curve Σ^{c} with only nodal singularities, of the form $\Sigma^{c} = \Sigma_{0} \cup \Sigma_{1} \cup \cdots \cup \Sigma_{K}$, where $\Sigma_{0} = \Sigma$ is called the principal component, and where the other components are rational curves \mathbb{CP}^{1} and are called bubbles; furthermore, two different components Σ_{i} and Σ_{j} meet at most at one point.
- 2. A S¹-principal bundle $E \to \Sigma_0$, a connection A on E, a section $\phi_0 : \Sigma_0 \to \mathcal{F} = E \times_{S^1} F$ and an element $c \in \mathbf{i}\mathbb{R}$ satisfying the equations

$$\begin{cases} \overline{\partial}_A \phi_0 = \rho_1\\ \Lambda F_A + \mu(\phi_0) = c + \rho_2. \end{cases}$$

3. For any $k \neq 0$, a holomorphic map $\phi_k : \Sigma_k \to \mathcal{F}$ whose image is inside a unique fibre \mathcal{F}_{x_k} of $\mathcal{F} \to \Sigma$ (note ϕ_k is holomorphic with respect to the complex structure on F). The maps $\phi_0, \phi_1, \ldots, \phi_K$ are required to glue together to give a map $\phi : X^c \to \mathcal{F}$.

We denote cusp ρ -THCs with tuples of the form $(\Sigma^{c}, E, A, \phi, c)$. Let $\iota : H_{*}(\mathcal{F}_{x}; \mathbb{Z}) \to H_{*}^{S^{1}}(F; \mathbb{Z})$ be the natural map from the homology of any fibre of \mathcal{F} to the equivariant homology of F (this map is well defined because, since S^{1} is connected, the action of S^{1} on $H_{*}(F; \mathbb{Z})$ is trivial). We will say that the ρ -THC ($\Sigma^{c}, E, A, \phi, c$) represents the class

$$(\phi_{0E})_*[\Sigma_0] + \sum_{k=1}^K \iota_* \phi_{k*}[\Sigma_k] \in H_2^{S^1}(F;\mathbb{Z}).$$

Theorem 3.4. Let $\beta \in H_2^{S^1}(F;\mathbb{Z})$ be any homology class. Consider a sequence of gauge equivalence classes $[A_j, \phi_j] \in \mathcal{M}_{\rho}(\beta, c_j)$, where $j \ge 1$, and where $\{c_j\} \subset \mathbf{i}\mathbb{R}$ is a bounded set. After passing to a subsequence, there exists a cusp ρ -THC (Σ^c, E, A, ϕ, c) and gauge transformations $g_j \in \mathscr{G}$ such that if $(A'_j, \phi'_j) = g_j(A_j, \phi_j)$ we have

1. $c_j \rightarrow c;$

- 2. $A'_j \to A \text{ in } C^{\infty};$
- 3. the ρ -THC ($\Sigma^{c}, E, A, \phi, c$) represents the class β ;
- 4. the images $\phi'_j(X) \subset \mathcal{F}$ converge pointwise to $\phi(X^c)$, that is, for any sequence $x_i \in X_j$ there exists $x \in X^c$ such that $\phi'_i(x_j) \to \phi(x)$.

Furthermore, the curve Σ^{c} is a tree, in the sense that the graph with one point for each irreducible component of Σ^{c} and with an edge joining two points exactly when these correspond to components which intersect, is a tree. And, finally, the limit ρ -THC curve ($\Sigma^{c}, E, A, \phi, c$) is stable, which means that if the map ϕ_{k} for $1 \leq k \leq K$ is constant, then the bubble Σ_{k} meets at least three other irreducible components of Σ^{c} .

Proof. We first take a subsequence of (A_j, ϕ_j, c_j) (and denote it with the same symbol) such that $c_j \to c \in \mathbf{i}\mathbb{R}$. Just as in the proof of Theorem 2.3, we take gauge transformations which put the connections in Coulomb gauge and then use the existing compactness theorem for pseudoholomorphic curves. So let $g_j \in \mathscr{G}$ such that $A'_j = g^*A_j$ satisfies $d^*(A'_j - A_0) = 0$, and let $\phi'_j = g^*\phi_j$. By gauge invariance, (A'_j, ϕ'_j) satisfies

$$\begin{cases} \overline{\partial}_A \phi_0 = \rho_1 \\ \Lambda F_A + \mu(\phi_0) = c + \rho_2 \end{cases}$$

Now, the second equation combined with the Coulomb condition gives a uniform bound $||A'_j - A_0||_{L^p_1} < C$, by ellipticity of $d + d^*$. Using Rellich theorem on the compactness of the embedding $L^p_1 \to C^0$, we deduce that there is a connection A satisfying $||A - A_0||_{L^p_1} < \infty$ such that, after restricting to a subsequence (which we denote also by $\{A'_j, \phi'_j\}$), $\{A'_j\}$ converges uniformly in C^0 to A. This implies that the complex structures $I_j = I(A_j)$ converge in C^0 to I = I(A). On the other hand, $\phi'_j : \Sigma \to \mathcal{F}$ is a perturbed I_j holomorphic map for any j (by Lemma 1.2). (These maps can be regarded also as holomorphic maps to $\Sigma \times \mathcal{F}$, giving to this space a complex structure of the form $\begin{pmatrix} I_{\Sigma} & 0 \\ \rho_1 & I_j \end{pmatrix}$, see for example [Gr, Ru1].) By Corollary 3.2 there are bounds on $||d_{A'_j}\phi'_j||^2_{L^2}$. But if we give to \mathcal{F} the Riemannian metric g_j obtained by summing the metric $\omega(\cdot, I \cdot)$ on F and the metric on Σ by means of the splitting of $T\mathcal{F}$ given by the connection A'_j , then we have

$$\|d\phi_j'\|_{L^2,q_i}^2 = \|d_{A_j'}\phi_j'\|_{L^2}^2 + \operatorname{Vol}(\Sigma),$$

so the energy of the maps ϕ'_j is bounded. On the other hand, the metrics g_j converge to a limit $g_j \to g$ in C^0 . This implies that for any fixed metric on \mathcal{F} the corresponding energies of ϕ'_j are also uniformly bounded.

At this point we may apply Theorem 1 in [IvSh] on Gromov compactness for pseudoholomorphic maps (which is valid for continuous complex structures on the target manifold converging uniforming to a limit) to deduce the existence (again, after restricting to a subsequence) of a cusp curve Σ^c together with a limit map $\phi : \Sigma^c \to \mathcal{F}$, which is I holomorphic. Their theorem gives an identification of the principal component $\Sigma_0 \simeq \Sigma$ such that the maps ϕ'_j converge pointwise to ϕ in $\Sigma_0 \setminus \{x_1, \ldots, x_p\}$, where $\{x_1, \ldots, x_p\}$ are the bubbling points of Σ_0 (i.e., the points where a bubble gets attached). From this we easily deduce that $\phi_0 : \Sigma_0 \simeq \Sigma \to \mathcal{F}$ is a section of the bundle \mathcal{F} . Now, using the *a priori* estimates given in §3 of [IvSh], and repeating the argument in the proof of Theorem 2.3 we deduce that for any compact $K \subset \Sigma_0 \setminus \{x_1, \ldots, x_p\}$ the restriction of (A, ϕ_0) to K is smooth and we have convergence in C^{∞} of (A'_j, ϕ'_j) to (A, ϕ_0) . This implies that the restriction of A to $\Sigma_0 \setminus \{x_1, \ldots, x_p\}$ is in Coulomb gauge. On the other hand, since $||A - A_0||_{L^p_1} < \infty$, we deduce that A is in Coulomb gauge in the whole Σ , so Theorem 2.3 implies that (A_0, ϕ_0) is smooth. Finally, statement (3) of the theorem follows from statement (3) of Theorem 1 in [IvSh].

Remark. In [Mu2] a proof of this theorem is given which applies to the case of compact connected structure group different from S^1 (see the remarks in Section 1.8). The idea consists of first proving a local result (essentially, combining Uhlenbeck's theorem on existence of local Coulomb gauge with and *equivariant* version of Gromov–Schwarz lemma, which is Lemma 4.2.1 in [Mu1]), and then using a standard patching argument.

4. INVARIANT COMPLEX STRUCTURES AND MODULI OF RATIONAL CURVES

In all this section Σ will be the Riemann sphere $S^2 = \mathbb{CP}^1$. It is well known that for a generic complex structure $I \in \text{End}(TF)$ compatible with ω the moduli space of simple I holomorphic maps $s : \Sigma \to F$ is a smooth manifold of dimension $2\langle c_1(TF), s_*[\Sigma] \rangle + 2n$ (see for example Theorem 3.1.2 in [McDS1]). (Recall that s is a simple map if it does not factor through a nontrivial ramified covering $\Sigma \to \Sigma$.)

However, in order for the equations (12) to be gauge invariant we need to chose S^1 -invariant complex structres on F, and these are in general far from being generic. In fact, most of the times the moduli of simple holomorphic maps with respect to S^1 -invariant complex structures will not be smooth or will not have the expected dimension. So to get smooth moduli of maps we will have to restrict ourselves to subsets of the set of simple holomorphic maps. We will take these subsets to be simple curves with fixed isotropy pair (see below).

The results in this section might be seen as a piece of Gromov–Witten theory for symplectic orbifolds. Such a theory should study in general pseudo-holomorphic maps from compact complex orbifolds of complex dimension 1 to orbifolds, and here we will study in particular pseudo-holomorphic maps from $\mathbb{CP}^1/\mathbb{Z}_m$ to F/\mathbb{Z}_m , where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ acts on \mathbb{CP}^1 by rotations through a fixed axis. Parts of Gromov–Witten theory for orbifolds in the algebraic category have been worked out by D. Abramovich and A. Vistoli [AbVi].

Let \mathscr{I}_{ω} be the set of complex structures on F which are compatible with ω , and let $\mathscr{I}_{\omega,S^1} \subset \mathscr{I}_{\omega}$ be the S^1 -invariant ones. (Recall that by Lemma 5.49 in [McDS2] \mathscr{I}_{ω,S^1} is a nonempty and contractible set in the C^{∞} topology.)

4.1. Isotropy pairs.

Definition 4.1. Let $s: \Sigma \to F$ be any smooth map. We define the isotropy pair of s to be the pair of closed subgroups $L(s) \subset H(s) \subset S^1$ defined as follows

$$H(s) := \{ \theta \in S^1 | \ \theta \cdot s(\Sigma^1) = s(\Sigma^1) \}$$
$$L(s) := \{ \theta \in H(s) | \ \theta|_{s(\Sigma^1)} = \mathrm{Id} \}.$$

(The dot \cdot means the action of S^1 on F.)

Theorem 4.2. Let $I \in \mathscr{I}_{\omega,S^1}$, and let $s : \Sigma \to F$ be a simple holomorphic map. Let H = H(s). There exists a disk $D \subset \Sigma$ such that

$$S^1 \cdot s(D) \cap s(\Sigma) = H \cdot s(D).$$

Proof. In order to prove the theorem we will use the following result on holomorphic curves (see Lemma 2.2.3 in [McDS1]).

Lemma 4.3. Let $I \in \mathscr{I}_{\omega}$, and let $s_1, s_2 : \Sigma \to F$ be two simple *I*-holomorphic maps. Let $K \subset \Sigma$ be a closed subset such that, for any $x \in K$, $ds_1(x) \neq 0 \neq ds_2(x)$. If the intersection $s_1(K) \cap s_2(K)$ contains infinite points, then $s_1 = s_2$.

