

Absolutely singular dynamical foliations

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Introduction

Let A_2 be the automorphism of the 2-torus, $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Let A_3 be the automorphism of the 3-torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ given by $\begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\text{Diff}_\mu^2(\mathbf{T}^3)$ be the set of C^2 diffeomorphisms of \mathbf{T}^3 that preserve Lebesgue-Haar measure μ .

In [SW1], M. Shub and A. Wilkinson prove the following theorem.

Theorem: *Arbitrarily close to A_3 there is a C^1 -open set $U \subset \text{Diff}_\mu^2(\mathbf{T}^3)$ such that for each $g \in U$,*

1. g is ergodic.
2. There is an equivariant fibration $\pi : \mathbf{T}^3 \rightarrow \mathbf{T}^2$ such that $\pi g = A_2 \pi$. The fibers of π are the leaves of a foliation \mathcal{W}_g^c of \mathbf{T}^3 by C^2 circles. In particular, the set of periodic leaves is dense in \mathbf{T}^3 .
3. There exists $\lambda^c > 0$ such that, for μ -almost every $w \in \mathbf{T}^3$, if $v \in T_w \mathbf{T}^3$ is tangent to the leaf of \mathcal{W}_g^c containing w , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|T_w g^n v\| = \lambda^c.$$

4. Consequently, there exists a set $S \subseteq \mathbf{T}^3$ of full μ -measure that meets every leaf of \mathcal{W}_g^c in a set of leaf-measure 0. The foliation \mathcal{W}_g^c is not absolutely continuous.

Additionally, it is shown that the diffeomorphisms in U are nonuniformly hyperbolic and Bernoullian. In this note, we prove:

Theorem I: *Let g satisfy conclusions 1.–3. of the previous theorem. Then there exist $S \subseteq \mathbf{T}^3$ of full μ -measure and $k \in \mathbf{N}$ such that S meets every leaf of \mathcal{W}_g^c in exactly k points. The foliation \mathcal{W}_g^c is absolutely singular.*

Remark: In A. Katok’s example of an absolutely singular foliation in [Mi], the leaves of the foliation meet the set of full measure in one point. In the [SW1] examples, the set S may necessarily meet leaves of \mathcal{W}_g^c in more than one point, as the following argument of Katok’s shows.

It follows from Theorem II in [SW2] that for $k \in \mathbf{Z}_+$ and for small $a, b > 0$, the map $g = j_{a,k} \circ h_b$ satisfies the hypotheses of Theorem I, where

$$h_b(x, y, z) = (2x + y, x + y, x + y + z + b \sin 2\pi y), \quad \text{and}$$

$$j_{a,k}(x, y, z) = (x, y, z) + a \cos(2\pi kz) \cdot (1 + \sqrt{5}, 2, 0).$$

For $k \in \mathbf{N}$, let ρ_k be the vertical translation that sends (x, y, z) to $(x, y, z + \frac{1}{k})$. Note that $h_b \circ \rho_k = \rho_k \circ h_b$ and $j_{a,k} \circ \rho_k = \rho_k \circ j_{a,k}$. Thus $g \circ \rho_k = \rho_k \circ g$.

The fibration $\pi : \mathbf{T}^3 \rightarrow \mathbf{T}^2$ was obtained in [SW1] by using the persistence of normally hyperbolic submanifolds under perturbations. In the present case the symmetries ρ_k preserve the fibers of the trivial fibration $P : \mathbf{T}^3 \rightarrow \mathbf{T}^2$ from which one starts, and also the maps g . Therefore the fibers of $\pi : \mathbf{T}^3 \rightarrow \mathbf{T}^2$ (i.e., the leaves of center foliation \mathcal{W}_g^c) are invariant under the action of the finite group $\langle \rho_k \rangle$.

Let S be the (full measure) set of points in \mathbf{T}^3 for which the center direction is a positive Lyapunov direction (i.e. for which conclusion 3 holds). Since $\rho_k(\mathcal{W}_g^c) = \mathcal{W}_g^c$, it follows that $\rho_k S = S$. If $p \in S \cap \mathcal{W}^c(p)$, then $\rho_k(p) \in \rho_k(S) \cap \rho_k(\mathcal{W}^c(p)) = S \cap \mathcal{W}^c(p)$; that is, $S \cap \mathcal{W}^c(p)$ contains at least k points.

