

On some claims in Ramanujan's 'unpublished' manuscript on the partition and tau functions

Pieter Moree

Abstract

We disprove various claims made by Ramanujan in his, until very recently, unpublished manuscript [1] on the partition and tau functions. Furthermore, the second part of a related paper by G.K. Stanley [17] is corrected (the first part of which was earlier corrected by D. Shanks [16]).

1 Introduction

In his first letter (16 Jan. 1913) to Hardy [2, p. 24], Ramanujan made various claims. The fourth of them reads as follows: "(4) 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, ... are numbers which are either themselves squares or which can be expressed as the sum of two squares. The number of such numbers greater than A and less than B

$$= K \int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x), \quad (1)$$

where $K = 0.764\dots$ and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found, though complicated...". (Note that $\theta(x)$ should be $\theta(B)$.) Answering an inquiry of Hardy [2, p. 49], Ramanujan in his second letter to Hardy [2, p. 56] claims: "The order of $\theta(x)$ which you asked in your letter is $\sqrt{x}/\sqrt{\log x}$ ". In his lectures [6] on Ramanujan's work, Hardy states that Ramanujan also gave the exact value of K , namely

$$K = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p^2}\right)^{-1/2}.$$

Note that Ramanujan's claim, if correct, would imply that

$$B(x) = \frac{Kx}{\sqrt{\log x}} \left(1 + \frac{C_2}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right),$$

P. Moree: KdV Institute, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands, e-mail: moree@science.uva.nl

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with $C_2 = 1/2$, where $B(x)$ denotes the number of integers $n \leq x$ of the form $u^2 + v^2$ with u and v integers. Landau proved in 1908, using contour integration, that asymptotically $B(x) \sim Kx/\sqrt{\log x}$. His method can be easily extended [14] to prove that $B(x)$ has an asymptotic expansion in the sense of Poincaré, namely for every integer $r \geq 2$, we have

$$B(x) = \frac{Kx}{\sqrt{\log x}} \left(1 + \sum_{j=2}^r \frac{C_j}{\log^{j-1} x} + O\left(\frac{1}{\log^r x}\right) \right),$$

with C_2, \dots, C_r constants. Shanks [16], correcting on Stanley [17], computed the second order constant C_2 to equal $0.5819486 \dots$ and thus disproved Ramanujan's claim. For an overview of further results in this direction the reader is referred to [10].

There is some evidence (but see [12, p. 92]) that along with his final letter (12 Jan. 1920) to Hardy, Ramanujan included a manuscript on congruence properties of $\tau(n)$ and $p(n)$. In this manuscript Ramanujan considers, for various special small primes q , the quantity $\sum_{n \leq x, q \nmid \tau(n)} 1$ and makes claims similar to (1). He defines $t_n = 1$ if $\tau(n) \not\equiv 0 \pmod{q}$ and $t_n = 0$ otherwise. He then typically writes: "It is easy to prove by quite elementary methods that $\sum_{k=1}^n t_k = o(n)$. It can be shown by transcendental methods that

$$\sum_{k=1}^n t_k \sim \frac{Cn}{(\log n)^\delta}; \tag{2}$$

and

$$\sum_{k=1}^n t_k = C \int_1^n \frac{dx}{(\log x)^\delta} + O\left(\frac{n}{(\log n)^r}\right). \tag{3}$$

where r is any positive number". Note that the truth of $\sum_{k=1}^n t_k = o(n)$ would imply that $q \nmid \tau(n)$ for almost all n . The values of δ can be found in the final column of Table 1. Except for $q = 5$ and $q = 691$ Ramanujan also writes down an Euler product for C . It is not difficult to check that these are correct, except when $q = 23$, in which case this is due to a factor $(1 - 23^{-s})^{-1}$ erroneously omitted in the generating function.

It appears from [17] that Hardy planned to have this manuscript published under Ramanujan's name after some editing. Indeed, he published some parts of it (see [1]). Unfortunately, he never published Ramanujan's full manuscript (which indeed needed some serious editing). Some of his results in connection with the unpublished manuscript were further worked out by his research student Geraldine Stanley and published in 1928 [17]. She claimed Ramanujan's assertion regarding C_2 and the second order coefficient for $\sum_{n \leq x, 5 \nmid \tau(n)} 1$ to be false. Unfortunately, her paper contains several typos and some mathematical errors, which were corrected by Shanks [16] for the $B(x)$ case and are corrected in this paper for the $5 \nmid \tau(n)$ case in Section 10.