From now on we fix a complex structure $I \in \mathscr{I}_{\omega,S^1}$, and we take on F the S^1 -invariant metric $\omega(\cdot, I \cdot)$. Let $\mathscr{X} \in \Gamma(TF)$ be the vector field generated by the infinitesimal action of $\mathbf{i} \in \mathbf{i}\mathbb{R} = \mathrm{Lie}(S^1)$. For any $x \in \Sigma$ and any smooth map $s : \Sigma \to F$ we define

$$\theta_s(x) := \operatorname{dist}(\mathscr{X}(s(x)), ds(T\Sigma)(x))$$

(note that $\mathscr{X}(s(x)) \in T_{s(x)}F$ and that $ds(T\Sigma)(x)$ is a 2-dimensional subspace of $T_{s(x)}F$). Suppose to begin with that $H = H(s) = \{1\}$. Assume that for any open set $U \subset \Sigma$ there exists a point $x \in U$ and $1 \neq \alpha \in S^1$ such that $\alpha \cdot s(x) \in s(\Sigma)$. We will see that this leads to a contradiction.

Let $Z = s^{-1}(\{s(z) | z \in \Sigma, ds(z) = 0\})$. This is a finite set (see Lemma 2.2.1 in [McDS1]). So $s(\Sigma)$ is not contained in $S^1 \cdot s(Z)$, since the latter is a disjoint union of points and circles (since s is simple, it is in particular non-constant). Let T be a small S^1 -invariant tubular neighbourhood of $S^1 \cdot s(Z)$, and put $\Sigma' = s^{-1}(F \setminus T)$. The set of noninjective points $Z' = \{z \in \Sigma | \sharp s^{-1}s(z) > 1\}$ can only accumulate at critical points (combine Lemma 2.2.3 and Proposition 2.3.1 in [McDS1]), so $Z'' = Z' \cap \Sigma'$ is finite. Hence, $s(\Sigma')$ is not wholly contained in $S^1 \cdot s(Z'')$ so we may take a small S^1 -invariant tubular neighbourhood T'' of $S^1 \cdot s(Z'')$ so that $\Sigma'' = s^{-1}(F \setminus T'') \cap \Sigma'$ has nonempty interior.

Let $Y = \{z \in \Sigma | \theta_s(z) = 0\}$. This is a closed set. If $\operatorname{int} Y \neq \emptyset$ then, for $\alpha \in S^1$ near $1 \in S^1$, $\alpha \cdot s(\Sigma)$ and $s(\Sigma)$ meet at an open set and hence, by Lemma 4.3, they coincide. But this is implies that $H \neq \{1\}$, in contradiction with our assumption. So we may suppose that there is a small open disk $D_a \subset \Sigma''$ such that $\inf \theta_s|_{D_a} = a > 0$. Suppose also that $S^1 \cdot s(D_a) \subset W \subset F$, where W is open and S^1 -invariant, and all points in W have the same stabiliser, so that W/S^1 is a smooth manifold.

The composition $D_a \xrightarrow{s_a} W \xrightarrow{\pi} W/S^1$ is an embedding (here $s_a = s|_{D_a}$) if D_a is small enough. Let $N \subset W/S^1$ be an open neighbourhood of $\pi s_a(D_a)$ with a submersion $p: N \to \pi s_a(D_a)$ which is a left inverse for the inclusion $\pi s_a(D_a) \hookrightarrow N$. Let $Y_N =$ $Y \cap (\pi s)^{-1}(N)$ (where $\pi: F \to F/S^1$ is the projection). The critical points of

$$\Sigma \cap (\pi s)^{-1}(N) \xrightarrow{s} N \xrightarrow{p} \pi s(D_a)$$

contain Y_N . Hence, by Sard's theorem $\pi s(Y_N) \subset \pi s(D_a)$ has measure zero. Since $\pi s(Y_N)$ is closed, its complementary contains a closed disk Σ_0 . Furthermore, there exists b > 0 such that for any $x \in \Sigma_0$ and $\alpha \in S^1$ if $\alpha \cdot s(x) = s(y) \in s(\Sigma)$, then $\theta_s(y) \geq b$.

From the construction of Σ_0 we deduce the following. There exist real positive numbers r, η , ϵ such that for any $x \in \Sigma_0$ and $\alpha \in S^1$ if $z = \alpha \cdot s(x) \in s(\Sigma)$, $s^{-1}(z)$ has a unique element $y \in \Sigma$ and if $D_y = D(y; r)$ is the disk centered at y of radius r, the following holds.

P1. If $w \in s(\Sigma)$ and $d(w, y) < \eta$, then $w \in s(D_y)$. P2. There exists an open neighbourhood $V \subset F$ of s(y) containing $s(D_y)$ and a chart $\phi = (\phi_1, \dots, \phi_{2n}) : V \to \mathbb{R}^{2n}$ with $\phi(s(y)) = 0$ such that P2a. For any $v \in D_y$, $\phi_3(s(v)) = \dots = \phi_{2n}(s(v))$. P2b. If $\beta \in [-\epsilon, \epsilon] \subset S^1$, then for any $v \in D_y$, $\beta \cdot s(v) \in V$ and $\phi(\beta \cdot s(v)) = \phi(s(v)) + (0, 0, \beta, 0, \dots, 0)$.

We assume for the rest of the argument that $\operatorname{diam}(s(\Sigma_0)) < \eta/2$. Let us identify $S^1 \simeq [0, 2\pi)$ so that 0 is the identity and consider

$$I = \{ (\alpha, x) \in (0, 2\pi) \times \Sigma_0 | \alpha \cdot s(x) \in s(\Sigma) \}.$$
(13)

Thanks to the inequality $\theta_s|_{\Sigma_0} \geq b$ we know that there exists $\delta > 0$ such that $I \subset [\delta, 2\pi - \delta] \times \Sigma_0$. Clearly I is closed. By our assumption the image of the projection $\pi_{\Sigma} : I \to \Sigma_0$ is dense and so (since it is also closed) coincides with Σ_0 . Let now $[0, \mu] \subset [-\epsilon, \epsilon]$ be a subset such that for any $\nu \in [0, \mu]$ and for any $x \in F$, $d(x, \nu \cdot x) < \eta/2$.

Cover $[\delta, 2\pi - \delta]$ with closed intervals A_1, \ldots, A_r of length $< \mu$ and let $I_k = I \cap A_k \times \Sigma_0$. Since $\pi_{\Sigma}(I_1) \cup \cdots \cup \pi_{\Sigma}(I_r) = \Sigma_0$ and $\pi_{\Sigma}(I_l)$ is closed for any l, there exists a $\pi_{\Sigma}(I_k)$ with nonempty interior. Let $D \subset \operatorname{int} \pi_{\Sigma}(I_k)$ be a disk, and take $x \in D$. By assumption there exists $\alpha \in A_k$ such that $\alpha \cdot s(x) = s(y), y \in \Sigma$. By P2 there exists an open set $V \subset F$ containing $s(D_y) = s(D(y, r))$ and a chart

 $\phi: V \to \mathbb{R}^{2n}.$

For any $z \in D$ there exists $\beta \in A_k$ such that $\beta \cdot s(Z) \in s(\Sigma)$. On the other hand, since $d(\alpha \cdot s(z), \alpha \cdot s(x)) = d(s(z), s(x)) < \eta/2$ and $|\alpha - \beta| < \mu$ we have

$$d(\beta \cdot s(z), s(y)) < \eta.$$

Hence, by P1, $\beta \cdot s(z) \in s(D_y)$. So by P2b, if w = s(z), then $\phi_3(w) = \alpha - \beta$, $\phi_4(w) = \cdots = \phi_{2n}(w) = 0$. This implies that for any $z \in D$, $\sharp I_k \cap \{z\} \times A_k = 1$. Let (z, h(z)) be the unique element of this set. The function $h: D \to A_k$ is $h(z) = \alpha - \phi_3(s(z))$ and so is continuous. Hence there exists $c \in A_k$ such that $\sharp h^{-1}(c) = \infty$ (this follows from this easy result: if $h: [0, 1]^2 \to [0, 1]$ is a continuous map, then there exists $c \in I$ such that $\sharp h^{-1}(c) = \infty$). From this we see that $c \cdot s(\Sigma) \cap s(\Sigma)$ has infinite points which do not accumulate on critical points (since $s(\Sigma_0)$ is at positive distance from the S^1 -orbit of the image of any critical point of s). Finally, using Lemma 4.3 we deduce that $c \cdot s(\Sigma) = s(\Sigma)$, in contradiction with the assumption $H = \{1\}$.

The case $H = S^1$ is trivial. Suppose to finish now that $1 < \sharp H < \infty$. We assume that for any open set $U \subset \Sigma$ there exists $x \in U$ and $\alpha \in S^1 \setminus H$ such that $\alpha \cdot s(x) \in s(\Sigma)$. We do exactly the same thing as in the case $H = \{1\}$ to get a subset $\Sigma_0 \subset \Sigma$ (note that the function $\theta_s(x)$ is equivariant under the action of H). Now, the set I defined in (13) is at positive distance from $H \times \Sigma_0$. So the element $c \in S^1$ found at the end of the reasoning does not belong to H, and hence the fact that $c \cdot s(\Sigma) = s(\Sigma)$ leads to a contradiction.

Let $s: \Sigma \to F$ be a simple map, and let $g \in H(s)$ be any element. Let Σ_i be the set of injective points of s, that is, $\Sigma_i = \{x \in \Sigma | ds(x) \neq 0, \ \sharp s^{-1}s(x) = 1\}$. The action of g on $s(\Sigma)$ induces a holomorphic bijection $\gamma_i(g): \Sigma_i \to \Sigma_i$ which can be extended to a homeomorphism $\gamma(g): \Sigma \to \Sigma$. Now, since the map s is simple, the noninjective points $\Sigma \setminus \Sigma_i$ can only accumulate at a finite set of points (namely, the critical points Ker ds), and hence the map $\gamma(q)$ is holomorphic by standard removability of singularities. This way we have defined a map $\gamma : H(s) \to \operatorname{Aut}(\Sigma) = \operatorname{PSL}(2; \mathbb{C})$. Obviously, Ker $\gamma = L(s)$. Let

$$\operatorname{Map}^{L}(\Sigma, F) = \{ s \in \operatorname{Map}(\Sigma, F^{L}) \mid L(s) = L \}.$$

Theorem 4.4. Let $I \in \mathscr{I}_{\omega,S^1}$, and let $s \in \operatorname{Map}^L(\Sigma, F^L)$ be a simple holomorphic map. Then the set $\{x \in \Sigma \mid L \neq (S^1)_{s(x)}\}$ (where $(S^1)_{s(x)}$ denotes the stabiliser of $s(x) \in F$) is finite.

Proof. Let $\Sigma' = \{x \in \Sigma \mid L \neq (S^1)_{s(x)}\}$ and suppose that $\sharp \Sigma' = \infty$. Since the set of different stabilizers of points of F is finite, we may assume that there exists a group L'' strictly containing L such that

$$\Sigma'' = \{ x \in \Sigma \mid (S^1)_{s(x)} = L'' \}$$

has infinite elements. Let now $\theta \in L'' \setminus L$. Then $s(\Sigma)$ and $\theta \cdot s(\Sigma)$ intersect at an infinite set Σ'' of points. Hence by Lemma 4.3 they coincide, and so $\theta \in H(s)$. But now $\gamma(\theta) \in \operatorname{Aut}(\Sigma)$ has infinitely many fixed points (all the points in Σ''), and so it must be the identity. But this implies that $\theta \in L$, which is a contradiction.