Thus Theorem I is “sharp” in the sense that we cannot say more about the value of k in general. We see no reason why $k = 1$ should hold even for a residual set in U .

Theorem I has an interesting interpretation. Recall that a G -extension of a dynamical system $f : X \rightarrow X$ is a map $f_\varphi : X \times G \rightarrow X \times G$, where G is a compact group, of the form $(x, y) \mapsto (g(x), \varphi(x)y)$. If f preserves

ν , and $\varphi : X \rightarrow G$ is measurable, then f_φ preserves the product of ν with Lebesgue-Haar measure on G . A $\mathbf{Z}/k\mathbf{Z}$ -extension is also called a k -point extension.

Let λ be an invariant probability measure for a k -point extension of $f : X \rightarrow X$, and $\{\lambda_x\}$ the family of conditional measures associated with the partition $\{\{x\} \times G\}$. We remark that if λ is ergodic, then each atom of λ_x must have the same weight $1/k$ (up to a set of λ -measure 0).

Now take $g \in U$. Choose a coherent orientation on the leaves of $\{\pi^{-1}(x)\}_{x \in T^2}$. Take $h : \mathbf{T}^3 \rightarrow \mathbf{T}^2 \times \mathbf{T}$ to be any continuous change of coordinates such that h restricted to $\pi^{-1}(x)$ is smooth and orientation preserving to $\{x\} \times \mathbf{T}$. We may then write $F = h \circ g \circ h^{-1} : \mathbf{T}^2 \times \mathbf{T} \rightarrow \mathbf{T}^2 \times \mathbf{T}$ in the form

$$F(x, p) = (A_2x, \varphi_x(p))$$

where $\varphi_x : \mathbf{T} \rightarrow \mathbf{T}$ is smooth and orientation preserving. If $P : \mathbf{T}^2 \times \mathbf{T} \rightarrow \mathbf{T}^2$ is the projection on the first factor of the product, we have $P \circ h = \pi$. Therefore, writing $\lambda = h^*\mu$, we have $P^*\lambda = \pi^*\mu$. Let $\{\lambda_x\}$ be the disintegration of the measure λ along the fibers $\{x\} \times \mathbf{T}$. By a further measurable change of coordinates, smooth along each $\{x\} \times \mathbf{T}$ fiber, we may assume that λ -almost everywhere, the atoms of λ_x are at l/k , for $l = 0, \dots, k-1$. But then φ_x permutes the atoms cyclically, and we obtain the following corollary.

Corollary: *For every $g \in U$ there exists $k \in \mathbf{N}$ such that (\mathbf{T}^3, μ, g) is isomorphic to an (ergodic) k -point extension of $(\mathbf{T}^2, \pi^*\mu, A_2)$.*

M. Shub has observed that if $g = j_{a,k} \circ h_b$, then $\pi^*\mu$ is actually Lebesgue measure on \mathbf{T}^2 .

1 Proof of Theorem I

The proof of Theorem I follows from a more general result about fibered diffeomorphisms. Before stating this result, we describe the underlying setup and assumptions.

Let X be a compact metric space with Borel probability measure ν , and let $f : X \rightarrow X$ be invertible and ergodic with respect to ν . Let M be a closed Riemannian manifold and $\varphi : X \rightarrow \text{Diff}^{1+\alpha}(M)$ a measurable map. Consider

the skew-product transformation $F : X \times M \rightarrow X \times M$ given by

$$F(x, p) = (f(x), \varphi_x(p)).$$

Assume further that there is an F -invariant ergodic probability measure μ on $X \times M$ such that $\pi_*\mu = \nu$, where $\pi : X \times M \rightarrow X$ is the projection onto the first factor.

For $x \in X$, let $\varphi_x^{(0)}$ be the identity map on M and for $k \in \mathbf{Z}$, define $\varphi_x^{(k)}$ by

$$\varphi_x^{(k+1)} = \varphi_{f^k(x)} \circ \varphi_x^{(k)}.$$

Since the tangent bundle to M is measurably trivial, the derivative map of φ along the M direction gives a cocycle $D\varphi : X \times M \times \mathbf{Z} \rightarrow GL(n, \mathbf{R})$, where $n = \dim(M)$:

$$(x, p, k) \mapsto D_p\varphi_x^{(k)}.$$

Assume that $\log^+ \|D\varphi\|_\alpha \in L^1(X \times M, \mu)$, where $\|\cdot\|_\alpha$ is the α -Hölder norm. Let $\lambda_1 < \lambda_2 \cdots < \lambda_l$ be the Lyapunov exponents of this cocycle; they exist for μ -a.e. (x, p) by Oseledec's Theorem and are constant by ergodicity. We call these the *fiberwise exponents* of F . Under the assumptions just described, we have the following result.