In 1928 Hardy passed on his Ramanujan materials to G.N. Watson. Watson's papers [20, 21, 22] are inspired by Ramanujan's 'unpublished' manuscript. In particular Watson [20] rigorously proved Ramanujan's assertion that $691 \nmid \tau(n)$ for almost all n by establishing (2) with $q = 691$ and $\delta = 1/690$. A remarkable

result, since for $1 \leq n \leq 5000$ we have, as Ramanujan computed, $691 \nmid \tau(n)$, except for the multiples of 1381 in that range. After Watson's death in 1965 a part of the manuscript came into the possession of the library of Trinity College, Cambridge. Watson's copy of the remaining part can be found in the library of Oxford's Mathematical Institute. A full version of the unpublished manuscript (with proofs and commentary) has only recently become available to the general public [1].

Ramanujan's claims are only the tip of an iceberg: it is now known [14, 15] that if $\sum_{n=1}^{\infty} a_n q^n$, with $q = e^{2\pi iz}$, is the Fourier expansion of a modular form of integral weight with integral coefficients for a congruence subgroup of the full modular group, then for every positive integer M , for almost all n we have $a(n) \equiv 0 \pmod{M}$. In particular, we have that (2) holds true for every odd prime q not in Table 1 with $\delta = q/(q^2 - 1)$. It can also be shown [18], using l -adic representations, that 2 and the primes in Table 1 are the only primes for which congruences for $\tau(n)$ exist.

The purpose of this paper is to establish the falsity of the various claims made by Ramanujan of the type (3). We do this by evaluating the relevant second order coefficients and compute them with at least a few decimals of precision. Moreover, we will correct the part of Stanley's paper pertaining to the $5|\tau(n)$ case. As far as possible we adopt Ramanujan's notation and the order in which we deal with the various primes follows the order in which they appear in Ramanujan's manuscript.

2 Some relevant general results

In this section we quote some general results that allow us to evaluate the second-order constants of the functions considered by Ramanujan.

The following result was obtained by Moree [9] using elementary methods. It can be deduced on analysing an appropriate functional equation (apparently first considered by Levin and Fainleib) for the relevant counting function. Given a multiplicative function f , we define $\Lambda_f(n)$ implicitly by

$$f(n) \log n = \sum_{d|n} f(d) \Lambda_f\left(\frac{n}{d}\right).$$

We denote the formal Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ associated to f by $L_f(s)$. It is easy to see that $-L'_f(s)/L_f(s) = \sum_{n=1}^{\infty} \Lambda_f(n)n^{-s}$. As usual the logarithmic integral $\int_2^x dt/\log t$ is denoted by $\text{Li}(x)$.

Theorem 1 [9]. *Let f be a multiplicative function satisfying*

$$0 \leq f(p^r) \leq c_1 c_2^r, \quad c_1 \geq 1, \quad 1 \leq c_2 < 2, \quad (4)$$

and $\sum_{p \leq x} f(p) = \tau \text{Li}(x) + O(x \log^{-2-\rho} x)$, where τ and $\rho > 1$ are positive real fixed numbers. Then, for some $\epsilon > 0$ and constant B_f we have

$$\sum_{n \leq x} \frac{\Lambda_f(n)}{n} = \tau \log x + B_f + O\left(\frac{x}{\log^{1+\epsilon} x}\right). \quad (5)$$

There exist constants $\epsilon' > 0$ and constants $C_1(f) > 0$ and $C_2(f)$ such that

$$\sum_{n \leq x} f(n) = \frac{C_1(f)x}{\log^{1-\tau} x} \left(1 + \frac{C_2(f)}{\log x} + O\left(\frac{1}{\log^{1+\epsilon'} x}\right) \right), \quad x \rightarrow \infty.$$

In particular, $C_2(f) = (1 - \tau)(1 + B_f)$.

Remark. Alternatively we can write

$$C_2(f) = (1 - \tau) \left(1 + \tau\gamma + \sum_{n=1}^{\infty} \frac{\Lambda_f(n) - \tau}{n} \right),$$

where γ denotes Euler's constant.

Corollary 1 *Let f satisfy the conditions of Theorem 1 with $\tau \neq 1$. Suppose that*

$$\sum_{n \leq x} f(n) = C_1(f) \int_1^x \frac{dt}{\log^{1-\tau} t} + O\left(\frac{x}{\log^r x}\right),$$

for some $r > 2 - \tau$, then we must have $B_f = 0$.

The constant B_f appearing in (5) can be computed using the following result.