4.2. The moduli of rational curves. Let $L \subset S^1$ be a closed group. The fixed point set $F^L \subset F$ is a compact symplectic submanifold (with possibly several connected components of different dimension). The action of S^{1} on F gives an action of the Lie group S^1/L on F^L . Fix a closed subgroup $\Gamma \subset \operatorname{Aut}(\Sigma)$, and assume that there is an injection $\rho: \Gamma \to S^1/L$. We will say that a map $s: \Sigma \to F^L$ is (Γ, ρ) -equivariant if $s(qx) = \rho(q)s(x)$ for any $x \in \Sigma$ and $q \in \Gamma$. Let us define

$$\operatorname{Map}(L, \Gamma, \rho) = \{ s \in \operatorname{Map}(\Sigma, F^L)_{L_1^p} \mid L(s) = L, s \text{ is } (\Gamma, \rho) \text{-equivariant} \}.$$
(14)

Let $I \in \mathscr{I}_{\omega,S^1}$. Define the moduli of (L, Γ, ρ) -equivariant curves with respect to I to be

$$\mathcal{M}_I(L,\Gamma,\rho) = \{s \in \operatorname{Map}(L,\Gamma,\rho) \mid \overline{\partial}_I s = 0, s \text{ simple}\}$$

 $\mathcal{M}_{I}(L,\Gamma,\rho) = \{s \in \operatorname{Map}(L,\Gamma,\rho) \mid \overline{\partial}_{I}s = 0, \ s \text{ simple}\}.$ For any $B \in H_{2}(F;\mathbb{Z})$, let also $\mathcal{M}_{I}(L,\Gamma,\rho;B) = \{s \in \mathcal{M}_{I}(L,\Gamma,\rho) \mid s_{*}[\Sigma] = B\}.$

Theorem 4.5. There is a subset $\mathscr{I}^{L,\Gamma,\rho} \subset \mathscr{I}_{\omega,S^1}$ of Baire second category (with respect to the C^{∞} topology on \mathscr{I}_{ω,S^1}) such that for any $I \in \mathscr{I}^{L,\Gamma,\rho}$ the moduli space $\mathcal{M}_I(L,\Gamma,\rho)$ is smooth and oriented. Furthermore, for any $I_0, I_1 \in \mathscr{I}^{L,\Gamma,\rho}$, there exists a path $[0,1] \ni \lambda \mapsto I_{\lambda} \in \mathscr{I}_{\omega,S^1}$ such that the space

$$\bigcup_{\lambda \in [0,1]} \mathcal{M}_{I_{\lambda}}(L,\Gamma,\rho)$$

has a natural structure of smooth oriented cobordism between $\mathcal{M}_{I_0}(L,\Gamma,\rho)$ and $\mathcal{M}_{I_1}(L,\Gamma,\rho)$.

Proof. The proof, with due modifications, is exactly like that of Theorem 2.2 or of Theorem 3.1.2 in [McDS1]. We will give a little more details, since in the course of the proof one needs to use Theorems 4.2 and 4.4. We start considering the completion $\mathscr{I}^l_{\omega,S^1}$ (resp. \mathscr{I}^l_{ω}) of \mathscr{I}_{ω,S^1} (resp. \mathscr{I}^l_{ω}) in the C^l norm, where l > 0 is a big enough integer, and we define for any $B \in H_2(F;\mathbb{Z})$

$$\mathcal{M}_{\mathscr{I}^{l}}(L,\Gamma,\rho;B) = \left\{ (s,I) \in \operatorname{Map}(L,\Gamma,\rho) \times \mathscr{I}^{l}_{\omega,S^{1}} \middle| \begin{array}{c} \overline{\partial}_{I}s = 0, \ s_{*}[\Sigma] = B, \\ \text{and } s \text{ simple} \end{array} \right\}.$$

We next prove that $\mathcal{M}_{\mathscr{I}^l}(L,\Gamma,\rho;B)$ is a smooth Banach manifold. Let $(s,I) \in \mathcal{M}_{\mathscr{I}^l}(L,\Gamma,\rho;B)$ be any point. We have to check that the linearisation

$$D\mathcal{E}(u,I): \Omega^0(s^*TF^L)_{L^p_1}^{\Gamma} \times T_I\mathscr{I}_{\omega,S^1}^{l} \to \Omega^{0,1}_I(s^*TF^L)_{L^p}^{\Gamma}$$

of the equation at (s, I) is surjective. Here we denote by $\Omega_I^{0,1}(s^*TF^L)^{\Gamma}$ the Γ -invariant sections of $\Lambda^{0,1}T\Sigma \otimes_{\mathbb{C}} s^*TF^L$ (the subscript stresses the fact that when tensoring over \mathbb{C} we use the complex structure I; on the other hand, this bundle has an action of Γ through the representation ρ). The tangent space $T_I\mathscr{I}_{\omega,S^1} \subset T_I\mathscr{I}_{\omega}^l$ is equal to the subspace of Γ -invariant elements in $T_I\mathscr{I}_{\omega}^l$. This latter space is the set of C^l sections of the bundle $\operatorname{End}(TF, I, \omega)$ whose fibre at $x \in F$ is the space of linear maps $Y: T_xF \to T_xF$ which satisfy

$$YI + IY = 0$$
 and $\omega(Y \cdot, \cdot) + \omega(\cdot, Y \cdot) = 0$

(see p. 34 in [McDS1]).

We now follow the notation (and the ideas) of the proof of Proposition 3.4.1 in [McDS1]. We may write the differential $D\mathcal{E}(s,I)(\xi,Y) = D_s\xi + \frac{1}{2}Y(s) \circ ds \circ j$, where j is the complex structure in Σ and D_s is a first order differential operator whose symbol coincides with that of Cauchy-Riemann operator. Hence D_s is elliptic and consequently Fredholm. So if $D\mathcal{E}(s,I)$ were not exhaustive there would exist a nonzero element $\eta \in \Omega_I^{0,1}(s^*TF^L)_{L^q}^{\Gamma}$ (where 1/p + 1/q = 1) such that for any $\xi \in \Omega^0(s^*TF^L)^{\Gamma}$ and for any $Y \in T_I \mathscr{I}_{\omega,S^1}^l$

$$\int_{\Sigma} \langle \eta, D_s \xi \rangle = 0 \text{ and } \int_{\Sigma} \langle \eta, Y(s) \circ ds \circ j \rangle = 0.$$
(15)

We now invoque Theorem 4.2 and obtain a disk $D \subset \Sigma$ such that

$$S^{1} \cdot s(D) \cap s(\Sigma) = H \cdot s(D).$$

Using theorem 4.4 we deduce that (after possibly shrinking D) all the elements in s(D) have stabiliser equal to L. Then η vanishes on an open subset of D. For suppose that $\eta(x) \neq 0$, where $x \in D$. One can always find an endomorphism $Y_0 \in \text{End}(T_{s(x)}F, I_{s(x)}, \omega_{s(x)})^L$ such that $\langle \eta(x), Y_0 \circ ds(x) \circ j(x) \rangle \neq 0$, since $\eta(x) \in T_{s(x)}F^L$. We extend Y_0 to $S^1 \cdot s(x)$ in a S^1 -equivariant way (we can do this because $(S^1)_{s(x)} = L$ and we took Y_0 to be L-invariant) and then we use a S^1 -invariant smooth cutoff function to extend Y_0 to a small neighbourhood of $S^1 \circ s(x)$. This can be done in such a way that the right hand side integral in (15) does not vanish. And this is a contradiction.

Consequently η vanishes in D. Since it also satisfies the left hand side equation in (15), Aronszajn's theorem [Ar] (see Theorem 2.1.2 in [McDS1]) implies that η vanishes identically. So $D\mathcal{E}(s, I)$ must be exhaustive, and this finishes the proof that $\mathcal{M}_{\mathscr{I}^l}(L, \Gamma, \rho; B)$ is smooth.

The proof of the first statement in Theorem 4.5 is resumed as Theorem 2.2 or in p. 36 in [McDS1]. One uses the Sard–Smale theorem (for that l has to be big enough, depending on the index of the linearisation $D\mathcal{E}$, which on its turn is a function of $B \in H_2(F;\mathbb{Z})$) to prove the existence of a subset $(\mathscr{I}^{L,\Gamma,\rho})^{B,l} \subset \mathscr{I}^l_{\omega,S^1}$ of the second category such that for any $I \in (\mathscr{I}^{L,\Gamma,\rho})^{B,l}$ the moduli space $\mathcal{M}_I(L,\Gamma,\rho;B)$ is smooth. Then a trick of Taubes allows to deduce from this that there exists a subset $(\mathscr{I}^{L,\Gamma,\rho})^B \subset$ \mathscr{I}_{ω,S^1} of the second category with the same property, but consisting of smooth complex structures and not of C^l ones as before. Since the set of homology classes $B \in H_2(F;\mathbb{Z})$ is countable, the intersection

$$\mathscr{I}^{L,\Gamma,\rho} = \bigcap_{B \in H_2(F;\mathbb{Z})} (\mathscr{I}^{L,\Gamma,\rho})^B$$

is again of the second category.

To finish the proof, note that the linearisation of the equations is, modulo a compact operator, the Cauchy–Riemann operator. Hence the cohomology groups of the deformation complex carry natural orientations (because they are complex vector spaces) and consequently so does the moduli space.

The last statement of the theorem on the independence of the cobordism class for generic I is proved analogously.

We now define

$$\mathscr{I}^{\mathrm{reg}}_{\omega,S^1} = \bigcap_{L,\Gamma,\rho} \mathscr{I}^{L,\Gamma,\rho},$$

where the intersection is taken for the triples (L, Γ, ρ) such that the moduli $\mathcal{M}_{\mathscr{I}}(L, \Gamma, \rho; B)$ is nonempty for some $B \in H_2(F; \mathbb{Z})$. Again, this is a Baire set of the second category.

4.3. Index computations. In this subsection we will compute the dimension of the moduli spaces $\mathcal{M}_I(L,\Gamma,\rho)$ for generic I. Let us fix a triple (L,Γ,ρ) . Recall that L is a subset of S^1 , Γ is a compact subgroup of $\operatorname{Aut}(\mathbb{CP}^1)$ and $\rho:\Gamma \to S^1/L$ is an injection. This latter condition implies that Γ is abelian. Since $\operatorname{Aut}(\mathbb{CP}^1) = \operatorname{PSL}(2;\mathbb{C})$, any element in $\operatorname{Aut}(\mathbb{CP}^1)$ which spans a compact subgroup must fix two points of \mathbb{CP}^1 . And since Γ is abelian, there must exist two points x_+ and x_- which are fixed by all the elements of Γ . Using one of the fixed points, say x_+ , we get an injection $\Gamma \to S^1 \subset \mathbb{C}^*$ by assigning to any $\gamma \in \Gamma$ the induced endomorphism $\iota(\gamma) \in \operatorname{GL}(T_{x_+}\mathbb{CP}^1)$. In the sequel we will identify Γ with its image in S^1 . There are two possibilities. Either Γ is a finite group or $\Gamma \simeq S^1$. When Γ is a finite group, the map ι fixes an isomorphism $\Gamma \simeq \mathbb{Z}/m\mathbb{Z}$, and when Γ is infinite ι gives an identification with S^1 .

If $\Gamma \neq \{1\}$ then, for any (Γ, ρ) -equivariant map $s : \mathbb{CP}^1 \to F$, the fixed points x_{\pm} are mapped by s to the fixed point set F^f (because by assumption the action on $F \setminus F^f$ is free). Let z be a holomorphic coordinate in \mathbb{CP}^1 centered at x_+ . Taking S^1 -equivariant coordinates in a neighbourhood of $s(x_+)$ the map s can be written (see p. 16 in [McDS1]) $s(z) = az^l + O(|z|^{l+1})$, and the constant a can be identified with an element of $T_{s(x_+)}F$. Let $T_{x_{\pm}}^P F$ (resp. $T_{x_{\pm}}^Z F, T_{x_{\pm}}^N Z$) be the subspace of $T_{x_{\pm}}F$ spanned by vectors of weight 1 (resp. 0, -1) under the action of S^1 . Since the action of S^1 on F is semi-free, there are no more weights, and hence a must lie in $T_{x_{\pm}}^P F \cup T_{x_{\pm}}^Z F \cup T_{x_{\pm}}^N F$ (otherwise the vector space it spans would not be invariant under

the action of Γ). Using the local expression of s(z) we may write, for any $\theta \in \Gamma$ and z near x_+ , $s(\theta z) = \rho(\theta) \cdot s(z) = \theta^l s(z)$ modulo $O(|z|^{l+1})$. The \cdot in the second term refers to the action of S^1 on $T_{x_+}F$. From this we deduce that a cannot belong to $T_{x_{\pm}}^Z F$ and that: if $a \in T_{x_{\pm}}^P F$ then $\rho(\theta) = \theta^l$, and if $a \in T_{x_{\pm}}^N F$ then $\rho(\theta) = \theta^{-l}$. In fact, after possibly composing s with the holomorphic map $r : \mathbb{CP}^1 \to \mathbb{CP}^1$ defined r([x : y]) = [y : x] in coordinates for which $x_+ = [0 : 1]$ and $x_- = [1 : 0]$, we may assume that $a \in T_{x_{\pm}}^P F$. Hence $\rho(\theta) = \theta^l$ for any $\theta \in \Gamma$, where l is a positive integer. If $\Gamma = S^1$, then l must be 1, and if $\Gamma = \mathbb{Z}/m\mathbb{Z}$ then l and m must be coprime and the representation ρ only depends on the class of l modulo m.