Theorem II: *Suppose that $\lambda_l < 0$. Then there exists a set $S \subseteq X \times M$ and an integer $k \geq 1$ such that*

- $\mu(S) = 1$
- For every $(x, p) \in S$, we have $\#(S \cap \{x\} \times M) = k$.

This has the immediate corollary:

Corollary: *Let $f \in \text{Diff}^{1+\alpha}(M)$. If μ is an ergodic measure with all of its exponents negative, then it is concentrated on the orbit of a periodic sink.*

The corollary has a simple proof using regular neighborhoods. Our proof is a fibered version. Theorem I is also a corollary of Theorem II. For this, the argument is actually applied to the inverse of g , which has negative fiberwise exponents, rather than to g itself, whose fiberwise exponents are positive. As we described in the previous remarks, there is a measurable change of

coordinates, smooth along the leaves of \mathcal{W}_g^c in which g^{-1} is expressed as a skew product of $\mathbf{T}^2 \times \mathbf{T}$.

Remark: Without the assumption that f is invertible, Theorem II is false. An example is described by Y. Kifer [Ki], which we recall here. Let $f : \mathbf{T} \rightarrow \mathbf{T}$ be a $C^{1+\alpha}$ diffeomorphism with exactly two fixed points, one attracting and one repelling. Consider the following random diffeomorphism of \mathbf{T} : with probability $p \in (0, 1)$, apply f , and with probability $1 - p$, rotate by an angle chosen randomly from the interval $[-\epsilon, \epsilon]$.

Let $X = (\{0, 1\} \times \mathbf{T})^{\mathbf{N}}$. To generate a sequence of diffeomorphisms f_0, f_1, \dots according to the above rule, we first define $\varphi : X \rightarrow \text{Diff}^{1+\alpha}(\mathbf{T})$ by

$$\varphi(\omega) = \begin{cases} f & \text{if } \omega(0) = (0, \theta), \\ R_\theta & \text{if } \omega(0) = (1, \theta), \end{cases}$$

where R_θ is rotation through angle θ . Next, we let ν_ϵ be the product of $p, 1-p$ -measure on $\{0, 1\}$ with the measure on \mathbf{T} that is uniformly distributed on $[-\epsilon, \epsilon]$. Then corresponding to $\nu_\epsilon^{\mathbf{N}}$ -almost every element $\omega \in X$ is the sequence $\{f_k = \varphi(\sigma^k(\omega))\}_{k=0}^\infty$, where $\sigma : X \rightarrow X$ is the one-sided shift $\sigma(\omega)(n) = \omega(n+1)$.

Put another way, the random diffeomorphism is generated by the (noninvertible) skew product $\tau : X \times \mathbf{T} \rightarrow X \times \mathbf{T}$, where $\tau(\omega, x) = (\sigma(\omega), \varphi(\omega)(x))$. An ergodic ν_ϵ -stationary measure for this random diffeomorphism is a measure μ_ϵ on \mathbf{T} such that $\mu_\epsilon \times \nu_\epsilon^{\mathbf{N}}$ is τ -invariant and ergodic. Such measures always exist ([Ki], Lemma I.2.2), but, for this example, there is an ergodic stationary measure with additional special properties.

Specifically, for every $\epsilon > 0$, there exists an ergodic ν_ϵ -stationary measure μ_ϵ on \mathbf{T} such that, as $\epsilon \rightarrow 0$, $\mu_\epsilon \rightarrow \delta_{x_0}$, in the weak topology, where δ_{x_0} is Dirac measure concentrated on the sink x_0 for f . From this, it follows that, as $\epsilon \rightarrow 0$, the fiberwise Lyapunov exponent for μ_ϵ approaches $\log |f'(x_0)| < 0$, which is the Lyapunov exponent of δ_{x_0} . Thus, for ϵ sufficiently small, the fiberwise exponent for τ with respect to μ_ϵ is negative. Nonetheless, it is easy to see that μ_ϵ for $\epsilon > 0$ cannot be uniformly distributed on k atoms; if μ_ϵ were atomic, then τ -invariance of $\mu_\epsilon \times \nu_\epsilon^{\mathbf{N}}$ would imply that, for every $x \in \mathbf{T}$,