Lemma 1 *Let f satisfy the conditions of Theorem 1 and let B_f and τ be defined as in Theorem 1. Then*

$$B_f = - \lim_{s \rightarrow 1+0} \left(\frac{L'_f(s)}{L_f(s)} + \frac{\tau}{s-1} \right).$$

Example. Take $f = \mathbf{1}$, that is $f(n) = 1$ for every $n \geq 1$. Then Λ_f equals the Von Mangoldt function and $L_f(s) = \zeta(s)$. Using the well-known Taylor expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + o(1) \tag{6}$$

around $s = 1$, we obtain that $B_f = -\gamma$.

Our proof of Lemma 1 will make use of the following result due to Landau [8, pp. 73-74].

Lemma 2 *Suppose that $\sum_{n \leq x} h(n) = \alpha x + O(g(x))$, where $g(x)$ is a positive function of x such that $g(x)/x^2$ is monotonically decreasing for every $x \geq x_0$, x_0 some fixed number, and where $\int_{x_0}^{\infty} g(t)dt/t^2 < \infty$. Then, for $\text{Re}(s) > 1$, we have the estimate*

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \frac{\alpha}{s-1} + \beta + o(1), \quad s \rightarrow 1+0,$$

for some constant β . Furthermore, we have

$$\sum_{n \leq x} \frac{h(n)}{n} = \alpha \log x + \beta + o(1), \quad x \rightarrow \infty.$$

We are now in the position to prove Lemma 1.

Proof of Lemma 1. By partial integration we deduce from (5) that $\sum_{n \leq x} \Lambda_f(n) = \tau x + O(x \log^{-1-\epsilon} x)$. We thus can invoke Lemma 2 with $h(n) = \Lambda_f(n)$, $\alpha = \tau$ and $g(x) = x \log^{-1-\epsilon} x$. On noting that $-L'_f(s)/L_f(s) = \sum_{n=1}^{\infty} \Lambda_f(n)n^{-s}$, the result then follows. \square

Serre [14] gave some beautiful applications of Landau's method to counting functions involving coefficients of modular forms. In order to formulate his result we have to define the concept of *Frobenius density*. A set of primes \mathcal{P} is called Frobenius of density δ , if there exists a finite Galois extension K/\mathbb{Q} and a subset H of $G := \text{Gal}(K/\mathbb{Q})$ such that H is stable under conjugation, $|H|/|G| = \delta$ and for every prime p , with at most finitely many exceptions, one has $p \in \mathcal{P}$ if and only if $\sigma_p(K/\mathbb{Q}) \in H$, where $\sigma_p(K/\mathbb{Q})$ denotes the Frobenius map of p in G (defined modulo conjugation in case p does not divide the discriminant of K). Serre [14], using Landau's contour integration method, established the following result, which we formulate here in a slightly less general form.

Theorem 2 [14, Théorème 2.8]. *Let $\alpha(n) : \mathbb{N} \rightarrow \mathbb{Z}$ be a multiplicative function. Let q be a fixed prime. Suppose that the set $P := \{p \text{ is prime} : q|\alpha(p)\}$ is Frobenian of density $0 < 1 - \tau < 1$. Let $h(s) = \sum_{q|\alpha(n)} n^{-s}$. Then around $s = 1$ we have*

$$\frac{h(s)}{s} = \frac{1}{(s-1)^\tau} (e_0 + e_1(s-1) + \cdots + e_k(s-1)^k + \cdots),$$

for certain numbers e_j with $e_0 \neq 0$ and we have, furthermore,

$$\sum_{\substack{n \leq x \\ q|\alpha(n)}} 1 = \frac{e_0 x}{\Gamma(\tau) \log^{1-\tau} x} \left(1 + \cdots + \frac{\Gamma(\tau) e_k}{\Gamma(\tau - k) e_0 \log^k x} + O\left(\frac{1}{\log^{k+1} x}\right) \right).$$

In particular the second order constant, $\Gamma(\tau)e_1/(\Gamma(\tau-1)e_0)$, equals

$$(1 - \tau) \lim_{s \rightarrow 1+0} \left(-\frac{L'_f(s)}{L_f(s)} - \frac{\tau}{s-1} + 1 \right).$$

It can be shown that if the conditions of Theorem 2 are satisfied, also the conditions of Theorem 1 are satisfied. Theorem 1 in combination with Lemma 1 and Theorem 2 both predict the same second order coefficient.