In the sequel we will write $\mathcal{M}_I(L, \Gamma, l; B)$ instead of $\mathcal{M}_I(L, \Gamma, \rho; B)$. When $\Gamma = 1$ we will write $\mathcal{M}_I(L; B)$ instead of $\mathcal{M}_I(L, \Gamma, \rho)$, and when $L = \Gamma = 1$ we will write $\mathcal{M}_I(B)$.

4.3.1. The deformation complex

Let $B \in H_2(F;\mathbb{Z})$, $I \in \mathscr{I}_{\omega,S^1}^{\mathrm{reg}}$, and $s \in \mathcal{M}_I(L,\Gamma,l;B)$. The deformation complex of the moduli $\mathcal{M}_I(L,\Gamma,l;B)$ at s is

$$D_s^{\Gamma}: \Omega^0(s^*TF^L)^{\Gamma} \to \Omega_I^{0,1}(s^*TF^L)^{\Gamma},$$

where D_s^{Γ} is equal to the Cauchy-Riemann operator modulo a compact operator (see p. 28 in [McDS1]). Since $I \in \mathscr{I}_{\omega,S^1}^{\mathrm{reg}}$, this operator is exhaustive and consequently the dimension of $\mathcal{M}_I^{L,\Gamma,l}(B)$ at s is equal to dim(Ker D_s^{Γ}). To compute this dimension we consider the natural extension of D_s^{Γ}

$$D_s: \Omega^0(s^*TF^L) \to \Omega_I^{0,1}(s^*TF^L)$$
(16)

(this is the deformation complex of the moduli of holomorphic curves in F^L). The operator D_s is Γ -equivariant, and hence acts on the cohomology groups H_s^i of the complex. We have Ker $D_s^{\Gamma} = (H_s^0)^{\Gamma}$ and Coker $D_s^{\Gamma} = (H_s^1)^{\Gamma} = 0$. So the complex dimension at s is equal to

$$\dim T_s \mathcal{M}_I(L,\Gamma,l;B) = \dim (H_s^0)^{\Gamma} - \dim (H_s^1)^{\Gamma}.$$
(17)

This dimension can be computed putting instead of D_u any equivariant Dolbeaut operator on s^*TF^L , since they have the same symbol. Because the action of S^1 on Fis almost-free, we need only distinguish these possibilities.

Case 1. $L = S^1$, $\Gamma = \{1\}$. Let $F^f = F_1 \cup \cdots \cup F_r$ be the connected components of the fixed point set. Suppose that $B \in H_2(F_k; \mathbb{Z}) \subset H_2(F; \mathbb{Z})$. Then by Riemann-Roch the moduli space has dimension

$$\dim \mathcal{M}_I(S^1; B) = 2\langle c_1(TF_k), B \rangle + \dim F_k.$$

Case 2. $L = \{1\}, \Gamma \neq \{1\}$. Let $x_{\pm} \in \mathbb{CP}^1$ be the two points which are fixed by Γ . Since the map s is (Γ, ρ) -equivariant, we have a natural lift of the action ρ of Γ on \mathbb{CP}^1 to $E = s^*TF \to \mathbb{CP}^1$. Let us write it $\gamma : \Gamma \to \operatorname{Aut}(E)$, where $\operatorname{Aut}(E)$ denotes the automorphisms of E as vector bundle. The map γ induces representations γ_{\pm} of Γ on the fibres $E_{x_{\pm}}$ over x_{\pm} . The weights of this representation are l times the weights of the representation of S^1 on $TF_{s(x_{\pm})}$ (which belong to $\{-1, 0, 1\}$). Let P_{\pm} (resp. Z_{\pm}, N_{\pm}) be the number of weights of the representation γ_{\pm} which are equal to 1 (resp. 0, -1). Denote $\operatorname{Ind}_{\gamma}(E) = \dim(H^0_s)^{\Gamma} - \dim(H^1_s)^{\Gamma}$ the Γ -invariant part of the index of the operator D_s on E. We will denote by $\operatorname{rk}(E)$ the complex rank of E.

4.3.2. Case $\Gamma = \mathbb{Z}/m\mathbb{Z}$

Theorem 4.6. Let P_{\pm} (resp. Z_{\pm} , N_{\pm}) be the number of weights of γ_{\pm} which are l (resp. 0, -l). Let l' = l + km for $k \in \mathbb{Z}$ such that $1 \leq l' \leq m - 1$. Then

$$\operatorname{Ind}_{\gamma}(E) = \frac{2}{m} (\deg(E) + m \operatorname{rk}(E) - m(P_{-} + N_{+}) + l'(P_{-} + N_{+} - P_{+} - N_{-})).$$

Proof. We may write

$$m \operatorname{Ind}_{\gamma}(E) = \sum_{k \in \mathbb{Z}/m\mathbb{Z}} \operatorname{Tr}(\gamma(k), H^{0}(E)) - \operatorname{Tr}(\gamma(k), H^{1}(E))$$
$$= 2(\operatorname{deg}(E) + \operatorname{rk}(E)) + \sum_{k=1}^{m-1} \operatorname{Tr}(\gamma(k), H^{0}(E)) - \operatorname{Tr}(\gamma(k), H^{1}(E)),$$

by Riemann–Roch, where $\text{Tr}(\gamma(k), H^i(E))$ denotes the trace of $\gamma(k)$ acting on $H^i(E)$. We will compute the value of

$$\operatorname{Tr}(\gamma(k), H^0(E)) - \operatorname{Tr}(\gamma(k), H^1(E))$$

for $1 \le k \le m - 1$ using Atiyah-Bott fixed point theorem (see [BeGeV]).

Theorem 4.7 (Atiyah-Bott). Let M be a compact complex manifold and $W \to M$ a holomorphic vector bundle. Let $g: M \to M$ be a complex diffeomorphism which lifts to $g: W \to W$. Suppose that the fixed points of g are isolated. Then

$$\sum_{i} (-1)^{i} \operatorname{Tr}_{\mathbb{C}}(g, H^{i}(W)) = \sum_{x_{0} \in M^{g}} \frac{\operatorname{Tr}_{\mathbb{C}}(g_{x_{0}}^{W})}{\det_{T_{x_{0}}^{1,0}M}(1 - g_{x_{0}}^{-1})},$$

where $H^i(W)$ is the *i*-th Dolbeaut cohomology group and $g_{x_0}^W : W_{x_0} \to W_{x_0}$ is the complex linear endomorphism of the fibres over the fixed points induced by g (we use the determinant of complex endomorphisms).

In our case we have for any $1 \leq k \leq m-1$ a complex diffeomorphism $\rho_m(k) \in \operatorname{Aut}(\mathbb{CP}^1)$ whose fixed points are x_{\pm} . Let $\theta = \exp(2\pi \mathbf{i}/m)$. We then have

$$\det(1 - \rho_m(k)_{x\pm}^{-1}) = (1 - \theta^{\pm 1}).$$

Let $N = \operatorname{rk}(E)$ and let $b_{\pm}^1, \ldots, b_{\pm}^N \in \mathbb{Z}/m\mathbb{Z}$ be the weights of γ_{\pm} . Then

$$\operatorname{Tr}_{\mathbb{C}}(\gamma(k)_{\pm}) = \sum_{j=1}^{N} \theta^{b_{\pm}^{j}k}.$$

So using Theorem 4.7 we conclude that

$$\operatorname{Ind}_{\gamma}(E) = \frac{2}{m} \left(\operatorname{deg}(E) + \operatorname{rk}(E) + \sum_{k=1}^{m-1} \sum_{j=1}^{N} \left(\frac{\theta^{b_{+}^{j}k}}{1 - \theta^{-k}} + \frac{\theta^{b_{-}^{j}k}}{1 - \theta^{k}} \right) \right).$$
(18)

Lemma 4.8. Let $\theta = \exp(2\pi \mathbf{i}/m)$. Then for $1 \le w \le m-1$

$$\sum_{k=1}^{m-1} \frac{1}{1-\theta^k} = \sum_{k=1}^{m-1} \frac{1}{1-\theta^{-k}} = \frac{m-1}{2}$$
$$\sum_{k=1}^{m-1} \frac{\theta^{wk}}{1-\theta^k} = \sum_{k=1}^{m-1} \frac{\theta^{-wk}}{1-\theta^{-k}} = -\frac{m-1}{2} + w - 1$$

Proof. Let $f(x) = \prod_{k=1}^{m-1} (x - \theta^k)$. We have $f(x) = 1 + x + \dots + x^{m-1}$ and $\sum_{k=1}^{m-1} \frac{1}{1 - \theta^k} = \frac{f'(1)}{f(1)} = \frac{m(m-1)/2}{m} = \frac{m-1}{2}.$

In general, for any $1 \le w \le m - 1$

$$\sum_{k=1}^{m-1} \frac{\theta^{wk}}{1-\theta^k} = \sum_{k=1}^{m-1} \left(-\frac{1-\theta^{wk}}{1-\theta^k} + \frac{1}{1-\theta^k} \right) = \sum_{k=1}^{m-1} -(1+\theta^k+\dots+\theta^{(w-1)k}) + \frac{m-1}{2}$$
$$= -\frac{m-1}{2} + w - 1,$$

since, for any $w \in \mathbb{Z}$, $\sum_{k=1}^{m-1} \theta^{wk}$ is m-1 if $m \mid w$ and -1 otherwise.

Now, combining the above lemma with (18) we get

$$\operatorname{Ind}_{\gamma}(E) = \frac{2}{m} (\deg(E) + m \operatorname{rk}(E) - m(P_{-} + N_{+}) + l'(P_{-} + N_{+} - P_{+} - N_{-})),$$

which is what we wanted to prove.

4.3.3. Case $\Gamma = S^1$

Theorem 4.9. Let P_{\pm} (resp. Z_{\pm} , N_{\pm}) be the number of weights of γ_{\pm} which are 1 (resp. 0, -1). Then

$$\operatorname{Ind}_{\gamma}(E) = 2(\operatorname{rk}(E) - (P_{-} + N_{+}))$$

Proof. For any $p \in \mathbb{N}$, let $\Gamma_p = \mathbb{Z}/2^p\mathbb{Z} \subset S^1$, and consider the action $\gamma_p : \Gamma_p \to \operatorname{Aut}(E)$ induced by γ . Then we clearly have $\operatorname{Ind}_{\gamma}(E) = \lim_{p \to \infty} \operatorname{Ind}_{\gamma_p}(E)$, and the equality follows then from Theorem 4.6.

Using the formula $\deg(E) = P_+ + N_- - P_- - N_+$, the above index can also be written $\operatorname{Ind}_{\gamma}(E) = 2(\deg(E) + \operatorname{rk}(E) - (P_+ + N_-)).$

4.3.4. An inequality

Lemma 4.10. Assume that $\Gamma \neq \{1\}$ and that $\deg(E) > 0$. Then we have

$$\operatorname{Ind}_{\gamma}(E) \le 2(\deg(E) + \operatorname{rk}(E)) - 4.$$

Proof. Suppose to begin that $\Gamma = \mathbb{Z}/m\mathbb{Z}$ and that $1 \leq l' \leq m-1$ is as in Theorem 4.6. Since $\operatorname{Ind}_{\gamma}(E)$ is an even integer (because both $H^0_s(E)$ and $H^1_s(E)$ are complex spaces and the action of Γ respects the complex structure, hence $H^0_s(E)^{\Gamma}$ and $H^1_s(E)^{\Gamma}$ are both complex spaces), it is enough for our purposes to prove

$$\frac{1}{2}\operatorname{Ind}_{\gamma}(E) \le \deg(E) + \operatorname{rk}(E) - (1 + 1/m).$$

Writing the value of $\operatorname{Ind}_{\gamma}(E)$ given by Theorem 4.6, multiplying by m and simplifying we arrive at the (equivalent) inequality

$$m+1 \le (m-1)\deg(E) + (m-l')(P_- + N_+) + l'(P_+ + N_-),$$

which is a consequence of $P_+ + N_+ + P_- + N_- \ge 2$ and $\deg(E) \ge 1$, taking into account that $m - l' \ge 1$ and $l' \ge 1$.