$$\begin{aligned} \mu_\epsilon(\{x\}) &= p\mu_\epsilon(\{f^{-1}(x)\}) + (1-p) \int_{-\epsilon}^\epsilon \mu_\epsilon(\{R_\theta(x)\}) d\theta \\ &= p\mu_\epsilon(\{f^{-1}(x)\}), \end{aligned}$$

which is impossible if μ_ϵ has finitely many atoms. In fact, μ_ϵ can be shown to be absolutely continuous with respect to Lebesgue measure (see [Ki], p. 173ff and the references cited therein). Hence invertibility is essential, and we indicate in the proof of Theorem II where it is used.

Proof of Theorem II: We first establish the existence of fiberwise “stable manifolds” for the skew product F . A general theory of stable manifolds for random dynamical systems is worked out in ([Ki], Theorem V.1.6; see also [BL]); since we are assuming that all of the fiberwise exponents for F are negative, we are faced with the simpler task of constructing fiberwise regular neighborhoods for F (see the Appendix by Katok and Mendoza in [KH]). We outline a proof, following closely [KH].

Theorem 1.1 (*Existence of Regular Neighborhoods*) *There exists a set $\Lambda_0 \subseteq X \times M$ of full measure such that for $\epsilon > 0$:*

- *There exists a measurable function $r : \Lambda_0 \rightarrow (0, 1]$ and a collection of embeddings $\Psi_{(x,p)} : B(0, q(x, p)) \rightarrow M$ such that $\Psi_{(x,p)}(0) = p$ and $\exp(-\epsilon) < r(F(x, p))/r(x, p) < \exp(\epsilon)$.*
- *If $\varphi_{(x,p)} = \Psi_{F(x,p)}^{-1} \circ \varphi_x \circ \Psi_{(x,p)} : B(0, r(x, p)) \rightarrow \mathbf{R}^n$, then $D_0\varphi_{(x,p)}$ satisfies*

$$\exp(\lambda_1 - \epsilon) \leq \|D_0\varphi_{(x,p)}^{-1}\|^{-1}, \|D_0\varphi_{(x,p)}\| \leq \exp(\lambda_1 + \epsilon).$$

- *The C^1 distance $d_{C^1}(\varphi_{(x,p)}, D_0\varphi_{(x,p)}) < \epsilon$ in $B(0, r(x, p))$.*
- *There exist a constant $K > 0$ and a measurable function $A : \Lambda_0 \rightarrow \mathbf{R}$ such that for $y, z \in B(0, r(x, p))$,*

$$K^{-1}d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)) \leq \|y - z\| \leq A(x)d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)),$$

with $\exp(-\epsilon) < A(F(x, p))/A(x, p) < \exp(\epsilon)$.

Proof: See the proof of Theorem S.3.1 in [KH]. \square

Decompose μ into a system of fiberwise measures $d\mu(x, p) = d\mu_x(p)d\nu(x)$. Invariance of μ with respect to F implies that, for ν -a.e. $x \in X$,

$$\varphi_{x*}\mu_x = \mu_{f(x)}.$$

Corollary 1.2 *There exists a set $\Lambda \subseteq X \times M$, and real numbers $R > 0$, $C > 0$, and $c < 1$ such that*

(1) $\mu(\Lambda) > .5$, and, if $(x, p) \in \Lambda$, then $\mu_x(\Lambda_x) > .5$, where $\Lambda_x = \{p \in M \mid (x, p) \in \Lambda\}$,

(2) If $(x, p) \in \Lambda$ and $d_M(p, q) \leq R$, then

$$d_M(\varphi_x^{(m)}(p), \varphi_x^{(m)}(q)) \leq Cc^m d_M(p, q),$$

for all $m \geq 0$.

