Random triangle in square: geometrical approach

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The classical problem of mean area of random triangle within the square is solved by a simple and explicit method. Some other related problems are also solved using Mathematica.

I. INTRO

We call our approach geometrical as instead of considering 6-fold integral in abstract space we consider random triangle (RT) inside the plane rectangle when all possible cases are explicitly apparent.

Area of triangle with vertices p1=(x1,y1), p2=(x2,y2), p3=(x3,y3) is equal to $s = \frac{1}{2}(x1(y2-y3)+x2(-y1+y3)+x3(y1-y2))$. (1)

Let points p1, p2, p3 are randomly (with constant differential probability function) distributed over the rectangle with sides A, B. What is the mean area of triangles with vertices p1,p2,p3?

Answer is evident: zero, as any given triangle corresponds to 6 cases of full permutation of three points at vertices of the triangle. Mean area of this 6 triangles, as given by (1), is zero. But if we take triangle as geometrical figure and if we consider an area of such a figure as positive value, then we must take absolute value of s in formula (1) and... calculation of relevant integrals become impossible even for Mathematica. So Michael Trott in his recent brilliant paper in Mathematica Journal [1] found 496 different integrals each over subregion with the same sign of s, and then used Mathematica to solve such an enormously difficult task. Needless to say that M.Trott's stunning skill of using Mathematica is far out of scope of ordinary reader (as me, e.g.), so I've spend some three weeks in searching a more simple solution. The result is most easily get by the explicit geometrical approach.

II. GEOMETRICAL APPROACH

Here we consider RT in square (=right rectangle) with side length A. First observation is that due to points symmetry we may simplify problem by considering particular relation between points. For example, as we will do here, we may consider only case x1 < x2 < x3with due account of normalizing condition.

First point (p1) may take any position inside the square, so the region of integration over x1, y1 is 0 < x1 < A; 0 < y1 < A at all cases considered further.

Now we should discriminate two cases of relation between ordinates of 1st and 2nd points: 1) $y_2 > y_1$ and 2) $y_2 < y_1$.

A. $y_2 > y_1$

In this case, important is relation between two angular coefficients k1 and k2: $k1 = (A - y1)/(A - x1); \quad k2 = (y2 - y1)/(x2 - x1).$

k2 > k1. In this case the line (p1, p2) crosses the upper side of square at the point (x31m, A) with x31m = (A - y2)/k2 + x2, see Fig.1, panel 1A.



FIG. 1. Ten different cases of integration regions, bounded by solid lines

The region of integration over x^2, y^2 is: $x^1 < x^2 < A; y^{21m} < y^2 < A, y^{21m} = k^1(x^2 - x^1) + y^1.$ The first region of integration over x^3, y^3 is : $x^2 < x^3 < x^{31m}; k^2(x^3 - x^2) + y^2 < y^3 < A,$ region 1+, panel 1A, Fig. 1.

At this region formula (1) gives positive value as points (p1, p2, p3) make right-handed system: moving in direction $(p1 \rightarrow p2 \rightarrow p3 \rightarrow p1)$ we make counter-clockwise rotation. Now we are ready to calculate first integral:

$$I1 = \int_0^A dx \int_0^A dy \int_{x_1}^A dx \int_{y_{21m}}^A dy \int_{x_2}^{x_{31m}} dx \int_{k_2(x_3-x_2)+y_2}^A (s) dy = \frac{A^8}{34560}$$

Second integral appears from region "under" the first integral's region, region 2-, panel 1A, Fig. 1, and here formula (1) should be taken with negative sign:

$$I2 = \int_0^A dx \int_0^A dy \int_{x_1}^A dx \int_{y_{21m}}^A dy \int_{x_2}^{x_{31m}} dx \int_0^{x_{2(x_3-x_2)+y_2}} (-s) dy = \frac{23A^8}{34560}$$

Note that 2nd integral differs from 1st one only by integration boundaries over y3 (and by sign of s).

Also, the interesting exact relation occurs between numerical values of two considered integrals : I2 = 23I1.