The case $\Gamma = S^1$ can be deduced from the previous one using the same limit trick as in the proof of Theorem 4.9.

4.4. Evaluation maps are submersions. Let $B \in H_2(F; \mathbb{Z})$. In this subsection we will generalise the result in §6.1 of [McDS1] for curves in $\mathcal{M}_I(L, \Gamma, \rho; B)$. For any $x \in \mathbb{CP}^1$ we have an evaluation map

$$\operatorname{ev}_x : \mathcal{M}_{\mathscr{I}} = \mathcal{M}_{\mathscr{I}}(L, \Gamma, \rho; B) \to F$$

which sends any $s \in \mathcal{M}_{\mathscr{I}}$ to $\operatorname{ev}_x(s) = s(x)$. When $\Gamma = 1$ theorem 6.1.1 in [McDS1] says that the map ev_x is a submersion. When $\Gamma \neq 1$ this need not hold any longer. In fact, we must distinguish two possibilities. If $x \neq x_{\pm}$, then the map $\operatorname{ev}_x : \mathcal{M}_{\mathscr{I}} \to F$ is a submersion, and if $x = x_{\pm}$ then the evaluation map ev_x takes values in F^f and the map $\operatorname{ev}_x : \mathcal{M}_{\mathscr{I}} \to F^f$ is a submersion. We state this in the following lemma.

Lemma 4.11. Suppose that $\Gamma \neq S^1$. Given $I \in \mathscr{I}_{\omega,S^1}$, a curve $s \in \mathcal{M}_I(L, \Gamma, \rho; B)$ and a point $x \in \mathbb{CP}^1$ different from x_{\pm} (resp. equal to x_{\pm}) there exists $\delta > 0$ such that for any $v \in T_{s(x)}F^L$ (resp. for any $v \in T_{s(x)}F^f$) and every $0 < \rho < r < \delta$ there exists a smooth Γ -equivariant vector field $\xi \in \Omega^0(s^*TF^L)^{\Gamma}$ and an infinitesimal variation of almost complex structure $Y \in T_I \mathscr{I}_{\omega,S^1}$ (see Theorem 4.5) such that the following holds

- i) $D_s\xi + \frac{1}{2}Y(s) \circ ds \circ j = 0$ (that is, the pair (ξ, Y) belongs to $T_{(s,I)}\mathcal{M}_{\mathscr{I}}$),
- *ii*) $\xi(x) = v$ and

iii) ξ is supported in $\Gamma \cdot B_{\delta}(x)$ and Y is supported in and arbitrarily small neighbourhood of $s(\Gamma \cdot (B_r(x) \setminus B_{\rho}(x)))$.

Proof. Since the proof is almost the same as that of Lemma 6.1.2 in [McDS1], we will just give a sketch and mention the differences. The first thing to do is to find a local solution ξ_0 of $D_s \xi = 0$ in $B_{\delta}(x)$ satisfying $\xi_0(x) = v$. This is done by solving a boundary value problem (see proposition 4.1 in [McD1] and the references therein). Then one multiplies ξ by a cutoff function with support in a neighbourhood of $B_r(x) \setminus B_{\rho}(x)$ to extend ξ_0 to a section of s^*TF^L . One then averages ξ_0 by the action of Γ and obtains a section $\xi \in \Omega^0(s^*TF^L)^{\Gamma}$. Finally, one must modify I by a suitable infinitesimal $Y \in T_I \mathscr{I}_{\omega,S^1}$ so that i) is satisfied (in order to take Y Γ -equivariant one needs to be careful with the fixed point locus of the action of S^1 ; this may be done using theorem 4.4, as was done in the proof of Theorem 4.5). This Y can be taken fulfilling property *iii*), repeating the argument in [McDS1] but taking into account Γ -equivariance. **Definition 4.12.** We will say that a point $x \in \mathbb{CP}^1$ is critical with respect to the tuple (L, Γ, ρ) if either $L = S^1$ or $\Gamma \neq 1$ and $x = x_{\pm}$.

5. Definition of the invariants

5.1. **Pseudo-cycles.** In this subsection we will review some basic facts about pseudo-cycles. All the results which we will use are taken from §7.1 in [McDS1].

Let X be a smooth compact m-dimensional manifold, and let $R \subset X$ be a subset. We will say that R has dimension at most k (and write dim $R \leq k$) if R is contained in the image of a smooth map $g: W \to X$, where W is a σ -compact k dimensional smooth manifold (recall that a space is σ -compact if it can be covered by countably many compact sets).

Given a map $f: M \to X$ we define the boundary Ω_f of f(M) to be

$$\Omega_f = \bigcap_{K \subset M} \overline{f(M \setminus K)},$$

where the intersection runs over all the compact subsets K of M. The set $\Omega_f \subset X$ coincides with the set of all points in X which are limit of sequences $f(m_j)$, where m_j has no convergent subsequence in M.

Definition 5.1. A k-dimensional pseudo-cycle is a smooth map $f : M \to X$, where dim M = k and M is oriented, such that $\Omega_f \leq k-2$. Two k-dimensional pseudo-cycles $f_0 : M_0 \to X$ and $f_1 : M_1 \to X$ are bordant if there exists an oriented cobordism Wbetween M_0 and M_1 and a smooth map $F : W \to X$ extending f_0 and f_1 such that dim $\Omega_F \leq k-1$.

Given two pseudo-cycles in X of complementary dimension there is a well defined intersection number between them (which coincides with the usual one if they are cycles, see Lemma 7.1.3 in [McDS1]). On the other hand, any homology class in Xcan be represented by a pseudo-cycle (see Remark 7.1.1 in [McDS1]). Using these two facts, one can define a pairing between pseudo-cycles and homology classes in X. The following is Lemma 7.1.4 in [McDS1].

Lemma 5.2. Every k-dimensional pseudo-cycle $f : M \to X$ defines canonically a map

 $\Psi_f: H_{m-k}(X; \mathbb{Z}) \to \mathbb{Z}.$

Furthermore, if f and f' are bordant, then $\Psi_f = \Psi_{f'}$.

5.2. Construction of a finite dimensional target. The strategy in [McDS1] to define Gromov–Witten invariants consists of proving that the evaluation map (which takes values in the compact symplect manifold) is a pseudo-cycle. Our aim is to follow the same idea.

Suppose that $\beta \in H_2^{S^1}(F;\mathbb{Z})$, $c \in i\mathbb{R} \setminus \mathscr{C}$, and $\rho \in \mathscr{P}_c^{\text{reg}}$ have been fixed. Let $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_{\rho}(\beta, c)$ and $\mathcal{M} = \mathcal{M}_{\rho}(\beta, c)$. Take a positive integer p. To give a rigorous sense to formula (6) when \mathcal{M} is not compact, we proceed as follows. Let

$$\mathbb{E}^p = \mathscr{A} \times \mathscr{S} \times E^p \to \mathscr{A} \times \mathscr{S} \times \Sigma^p$$

be the $(S^1)^p$ -bundle pullback of $E^p \to \Sigma^p$ by the projection $\pi_{\Sigma} : \mathscr{A} \times \mathscr{S} \times \Sigma^p \to \Sigma^p$. This bundle has a natural action of \mathscr{G} , and its restriction to $\widetilde{\mathcal{M}} \times \Sigma^p$ descends to give a $(S^1)^p$ -bundle $\mathbb{E}^p \to \mathcal{M} \times \Sigma^p$. Furthermore, the canonical section

$$\widetilde{\Phi} : \mathscr{A} \times \mathscr{S} \times \Sigma^p \to \widetilde{\mathbb{E}}^p \times_{(S^1)^p} F^p = \mathscr{A} \times \mathscr{S} \times F^p (A, \phi, x_1, \dots, x_p) \mapsto (A, \phi, \phi(x_1), \dots, \phi(x_p))$$

is \mathscr{G} -equivariant and hence descends to give a section Φ of the bundle $\mathbb{E}^p \times_{(S^1)^p} F^p$. Now, Lemma 1.3 allows to obtain from Φ a map

$$\operatorname{ev}_{S^1}^p := \Phi_{\mathbb{E}^p} : \mathcal{M} \times \Sigma^p \to (F^p)_{(S^1)^p} = (F_{S^1})^p.$$

We would like to treat $ev_{S^1}^p$ as a pseudo-cycle.

However, $(F_{S^1})^p$ is not a finite dimensional manifold. To solve this problem, we will construct a compact oriented smooth manifold T with a $(S^1)^p$ principal bundle $E_T^p \to T$ and a Cartesian diagram

We will call such a diagram a smooth compact model of \mathbb{E}^p . Let

$$s_{F^p}^T : \mathbb{E}^p \times_{(S^1)^p} F^p \to E_T^p \times_{(S^1)^p} F^p$$

be the induced map. We will prove that $s_{F^p}^T \Phi : \mathcal{M} \times \Sigma^p \to E_T^p \times_{(S^1)^p} F^p$ is a pseudocycle. That this is consistent with our definition, and that the result is independent of the approximation $E_T^p \to T$ is proved by formula (4) in the remark after Lemma 1.3.

In the rest of this subsection we will construct the smooth compact model $E_T^p \to T$ and in the next one we will prove that $s_{F^p}^T \Phi$ is a pseudo-cycle.

5.2.1. For any finite subset $P = \{p_1, \ldots, p_N\} \subset \Sigma$, we let $F_P = \prod_{p \in P} \mathcal{F}_p$, and we define the map

$$e_P: \mathscr{A} \times \mathscr{S} \to F_P \\ (A, \phi) \mapsto (\phi(p_1), \dots, \phi(p_N)).$$

Consider the action of S^1 on F_P induced by the action of the constant gauge transformations on \mathcal{F} , and let F_P^f denote the fidxed point set of this S^1 -action (in fact $F_P \subset \mathscr{S}$ supports an action of the full gauge group \mathscr{G}). We have $F_P^f = \prod_{p \in P} \mathcal{F}_p^f$, where $\mathcal{F}^f = E \times_{S^1} F^f$.

Lemma 5.3. One can take $P \subset \Sigma$ such that $\overline{e_P(\widetilde{\mathcal{M}})} \cap F_P^f = \emptyset.$

Proof. For any $\epsilon > 0$ we will denote $P_{\epsilon} \subset \Sigma$ any finite subset such that the disks of radius ϵ centered at the points $p \in P_{\epsilon}$ cover Σ . Suppose that the claim of the lemma is not true. Then there exists a sequence $\epsilon_j \to 0$, sets P_{ϵ_j} and ρ -THCs $(A_j, \phi_j) \in \widetilde{\mathcal{M}}_{\rho}(\beta, c)$ so that for any j the image of the points in P_{ϵ_j} by the section ϕ_j is contained in $\mathcal{F}^f = E \times_{S^1} F^f$. By the compactness Theorem 3.4 one may take a subsequence of (A_j, ϕ_j) which, after suitably regauging, converge pointwise to a cusp ρ -THC. Now,

by construction, the image of the principal component Σ_0 of this limit cusp must be inside \mathcal{F}^f . But this is in contradiction with our assumption that $\rho \in \mathscr{P}_c$.