Proof: This follows in a standard way from the Mean Value Theorem and Lusin's Theorem. \square

To prove Theorem II, it suffices to show that there is a positive ν -measure set $B \subseteq X$, such that for $x \in B$, the measure μ_x has an atom, as the following argument shows. For $x \in X$, let $d(x) = \sup_{p \in M} \mu_x(p)$. Clearly d is measurable, f -invariant, and positive on B . Ergodicity of f implies that $d(x) = d > 0$ is positive and constant for almost all $x \in X$. Let $S = \{(x, p) \in X \times M \mid \mu_x(p) \geq d\}$. Observe that S is F -invariant, has measure at least d , and hence has measure 1. The conclusions of Theorem II follow immediately.

Let Λ , $R > 0$, $C > 0$, and $c < 1$ be given by Corollary 1.2, and let $B = \pi(\Lambda)$. Let N be the number of $R/10$ -balls needed to cover M . We now show that for ν -almost every $x \in B$, the measure μ_x has at least one atom.

For $x \in X$, let

$$m(x) = \inf \sum \text{diam}(U_j),$$

where the infimum is taken over all collections of closed balls U_1, \dots, U_k in M such that $k \leq N$ and $\mu_x(\bigcup_{j=1}^k U_j) \geq .5$. Let $m = \text{ess sup}_{x \in B} m(x)$.

We now show that $m = 0$. If $m > 0$, then there exists an integer J such that

$$C\Delta c^J N < m/2, \tag{1}$$

where Δ is the diameter of M . Let \mathcal{U} be a cover of M by N closed balls of radius $R/10$. For $x \in B$, let $U_1(x), \dots, U_{k(x)}(x)$ be those balls in \mathcal{U}

that meet Λ_x . Since these balls cover Λ_x , and $\mu_x(\Lambda_x) > .5$, it follows that $\mu_x(\bigcup_{j=1}^{k(x)} U_j(x)) \geq .5$. But $\varphi_x^{(i)*}\mu_x = \mu_{f^i(x)}$, and so it's also true that

$$\mu_{f^i(x)}\left(\bigcup_{j=1}^{k(x)} \varphi_x^{(i)}(U_j(x))\right) \geq .5, \quad (2)$$

for all i .

We now use the fact that $\varphi_x^{(i)}$ contracts regular neighborhoods to derive a contradiction. The balls $U_j(x)$ meet Λ_x and have diameter less than $R/10$, and so by Corollary 1.2, (2), we have

$$\text{diam}(\varphi_x^{(i)}(U_j(x))) \leq C\Delta c^i. \quad (3)$$

Let $\tau : B \rightarrow \mathbf{N}$ be the first-return time of f^J to B , so that $f^{J\tau(x)}(x) \in B$, and $f^{Ji}(x) \notin B$, for $i \in \{1, \dots, \tau(x) - 1\}$. Decompose the set B according to these first return times:

$$B = \bigcup_{i=1}^{\infty} B_i \pmod{0},$$

where $B_i = \tau^{-1}(i)$. Because f is invertible and f^{-1} preserves measure, we also have the mod 0 equivalence:

$$B' := \bigcup_{i=1}^{\infty} f^{Ji}(B_i) = B \pmod{0}.$$

Let $y \in B'$. Then $y = f^{Ji}(x)$, where $x \in B_i \subseteq B$, for some $i \geq 1$. It follows from the definition of $m(y)$ and inequalities (2), (3) and (1) that

$$\begin{aligned} m(y) &\leq \sum_{j=1}^{k(x)} \text{diam}(\varphi_x^{(Ji)}(U_j(x))) \\ &\leq Ck(x)\Delta c^{Ji} \\ &\leq CN\Delta c^J \\ &< m/2. \end{aligned}$$

But then

$$\begin{aligned} m &= \text{ess sup}_{x \in B} m(x) \\ &= \text{ess sup}_{y \in B'} m(y) \\ &< m/2, \end{aligned}$$

contradicting the assumption $m > 0$.

Thus $m = 0$, and, for ν -almost every $x \in B$, we have $m(x) = 0$. If $m(x) = 0$, then there is a sequence of closed balls $U^1(x), U^2(x), \dots$ with $\lim_{i \rightarrow \infty} \text{diam}(U^i(x)) = 0$ and $\mu_x(U^i(x)) \geq .5/N$, for all i . Take $p_i \in U^i(x)$; any accumulation point of $\{p_i\}$ is an atom for μ_x . Since we have shown that μ_x has an atom, for ν -a.e. $x \in B$, the proof of Theorem II is complete. \square

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