Last integral at the case $k_2 > k_1$, corresponding to region 3-, panel 1A, Fig. 1, is;

$$I3 = \int_0^A dx \int_0^A dy \int_{x_1}^A dx \int_{y_{21m}}^A dy \int_{x_{31m}}^A dx \int_0^A (-s) dy = \frac{7A^8}{1728} = 140I1.$$

Note that all three multiple integrals have the same first four particular integral regions and we may write down them in a more compact form, but we will not do this pure "decorative" operation.

k2 < k1. In this case and still at y1 < y2, the line (p1, p2) crosses the right side of square; now the region of integration over y2 is y1 < y2 < y21m, and we have two different regions of integration over p3:

 $x2 < x3 < A, \, k2(x3-x2) + y2 < y3 < A,$ with positive s, region 4+, panel 1B, Fig. 1, which gives I4, and

x2 < x3 < A, 0 < y3 < k2(x3 - x2) + y2, with negative s, r. 5-. p. 1A, Fig. 1, which gives I5.

Therefore we have two additional integrals:

$$I4 = \int_0^A dx \int_0^A dy \int_{x_1}^A dx \int_{y_1}^{y_2 + m} dy \int_{x_2}^A dx \int_{k_2(x_3 - x_2) + y_2}^A (s) dy = \frac{19A^8}{34560} = 19I1.$$

$$I5 = \int_0^A dx \int_0^A dy \int_{x_1}^A dx \int_{y_1}^{y_2 + m} dy \int_{x_2}^A dx \int_0^{k_2(x_3 - x_2) + y_2} (-s) dy = \frac{37A^8}{34560} = 37I1.$$

B. y2 < y1

Now important is relation between two coefficients k3 and k4: k3 = y1/(A - x1); k4 = (y1 - y2)/(x2 - x1).

 $\mathbf{k4} < \mathbf{k3}$. If k4 < k3, then the line (p1, p2) crosses the right side of square, see panel 1C, Fig. 1, and we have the case completely analogous to the case considered in the previous section (see also panel 1B) and two integrals, I6 with positive s and I7 with negative s, are equal to I4 and I5 respectively. Here our geometrical approach is particularly explicitly demonstrate his power: it is sufficient to look at panels 1A - 1D of the Fig. 1 to be convinced that actually we have only two different cases, one case when line (p1,p2) crosses two opposite sides of square and second case when line (p1,p2) crosses two adjacent sides of square.

k4 > k3. Now the line (p1,p2) crosses lower side of square, panel 1D, Fig. 1, and we have in essence the case coinciding with previous one, panel 1A, Fig. 1, so we have another three integrals with values actually found before: I8=I1, I9=I2, and I10=I3.

Now sum of all 10 integrals is equal to $II = 11A^8/864$. The normalizing coefficient is found by calculating a sum of abovementioned 10 integrals with (+/-s)=1 in all cases which gives $JJ = A^6/6$, that is 1/6 of volume of hypercube with side A. So the mean area of random triangle inside the square is II/JJ = 11/144 of host-figure's square, A^2 .

III. RT IN RECTANGLE

Our geometrical approach allows easily to calculate also the mean area of random triangle when host-figure is rectangle. Being experienced with the square case we consider here only two cases leading to 5 integrals.

As result, we present a simple and transparent Mathematica's code for calculating the mean area of random triangle in rectangle with sides A and B.

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(* y2\,>\,y1 *)
k1=(B-y1)/(A-x1); k2=(y2-y1)/(x2-x1); Y2=k1*(x2-x1)+y1; Y3=k2*(x3-x1)+y1;
(* k2\,>\,k1 *)
X = (B - y2)/k2 + x2;
I1:=Integrate[s,x1,0,A,y1,0,B,x2,x1,A,y2,Y2,B, x3,x2,X,y3,Y3,B];
(* I1=A^4*B^4/34560 *)
I2:=Integrate[-s,x1,0,A,y1,0,B,x2,x1,A,y2,Y2,B, x3,x2,X,y3,0,Y3];
(* I2=23*I1 *)
I3:=Integrate[-s,x1,0,A,y1,0,B,x2,x1,A,y2,Y2,B, x3,X,A,y3,0,B];
(* I3=140 I1 *)
(* k2\,<\,k1 *)
I4:=Integrate[s,x1,0,A,y1,0,B,x2,x1,A,y2,y1,Y2, x3,x2,A,y3,Y3,B];
(* I4=19*I1 *)
I5:=Integrate[-s,x1,0,A,y1,0,B,x2,x1,A,y2,y1,Y2, x3,x2,A,y3,0,Y3];
(* I5=37*I1 *)
I15=I1+I2+I3+I4+I5; (* I15=11*A^4*B^4/1728 *)
(*calculation of normalizing coefficient*)
s=1;J1=I1;(* J1=A^3*B^3/432 *);s=-1;J2=I2;(* J2=5*J1 *)
s=-1;J3=I3;(* J3=18*J1 *);s=1;J4=I4;(* J4=5*J1 *)
s=-1;J5=I5;(* J5=7*J1 *); J15=J1+J2+J3+J4+J5; (*J15=A^3*B^3/12*);
MeanSquareOfRandomTriangleInRectangle=I15/J15;(* = (11/144)*A*B *)
(*MeanSquareOfRandomTriangleInRectangle = 11/144 of rectangle's square*)
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IV. RT IN SQUARE FRAME