Let us take a subset $P \subset \Sigma$ as given by the preceding lemma. Let N be a \mathscr{G} invariant tubular neighbourhood of the fixed point set F_P^f which does not meet the
closure of $e_P(\widetilde{\mathcal{M}}_{\rho}(\beta, c))$. Take S to be two copies of $F_P \setminus N$ glued along ∂N :

$$S = (F_P \setminus N) \cup_{\partial N} - (F_P \setminus N).$$

Then S supports an action of \mathscr{G} , and no point in S is fixed by a nontrivial constant gauge transformation. Furthermore, the map e_P gives a \mathscr{G} -equivariant map $s_P: \widetilde{\mathcal{M}}_{\rho}(\beta, c) \to S.$

5.2.2. Let d be the degree of E, and let

$$\mathscr{A}_{\mathrm{Fl}} = \{ A \in \mathscr{A} \mid F_A = -\mathbf{i} 2d\pi \omega_{\Sigma} / \operatorname{Vol}(\Sigma) \}$$

be the set of projectively flat connections (here ω_{Σ} is the symplectic form in Σ). Then

$$\operatorname{Jac}_d(X) := \mathscr{A}_{\operatorname{Fl}}/\mathscr{G}$$

is a torus of real dimension twice the genus of Σ . We will construct a retraction $\mathscr{A}/\mathscr{G} \to \operatorname{Jac}_d(\Sigma)$. Recall that we have a metric on Σ , which induces metrics on the exterior algebra of forms $\Omega^*(\Sigma)$. Let \mathcal{H}^j be the space of harmonic *j*-forms with respect to this metric.

Let

$$G: \mathscr{A}/\mathscr{G} \to \Omega^2(\mathbf{i}\mathbb{R})$$

be the map which sends any [A] to $F_A + \mathbf{i}2\pi d\omega_{\Sigma}/\operatorname{Vol}(\Sigma)$. It is easy to see, using Hodge theory, that the image of G is the orthogonal of $\mathbf{i}\mathcal{H}^2(\Sigma)$ in $\Omega^2(\mathbf{i}\mathbb{R})$. The preimage of $0 \in \Omega^2(\mathbf{i}\mathbb{R})$ is precisely $\operatorname{Jac}_d(\Sigma)$. In fact, $G : \mathscr{A}/\mathscr{G} \to \mathbf{i}\mathcal{H}^2(\Sigma)^{\perp}$ is a smooth fibration with fibres diffeomorphic to $\operatorname{Jac}_d(\Sigma)$. We will construct a connection on this fibration by specifying its horizontal distribution.

Given any $[A] \in \mathscr{A}/\mathscr{G}$, the tangent space $T_{[A]}\mathscr{A}/\mathscr{G}$ can be canonically identified with Ker d_1^* , where $d_1 : \Omega^0(\mathbf{i}\mathbb{R}) \to \Omega^1(\mathbf{i}\mathbb{R})$ is the exterior derivation. Then we set the horizontal space at [A] to be

$$(T_{[A]}\mathscr{A}/\mathscr{G})_h := \operatorname{Ker} d_1^* \cap (\operatorname{Ker} d_2)^{\perp},$$

where $d_2: \Omega^1(\mathbf{i}\mathbb{R}) \to \Omega^2(\mathbf{i}\mathbb{R})$ is the exterior derivation. Now, using parallel transport along lines going through $-\mathbf{i}2\pi d\omega_{\Sigma}/\operatorname{Vol}(\Sigma) \in \Omega^2(\mathbf{i}\mathbb{R})$ we get the desired retraction

$$R: \mathscr{A}/\mathscr{G} \to \operatorname{Jac}_d(\Sigma)$$

5.2.3. By the definition of S, the group \mathscr{G} acts freely on $\mathscr{A} \times S$ (since the stabiliser of any connection is the set of constant gauge transformations). We define

$$T := (\mathscr{A}_{\mathrm{Fl}} \times S) / \mathscr{G} \times \Sigma^p$$

To define the bundle $E_T^p \to T$ we do the following construction. Let

$$\widetilde{E}_{S}^{p} = \mathscr{A} \times S \times E^{p} \to \mathscr{A} \times S \times \Sigma^{p},$$

and let $E_S^p = \widetilde{E}_1^p / \mathscr{G} \to (\mathscr{A} \times S) / \mathscr{G} \times \Sigma^p$ be the quotient bundle (that \widetilde{E}_S^p descends follows from the fact that the action of \mathscr{G} on $\mathscr{A} \times S$ is free). Let $j_1 : \operatorname{Jac}_d(\Sigma) \to \mathscr{A} / \mathscr{G}$ and $j_2: T \to (\mathscr{A} \times S)/\mathscr{G} \times \Sigma^p$ be the inclusions, and set $E_T^p := j_2^* E_S^p$. Let $j_3: E_T^p \to E_S^p$ be the inclusion. We have the following diagram:



The arrows in the second column are fibrations. Taking a connection on the total fibration $E_S^p \to \mathscr{A}/\mathscr{G}$ we extend the retraction R to obtain retractions R_2 and R_3 of the inclusions j_2 and j_3 . Then we get a Cartesian diagram



On the other hand, we have a \mathscr{G} -equivariant map

$$\begin{array}{rccc} \mathcal{M} \times \Sigma^p & \to & \mathscr{A} \times S \times \Sigma^p \\ (A, \phi, x_1, \dots, x_p) & \mapsto & (A, \phi(p_1), \dots, \phi(p_N), x_1, \dots, x_p), \end{array}$$

and similarly a lift $\widetilde{\mathcal{M}} \times E^p \to \mathscr{A} \times S \times E^p$. Dividing out by the action of \mathscr{G} and composing with R_3 and R_2 we get the desired Cartesian diagram (19).

5.3. $s_F^T \circ \Phi$ is a pseudo-cycle. There are now two things to prove. The first one is that, for a generic choice of complex structure and perturbation ρ , the map which we obtain using the finite dimensional approximation (19) is a pseudo-cycle. The second one is that the bordism class of this pseudo-cycle is independent of the complex structure and the perturbation, and that it only depends on the connected component of $\mathbf{i}\mathbb{R} \setminus \mathscr{C}$ in which c lies.

Theorem 5.4. (i) Let $\beta \in H_2^{S^1}(F;\mathbb{Z})$ and $c \in i\mathbb{R} \setminus \mathscr{C}$. Let $\mathscr{Q} = \mathscr{P}_c \times \mathscr{I}_{\omega,S^1}$. There is a subset $\mathscr{Q}^{\text{reg}} \subset \mathscr{Q}$ of Baire of the second category (with respect to the C^{∞} topology on \mathscr{Q}) such that if $(\rho, I) \in \mathscr{Q}^{\text{reg}}$, $\mathcal{M} = \mathcal{M}_{\rho,I}(\beta, c)$, and $s_F^T : \mathbb{E}^p \times_{(S^1)^p} F^p \to E_T^p \times_{(S^1)^p} F^p$ is the map induced by s^T , then $s_F^T \circ \Phi : \mathcal{M} \times \Sigma^p \to E_T^p \times_{(S^1)^p} F^p$ is a pseudo-cycle.

(ii) If $c' \in \mathbf{i}\mathbb{R} \setminus \mathscr{C}$ belongs to the same connected component of $\mathbf{i}\mathbb{R} \setminus \mathscr{C}$ as $c, \mathscr{Q}' = \mathscr{P}_{c'} \times \mathscr{I}_{\omega,S^1}, \ (\rho',I') \in \mathscr{Q}'^{\text{reg}}$ then, denoting $\mathcal{M}' = \mathcal{M}_{\rho',I'}(\beta,c')$ and $\mathbb{E}'^p \to \mathcal{M}' \times \Sigma^p$ the universal bundle, we can chose $E_T^p \to T$ such that there is a diagram like (19) and another one with $\mathbb{E}^p \to \mathcal{M} \times \Sigma^p$ replaced by $\mathbb{E}'^p \to \mathcal{M}' \times \Sigma^p$, and the two resulting psedo-cycles are bordant.

Proof. Recall that we write F_1, \ldots, F_r for the connected components of the fixed point set F^f . For any natural number K, \mathbf{K} will denote the set $\{1, \ldots, K\} \subset \mathbb{N}$ and $\mathbf{K}_0 = \{0\} \cup \mathbf{K}$. We define the framing $D = D(\Sigma^c, E, A, \phi, c)$ of the cusp ρ -THC (Σ^c, E, A, ϕ, c) to be the following set of data:

- 1. The element $c \in \mathbf{i}\mathbb{R}$.
- 2. The class $\beta_0 = \rho_{E_*} \phi_{0_*}[\Sigma_0] \in H_2^{S^1}(F; \mathbb{Z}).$ 3. The number K of bubbles in Σ^c .
- 4. Homology classes $B_1, \ldots, B_K \in H_2(F;\mathbb{Z})$ describing the image of the bubbles Σ_k in \mathcal{F} (these classes are well defined because the action of S^1 on $H_*(F;\mathbb{Z})$ is trivial).
- 5. A set $S \subset \mathbf{K}^2$ containing the pairs (i, j) such that $\phi_i(X_i) = \phi_j(X_j)$.
- 6. For every $k \in \mathbf{K}$ the tuple (L_k, Γ_k, ρ_k) such that, after identifying $\mathcal{F}_{x_k} \simeq F$, the bubble map ϕ_k belongs to $\mathcal{M}(L_k, \Gamma_k, \rho_k; B_k)$.
- 7. For any $k \in \mathbf{K}$ such that $L_k = S^1$, the number c(k) such that $\phi(\Sigma_k) \subset F_{c(k)}$; and for any $k \in \mathbf{K}$ such that $\Gamma_k \neq 1$, numbers $c(k)_+$ and $c(k)_-$ such that $\phi(x_{\pm}^k) \in$ $F_{c(k)_{\pm}}$, where $x_{\pm}^k \in \Sigma_k$ are the critical points of the bubble Σ_k .
- 8. A set $C \subset \mathbf{K}_0^2$ containing the pairs (i, j) such that i < j and $X_i \cap X_j \neq \emptyset$.
 - (a) A partition $C = C_{00} \cup C_{01} \cup C_{10} \cup C_{11}$ defined as follows. For any pair $(i, j) \in C$, let $x = X_i \cap X_j$. Put $\epsilon(i)$ to be 1 if $x \in X_i$ is a critical point with respect to (L_i, Γ_i, ρ_i) and 0 otherwise (see Definition 4.12), and define $\epsilon(j)$ similarly. Then (i, j) belongs to $C_{\epsilon(i)\epsilon(j)}$.
 - (b) A set $C'' \subset \mathbf{K}^3$ containing the sets (i, j, k) such that $\Gamma_i(X_i \cap X_j) \cap (X_i \cap X_k) \neq \emptyset$ (this is a subset of X_i).

Definition 5.5. We will say that two cusp ρ -THCs (Σ^c , E, A, ϕ , c) and (${\Sigma'}^c$, E', A', ϕ' , c) are equivalent if

- 1. they have the same framing;
- 2. there is an isomorphism $g: E \xrightarrow{\simeq} E'$ of bundles over $\Sigma_0 = \Sigma = \Sigma'_0$ such that $g^*A' = A \text{ and } g^*\phi'_0 = \phi_0;$
- 3. let K be the number of bubbles in both ρ -THCs; for any $1 \le k \le K$ there is an isomorphism $g_k : \Sigma_k \to \Sigma'_k$ such that $\phi_k = \phi'_k g_k$; furthermore, if $\Gamma_k \neq \{1\}$, then g_k maps the critical points of Σ_k to those of Σ'_k .

For example, if two cusp ρ -THCs are ρ -THCs (i.e., they have no bubbles) then they are equivalent if and only if they are gauge equivalent.

Definition 5.6. We denote by $\mathcal{M}'_{\rho,I}(D)$ the set of equivalence classes of cusp ρ -THCs with framing D. We define the total homology class of the frame D to be the equivariant homology class $\beta(D) = \beta_0 + \iota_* B_1 + \dots + \iota_* B_K \in H_2^{S^1}(F; \mathbb{Z}).$

5.3.1. The maps ϕ_k in a cusp curve may be multicovered. This means that ϕ_k factors as $\phi'_k \circ r_k$, where $r_k : \Sigma_k \to \Sigma_k$ is a ramified covering. If r_k has maximal degree, we will call ϕ'_k the simplification of ϕ . This will be a simple map.

For any cusp ρ -THC with frame D we make the following reduction process. First we forget the bubbles whose map to \mathcal{F} is constant, then we substitute the bubble maps ϕ_k by their simplifications ϕ'_k , and then we identify bubbles with the same image in \mathcal{F} . Finally, if necessary we forget some intersection points in order that no two irreducible components of the cusp we have obtained intersect at more than one point. After this process we end up with another cusp ρ -THC with frame D. We call the resulting cusp ρ -THC a reduced cusp. We will denote $\mathcal{M}_{\rho,I}(\overline{D}) \subset \mathcal{M}'_{\rho,I}(\overline{D})$ the set of equivalence classes of reduced cusp ρ -THCs with framing \overline{D} .

Note that the total homology class of \overline{D} will not necessarily be equal to that of D. If $\beta_0, \overline{B}_1, \ldots, \overline{B}_K$ are the homology classes of \overline{D} we will have

$$\beta(D) = \beta_0 + r_1 \iota_* \overline{B}_1 + \dots + r_K \iota_* \overline{B}_K,$$

where $r_k \geq 1$ are integers. This motivates the following definition.

Definition 5.7. If $\beta \in H_2(F_{S^1}; \mathbb{Z})$ and the homology classes $\beta_0, B_1, \ldots, B_K$ of a frame D satisfy $\beta = \beta_0 + r_1 \iota_* B_1 + \cdots + r_K \iota_* B_K$ for some integers $r_k \ge 1$, then we will say that the frame D is β -admisible.

We will denote $\mathcal{D}(B, c)$ the set of β -admisible framings D such that c(D) = c. This is obviously a numerable set. It contains a distinguished element D^T which represents the cusp curve with no bubbling. We will call D^T the top framing of B.

Through all the rest of the proof dim and codim will denote real dimension and codimension. Recall that the dimension of F is 2n.

5.3.2. For any tuple (L, Γ, ρ) denote by $\operatorname{Map}_{\operatorname{fibr}}(L, \Gamma, \rho)$ the set of maps $\phi : \mathbb{CP}^1 \to \mathcal{F}$ whose image is included in a single fibre \mathcal{F}_x and such that $\phi \in \operatorname{Map}(L, \Gamma, \rho)$. We have

$$\operatorname{Map}_{\operatorname{fibr}}(L,\Gamma,\rho) = E \times_{S^1} \operatorname{Map}(L,\Gamma,\rho).$$

Define for any homology class $B \in H_2(F; \mathbb{Z})$

$$\mathcal{M}_{\mathscr{I},\mathrm{fibr}}(L,\Gamma,\rho;B) = \left\{ (\phi,I) \in \mathrm{Map}_{\mathrm{fibr}}(L,\Gamma,\rho) \times \mathscr{I}_{\omega,S^1} \middle| \begin{array}{l} \overline{\partial}_I \phi = 0, \ \phi_*[\Sigma] = B \\ \phi \text{ simple} \end{array} \right\}.$$

Incidentally, this is the moduli space used to defined fibrewise and equivariant quantum cohomology by Givental, Kim and Lu (see [GiKm, Lu]). Note that we have $\mathcal{M}_{\mathscr{I},\text{fibr}}(L,\Gamma,\rho;B) = E \times_{S^1} \mathcal{M}_{\mathscr{I}}(L,\Gamma,\rho;B)$. Let $\pi : \mathcal{M}_{\mathscr{I},\text{fibr}}(L,\Gamma,\rho;B) \to \mathscr{I}_{\omega,S^1}$ be the projection. Just as in the proof of Theorem 4.5 one can check that, for generic $I \in \mathscr{I}_{\omega,S^1}$, the preimage $\mathcal{M}_{I,\text{fibr}}(L,\Gamma,\rho;B) = \pi^{-1}(I)$ is a smooth manifold of dimension

$$\dim \mathcal{M}_{I,\text{fibr}}(L,\Gamma,\rho;B) = \dim \mathcal{M}_{I}(L,\Gamma,\rho;B) + 2.$$

5.3.3. Fix a β -admisible frame D and suppose that the element $c = c(D) \in \mathbf{i}\mathbb{R}$ lies in the complementary of \mathscr{C} . Let K be the number of bubbles, $C \subset \{0, 1, \ldots, K\}^2$ the set of pairs describing which irreducible components intersect, $\beta_0, B_1, \ldots, B_K$ the homology classes of D, and (L_k, Γ_k, ρ_k) the tuples telling the moduli in which ϕ_k sits. We denote $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_K$ the irreducible components of the cusps with frame D. Define for any k the group G_k to be \mathbb{C}^* if $\Gamma_k \neq 1$ and $\mathrm{PSL}(2; \mathbb{C})$ if $\Gamma_k = 1$. When $\Gamma_k \neq 1$, we make G_k act on \mathbb{CP}^1 by rotations keeping fixed the points [1:0]and [0:1], and in the other case we consider the usual action of $G_k = \mathrm{PSL}(2; \mathbb{C})$ on \mathbb{CP}^1 . The group G_k acts effectively on $\mathcal{M}_I(L_k, \Gamma_k, \rho_k; B_k)$ by reparametrization: any $s \in \mathcal{M}_I(L_k, \Gamma_k, \rho_k; B_k)$ is mapped by $g \in G_k$ to $g(s) := s \circ g : \mathbb{CP}^1 \to F$.

5.3.4. Let us write

$$\mathcal{M}^*_{\mathscr{I}}(D) = \prod_{k=1}^K \mathcal{M}_{\mathscr{I}, \text{fibr}}(L_k, \Gamma_k, \rho_k; B_k) \setminus \Delta,$$

where Δ is the multidiagonal, that is, the set of elements (s_1, \ldots, s_K) such that $s_i = s_j$ for some $i \neq j$. $\mathcal{M}^*_{\mathscr{I}}(D)$ parameterizes tuples of K different holomorphic maps $\phi_k : \mathbb{CP}^1 \to \mathcal{F}$ whose image is contained in any fibre. Reasoning exactly like in the proof of Theorem 4.5 one proves that $\mathcal{M}^*_{\mathscr{I}}(D)$ is a smooth Banach manifold. Similarly, if $I \in \mathscr{I}_{\omega,S^1}$, we define

$$\mathcal{M}_{I}^{*}(D) = \prod_{k=1}^{K} \mathcal{M}_{I, \text{fibr}}(L_{k}, \Gamma_{k}, \rho_{k}; B_{k}) \setminus \Delta,$$

where Δ is the multidiagonal (and does not coincide with the previous one).

5.3.5. Let $\mathcal{F}^f = E \times_{S^1} F^f$. For any pair $e = (j, k) \in C$ we define

$$\mathcal{F}(e) = \begin{cases} \mathcal{F} \times \mathcal{F} & \text{if } e \in C_{00}, \\ \mathcal{F} \times \mathcal{F}^{f} & \text{if } e \in C_{01}, \\ \mathcal{F}^{f} \times \mathcal{F} & \text{if } e \in C_{10}, \\ \mathcal{F}^{f} \times \mathcal{F}^{f} & \text{if } e \in C_{11}, \end{cases}$$

and we write $\Delta(e) \subset \mathcal{F}(e)$ for the diagonal in $\mathcal{F}(e)$. We also set $\Sigma(e) = \Sigma(e, j) \times \Sigma(e, k)$, where $\Sigma(e, j)$ is defined as follows:

- if j = 0 or $\Gamma_j = \{1\}$, then $\Sigma(e, j) := \mathbb{CP}^1$; • if j > 0 and $\Gamma_j \neq \{1\}$, then
- if j > 0 and $\Gamma_j \neq \{1\}$, then - if $\Sigma_j \cap \Sigma_k = x^j_+$ (resp. x^j_-) then $\Sigma(e, j) = [1:0] \in \mathbb{CP}^1$ (resp. $[0:1] \in \mathbb{CP}^1$); - if $\Sigma_j \cap \Sigma_k \neq x^j_\pm$ then $\Sigma(e, j) := \mathbb{CP}^1$,

and $\Sigma(e, k)$ is defined similarly. Observe that the $\Sigma(e)$ depends on the framing D (in particular, on the isotropy pairs of the bubbles of Σ^{c}), and not only on Σ^{c} .

Define

$$\widetilde{\mathcal{M}}_{\mathscr{P},\mathscr{I}}(\beta,c) = \{ (A,\phi,\rho,I) \in \mathscr{A} \times \mathscr{S} \times \mathscr{Q} \mid \phi_{E*}[\Sigma] = \beta \text{ and } (A,\phi,\rho,I) \text{ satisfies (12)} \}.$$

We then have an evaluation map

$$\operatorname{ev}_C: \widetilde{\mathcal{M}}_{\mathscr{P},\mathscr{I}}(\beta_0, c) \times \mathcal{M}^*_{\mathscr{I}}(D) \times \prod_{e \in C} \Sigma(e) \to \prod_{e \in C} \mathcal{F}(e)$$

and a projection

$$\Theta: \widetilde{\mathcal{M}}_{\mathscr{P},\mathscr{I}}(\beta_0, c) \times \mathcal{M}^*_{\mathscr{I}}(D) \times \prod_{e \in C} \Sigma(e) \to \mathscr{I}^{K+1}_{\omega, S^1}$$

Let $\Delta_{\mathscr{I}}$ be the diagonal in $\mathscr{I}_{\omega,S^1}^{K+1}$. Since $c \in i\mathbb{R} \setminus \mathscr{C}$, Theorems 2.2 and 4.5 imply that $\Theta^{-1}(\Delta_{\mathscr{I}})$ is a smooth Banach manifold (this would not be true if we had not removed the multidiagonal in the definition of $\mathcal{M}_{\mathscr{I}}^*(D)$ — a similar thing occurs in Lemma 4.9 in [RuTi]). Let us define

$$\widetilde{\mathcal{R}}_{\mathscr{P},\mathscr{I}}(D) = \Theta^{-1}(\Delta_{\mathscr{I}}) \cap \operatorname{ev}_{C}^{-1}(\prod \Delta(e)).$$

Lemma 4.11, together with its nonequivariant version Lemma 6.1.2 in [McDS1], implies that the restriction of ev_C to $\Theta^{-1}(\Delta_{\mathscr{I}})$ is a submersion. Hence, $\widetilde{\mathcal{R}}_{\mathscr{P},\mathscr{I}}(D)$ is a smooth Banach manifold. On the other hand, the gauge group \mathscr{G} acts freely on $\widetilde{\mathcal{R}}_{\mathscr{P},\mathscr{I}}(D)$ (because it acts freely on $\widetilde{\mathcal{M}}(\beta_0, c)$), so

$$\mathcal{R}_{\mathscr{P},\mathscr{I}}(D) := \widetilde{\mathcal{R}}_{\mathscr{P},\mathscr{I}}(D)/\mathscr{G}$$

is also a Banach manifold.

Consider the projection

$$q: \mathcal{R}_{\mathscr{P},\mathscr{I}}(D) \to \mathscr{Q} = \mathscr{P}_c \times \mathscr{I}_{\omega,S^1}$$

This is a Fredholm map, to which we may apply Sard–Smale theorem. We deduce that there exists a subset $\mathscr{Q}^{\mathrm{reg}}(D) \subset \mathscr{Q}$ of Baire of the second category such that for any $(\rho, I) \in \mathscr{Q}^{\mathrm{reg}}(D)$ the set

$$\mathcal{R}_{\rho,I} := q^{-1}(\rho,I)$$

is a smooth Banach manifold of dimension equal to the index of the differential of q, i.e., to the virtual dimension.