Now we consider the related problem of random triangle in square frame. Let three points are randomly (with constant differential probability function) distributed along the sides of unit square (side length and square being 1). What is the mean area of triangles formed by these points as vertices?

The solution is elementary, but we consider it for pure pedagogical purposes. First observation is that due to symmetry of square and due to symmetry of points it is sufficient to assume that 1st particle is at bottom side of square. Then four different cases should be considered.

2nd particle is also at bottom side

Let 3rd particle moves with constant linear velocity along all sides of the square. We are looking for value of integral over coordinates of 3rd particle and then over coordinates

of 2nd particle which is allowed to move only along the bottom side of square. In the Mathematica's language we should calculate the following path integral:

s[x1_,y1_,x2_,y2_,x3_,y3_]:=(1/2)*Abs[(x1*(y2-y3)+ x2*(-y1+y3)+x3*(y1-y2))]: I1:=Integrate[(Integrate[s[x1,0,x2,0,x3,0],{ x3,0,1}]+ Integrate[s[x1,0,x2,0,1,y3],{ x2, 0,1},{y 3,0,1}]+ Integrate[s[x1,0,x2,0,x3,1],{ x2, 0,1},{ x3,0,1}]+ Integrate[s[x1,0,x2,0,0,y3],{ x2, 0,1},{ y3,0,1}]),{x2,0,1}]

Result is $I1 = \frac{1}{2} - x1 + x1^2$.

2nd particle is at the right side of the square

The relevant integral which we do not write down is equal to $I2 = \frac{11-8x1+3x1^2}{12}$

2nd particle is at the upper side of the square. $I3 = \frac{11-6x1+6x1^2}{12}$. 2nd particle is at the left side of the square. $I4 = \frac{6+2x1+3x1^2}{12}$. Sum of these for integrals gives $I14 = I1 + I2 + I3 + I4 = \frac{17}{6} - 2x1 + 2x1^2$. Now Integrate [I14, $\{x1, 0, 1\}$] gives 5/2. The normalizing coefficient is evidently 16, as we calculate 4 path integrals (over 3rd particle) which of them has path length equal to 4. Final result is: the mean area of random triangle inscribed in unit square is 5/32.

Let us look at this value (and check it!) from another point of view. We divide all sides of unit square to 10 equal parts and let each of three particles take all mid-points of these 40 parts. Then we have 40x40x40x40=640,000 triangles with mean area (as calculated by Mathematica) equal to 249/1600 = 498/3200 that is very close to 5/32.

So we understand more vividly in what sense the mean area of random triangles inscribed in the square is 5/32.

It is very interesting to compare these two "mean" values 11/144 and 5/32. First value is 22/45, that is almost exactly 1/2, of the second one. That is mean square of triangles with vertices randomly distributed all over the square is almost exactly half of mean square of triangles vertices of which are allowed to occur only at sides of square.

The reason of considering this last problem is originally related to my attempts to find simple solution of first problem. Is seemed to me that by solving the problem of inscribed triangles I could somehow solve the first problem also. Some hazy ideas about differentiation/integration connection between two problems unfortunately gave no yield and I solved these two problems separately.

What is left is the problem of mean volume of tetrahedron in the cube (M.Trott, personal communication). I hope that geometrical approach will also help in this much more difficult problem. But if the geometrical approach managed to reduce the number of integrals from 496 cumbersome ones in original solution by M. Trott to 5 very simple integrals, hopefully it will help in tetrahedron-in-cube problem as well.

Numerical value get by Mathematica gives 1/72, but this is not exact value, this is value get by Mathematica's command Rationalize[NumericalValue, 10⁻⁴].

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REFERENCES

[1] M. Trott, Mathematica Journal, v7 i2, 189-197,1998.