Lemma 5.8. Suppose that $\mathcal{D}(\beta, c) \ni D \neq D^T$. If $(\rho, I) \in \mathscr{Q}^{\mathrm{reg}}(D)$ then

$$\dim \mathcal{R}_{\rho,I}(D) \leq \dim \mathcal{M}(\beta,c) - 2 + \sum_{k=1}^{K} \dim G_k$$

Proof. Let $\widetilde{\mathcal{R}}_{\rho,I}(D)$ be the preimage of $\mathcal{R}_{\rho,I}(D)$ by the projection map $\widetilde{\mathcal{R}}_{\mathscr{P},\mathscr{I}}(D) \to \mathcal{R}_{\mathscr{P},\mathscr{I}}(D)$. We then have a commutative diagram

Since \mathscr{G} acts smoothly and freely on the top rows and the vertical arrows are projection to \mathscr{G} orbits, it follows that the codimension of ι is the same as the codimension of $\tilde{\iota}$. Now, $\widetilde{\mathcal{R}}_{\rho,I}(D)$ is the preimage of $\prod \Delta(e)$ by the evaluation map

$$\operatorname{ev}_{C}(\rho, I) : \widetilde{\mathcal{M}}_{\rho, I}(\beta_{0}, c) \times \mathcal{M}_{I}^{*}(D) \times \prod \Sigma(e) \to \prod \mathcal{F}(e),$$

and since $(\rho, I) \in \mathscr{Q}^{reg}(D)$, it follows that this is a submersion. Consequently, the codimension of $\tilde{\iota}$ is equal to the codimension of the inclusion $\prod \Delta(e) \subset \prod \mathcal{F}(e)$. Hence,

$$\dim \mathcal{R}_{\rho,I}(D) - \sum \dim G_k = \dim \mathcal{M}_{\rho,I}(\beta_0, c) + \dim \mathcal{M}_I^*(D) + \sum \dim \Sigma(e) + \sum \dim \Delta(e) - \sum \dim \mathcal{F}(e) - \sum \dim G_k.$$

To bound this dimension we divide the set of bubbles **K** in three subsets. Let **S** (resp. **T** and **U**) denote the set of $k \in \mathbf{K}$ such that $L_k = 1$, $\Gamma_k = 1$ (resp. $L_k = 1$, $\Gamma_k \neq 1$ and $L_k = S^1$, $\Gamma_k = 1$). Let $S = |\mathbf{S}|$, $T = |\mathbf{T}|$ and $U = |\mathbf{U}|$. Theorems 4.6 and 4.9 imply the following.

• If $k \in \mathbf{S}$ then dim $G_k = 6$ and

$$\dim \mathcal{M}_{I,\text{fibr}}(L_k, \Gamma_k, \rho_k; B_k) = 2 + 2\langle c_1(TF), B_k \rangle + 2n$$
$$= 2 + 2\langle c_1^{S^1}(TF), \iota_* B_k \rangle + 2n.$$

• If $k \in \mathbf{T}$ then dim $G_k = 2$ and

$$\dim \mathcal{M}_{I,\text{fibr}}(L_k, \Gamma_k, \rho_k; B_k) \leq 2 + 2\langle c_1(TF), B_k \rangle + 2n - 4$$
$$= 2\langle c_1^{S^1}(TF), \iota_* B_k \rangle + 2n - 2,$$

by Lemma 4.10 and Condition (7).

• If $k \in \mathbf{U}$ then dim $G_k = 6$ and

$$\dim \mathcal{M}_{I,\text{fibr}}(L_k, \Gamma_k, \rho_k; B_k) = 2 + 2\langle c_1(TF^J), B_k \rangle + \dim F_{c(k)}$$
$$\leq 2 + 2\langle c_1(TF), B_k \rangle + \dim F_{c(k)}$$
$$= 2 + 2\langle c_1^{S^1}(TF), \iota_* B_k \rangle + \dim F_{c(k)},$$

by Condition (9).

On the other hand, since D is B-admissible, using Condition (8) we obtain

$$\langle c_1^{S^1}(TF), \beta_0 \rangle + \sum_{k=1}^K \langle c_1^{S^1}(TF), \iota_* B_k \rangle \le \langle c_1^{S^1}(TF), \beta \rangle.$$

Hence,

$$\dim \mathcal{R}_{\rho,I}(D) - \sum \dim G_k \leq 2\langle c_1^{S^1}(TF), \beta \rangle + 2(n-1)(1-g) + (S+T)2n + \sum_{k \in \mathbf{U}} \dim F_{c(k)} - 4K + \sum_{e \in C} \dim \Sigma(e) + \dim \Delta(e) - \dim \mathcal{F}(e).$$

To find an upper bound for the last two terms we proceed as follows. Since for any e we have

 $\dim \Sigma(e) + \dim \Delta(e) - \dim \mathcal{F}(e) \le 0,$

an upper bound for $\sum_{e \in C'}$ where $C' \subset C$ will also give a bound on dim $\mathcal{R}_{I,\sigma}(D)$. So we take any subset $C' \subset C$ of K elements with the following property. The graph whose vertices are the elements of \mathbf{K}_0 and which has an edge joining i to j if either (i, j) or (j, i) belong to C' is connected. This implies that $C' \nsubseteq C_{11}$ (because otherwise the vertex $0 \in \mathbf{K}_0$ would be disconnected from the rest). Take an injective map

$$v: C' \to \mathbf{K}$$

which assigns to (i, j) either *i* or *j*. Let $k \in \mathbf{K}$ and $e = v^{-1}(k)$.

- If $k \in \mathbf{S}$ then dim $\Sigma(e)$ + dim $\Delta(e)$ dim $\mathcal{F}(e) = -2n + 2$.
- If $k \in \mathbf{T}$ and $e \notin C_{11}$ then $\dim \Sigma(e) + \dim \Delta(e) \dim \mathcal{F}(e) \leq -2n+2$ and if $e \in C_{11}$ then

 $\dim \Sigma(e) + \dim \Delta(e) - \dim \mathcal{F}(e) \le \max\{-\dim F_{c(k)_+}, -\dim F_{c(k)_+}\} - 2 \le -2n + 4,$ by Condition (7).

• If $k \in \mathbf{U}$ then $\dim \Sigma(e) + \dim \Delta(e) - \dim \mathcal{F}(e) = -\dim F_{c(k)} + 2.$

Since $C' \not\subseteq C_{11}$, we get

$$\dim \mathcal{R}_{\rho,I}(D) - \sum \dim G_k \leq 2\langle c_1^{S^1}(TF), B \rangle + 2(n-1)(1-g) - 2$$
$$= \dim \mathcal{M}_{\rho,I}(B,c) - 2,$$

which is what we wanted to prove.

On the other hand, note that $\widetilde{\mathcal{R}}_{\rho,I}(D)$ is invariant under the action of $\prod G_k$ in $\widetilde{\mathcal{M}}_{\rho,I}(\beta_0, c) \times \mathcal{M}_I^*(D) \times \prod \Sigma(e)$. This action is free and commutes with the action of \mathscr{G} (indeed, $\prod G_k$ acts on $\mathcal{M}_I^*(D)$ by reparametrization of bubbles, while \mathscr{G} acts on it by acting on its image —hence the first action is on the right and the second one is on the left). So $\mathcal{R}_{\rho,I}(D)$ inherits an action of $\prod G_k$, and it is not difficult to see that this action is free.

Let us define

$$\mathscr{Q}^{\mathrm{reg}} = \bigcap_{D \in \mathcal{D}(\beta,c)} \mathscr{Q}^{\mathrm{reg}}(D),$$

and assume that $(\rho, I) \in \mathscr{Q}^{\text{reg}}$. Just as in Subsection 5.2 we have, for any framing $D \neq D^T$, an evaluation map taking values in the compact smooth model of the universal bundle

$$\operatorname{ev}_D^0 : \mathcal{R}_{\rho,I}(D) \times (\bigcup \Sigma_k)^p \to E_T^p \times_{(S^1)^p} F^p.$$

If we make $\prod G_k$ act on $(\bigcup \Sigma_k)^p$ as well as on $\mathcal{R}_{\rho,I}$ then the map ev_D^0 is invariant, so it descends to a map

$$\operatorname{ev}_D: (\mathcal{R}_{\rho,I}(D) \times (\bigcup \Sigma_k)^p) / \prod G_k \to E_T^p \times_{(S^1)^p} F^p.$$

By Lemma 5.8, the domain of this map has dimension

$$\dim(\mathcal{R}_{\rho,I}(D) \times (\bigcup \Sigma_k)^p) / \prod G_k \le \dim(\mathcal{M}(\beta, c) \times (\bigcup \Sigma_k)^p) - 2.$$

Finally, Theorem 3.4 on compactness tells us that

$$\overline{s_{F^p}^T \Phi(\mathcal{M}(\beta, c) \times \Sigma^p)} \subset s_{F^p}^T \Phi(\mathcal{M}(\beta, c) \times \Sigma^p) \cup \bigcup_{D^T \neq D \in \mathcal{D}(\beta, c)} \operatorname{Im} \operatorname{ev}_D.$$

Indeed, $\mathcal{R}_{\rho,I}(D)/\prod G_k$ parametrizes the set $\mathcal{M}_{\rho,I}(D)$ of equivalence classes of cusp ρ -THCs with framing D. This finishes the proof that $s_{F^p}^T \Phi$ is a pseudo-cycle.

The second part of the theorem is proved following exactly the same method. \Box

5.4. Definition of the invariants. Proceeding as in Subsection 1.3 with $P = E_T^p$, $K = (S^1)^p$ and $V = F^p$, we get a map

$$c_T^p : E_T^p \times_{(S^1)^p} F^p \to (F^p)_{(S^1)^p}.$$

On the other hand, there is a projection map

$$\nu_T: T = (\mathscr{A}_{\mathrm{Fl}} \times S)/\mathscr{G} \times \Sigma^p \to \mathscr{A}_{\mathrm{Fl}}/\mathscr{G} = Jac_d(\Sigma)$$

Note that $(ES^1)^p$ is a contractible space on which $(S^1)^p$ acts freely. Hence

$$(F^p)_{(S^1)^p} = ((ES^1)^p \times F^p) / (S^1)^p = ((ES^1 \times F) / S^1)^p = (F_{S^1})^p.$$
(20)

Finally, let

$$\psi_{\Sigma,\beta,c}: H_*(E^p_T \times_{(S^1)^p} F^p; \mathbb{Z}) \to \mathbb{Z}$$

be the map induced by the pseudo-cycle $s_{F^p}^T$ using Lemma 5.2.

Now let $\alpha_1, \ldots, \alpha_p \in H^*_{S^1}(F; \mathbb{Z})$. Combining Künneth with the isomorphism (20) we may view

$$\alpha_1 \otimes \cdots \otimes \alpha_p \in H^*_{(S^1)^p}(F^p;\mathbb{Z}).$$

Let also

$$\gamma \in H^*(\mathscr{A}/\mathscr{G};\mathbb{Z}) = H^*(\operatorname{Jac}_d(\Sigma);\mathbb{Z}).$$

We define the Hamiltonian Gromov–Witten invariant of $\alpha_1, \ldots, \alpha_p, \gamma$ at β, c to be

$$\Psi_{\Sigma,\beta,c}(\alpha_1,\ldots,\alpha_p,\gamma) := \psi_{\Sigma,\beta,c}(PD(c_T^{p*}(\alpha_1\otimes\cdots\otimes\alpha_p)\cup\nu_T^*\gamma))$$

where PD denotes Poincaré Dual. It follows from Lemma 5.2 and Theorem 5.4 that $\Psi_{\Sigma,\beta,c}(\alpha_1,\ldots,\alpha_p,\gamma)$ only depends on the deformation class of ω as a S^1 -invariant symplectic structure, on he connected component in $i\mathbb{R}\backslash\mathscr{C}$ in which c lies, the (co)homology classes $\beta, \alpha_1, \ldots, \alpha_p, \gamma$, and the Riemann surface Σ (in fact, it only depends on the genus of Σ and not on its conformal class, as can be proved using the same cobordism methods as above). By Subsection 2.4 the invariants are independent of the Sobolev norm which was used to complete \mathscr{A}, \mathscr{S} and \mathscr{G} .

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