Let $T(s) = \sum_{n=1}^{\infty} t_n/n^s$. The approach followed in disproving Ramanujan's claims of the format (3) is to write $T(s) = \zeta(s)^\tau g(s)$ with $g(s)$ a regular function for $\text{Re}(s) > 1/2$. By Lemma 1 and (6) it then follows that $B_t = -\tau\gamma - g'(1)/g(1)$. We have $\tau = 1 - \delta$. The numerical work (carried out in Section 9) shows that $B_t \neq 0$. From Corollary 1 the falsity of Ramanujan's claim then follows for every $r > 2 - \tau$.

3 Divisibility of tau by 2

Ramanujan shows that $\tau(n)$ is odd or even according as n is an odd square or not. It thus follows that $\sum_{n \leq x, 2 \nmid \tau(n)} 1 = \left[\frac{1+\sqrt{x}}{2} \right]$.

4 Divisibility of tau by 5

At [1, p. 47] Ramanujan makes a statement of the form (3) with $q = 5$ and $\delta = 1/4$. Put $t_n = 0$ if $5|\tau(n)$ and $t_n = 1$ otherwise. Let $T(s) = \sum_{n=1}^{\infty} t_n/n^s$. Denote $\sum_{d|n} d^r$ by $\sigma_r(n)$. On using that $\tau(n) \equiv n\sigma_1(n) \pmod{5}$, it is easily seen that $T(s)$ equals

$$\prod_{p \equiv 1 \pmod{5}} \frac{1 - p^{-4s}}{(1 - p^{-s})(1 - p^{-5s})} \prod_{p \equiv \pm 2 \pmod{5}} \frac{1 - p^{-3s}}{(1 - p^{-s})(1 - p^{-4s})} \prod_{p \equiv 4 \pmod{5}} \frac{1}{1 - p^{-2s}}.$$

Let χ_c be the character of $(\mathbb{Z}/5\mathbb{Z})^*$ that is determined by $\chi_c(\bar{2}) = i$ and χ_5 be the character that is determined by $\chi_5(\bar{2}) = -1$. Denote

$$\prod_{p \equiv 1 \pmod{5}} \left(\frac{1 - p^{-4s}}{1 - p^{-5s}} \right)^4 \prod_{p \equiv \pm 2 \pmod{5}} \frac{(1 - p^{-3s})^4}{(1 - p^{-2s})^2(1 - p^{-4s})^3} \prod_{p \equiv 4 \pmod{5}} (1 - p^{-2s})^{-2}$$

by $H(s)$. Note that

$$T(s)^4 = (1 - 5^{-s})^3 H(s) \zeta(s)^3 \frac{L(s, \chi_c) L(s, \bar{\chi}_c)}{L(s, \chi_5)}. \quad (7)$$

Put

$$D = \prod_{p \equiv 1 \pmod{5}} \frac{1 - p^{-4}}{1 - p^{-5}} \prod_{p \equiv \pm 2 \pmod{5}} \frac{1 - p^{-3}}{(1 - p^{-2})^{1/2}(1 - p^{-4})^{3/4}} \prod_{p \equiv 4 \pmod{5}} \frac{1}{\sqrt{1 - p^{-2}}}.$$

Thus the first order Landau-Ramanujan constant C in this case, which was not written down by Ramanujan, equals

$$C = \frac{1}{\Gamma(\frac{3}{4})} \left(\frac{64L(1, \chi_c)L(1, \bar{\chi}_c)}{125L(1, \chi_5)} \right)^{1/4} D.$$

At [5, p. 388] the L -values above are given (for an excellent discussion of how to compute these values see [5, 6.5]); we have

$$L(1, \chi_c) = \overline{L(1, \bar{\chi}_c)} = \frac{2\pi}{25}(3 - i) \left(\sin\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right)$$

and $L(1, \chi_5) = \log(\frac{3+\sqrt{5}}{2})/\sqrt{5}$. On using that $\sin^2(2\pi/5) = (5 + \sqrt{5})/8$ and $\sin^2(4\pi/5) = (5 - \sqrt{5})/8$, we deduce that $L(1, \chi_c)L(1, \bar{\chi}_c) = 2\pi^2/25$. Alternatively we may deduce the latter equality by noting that

$$\frac{h(K)2^{r_1}(2\pi)^{r_2}R(K)}{\sqrt{|d(K)|}w(K)} = \text{Res}_{s=1} \zeta_{\mathbb{Q}(\zeta_5)}(s) = L(1, \chi_c)L(1, \bar{\chi}_c)L(1, \chi_5),$$

where $\zeta_{\mathbb{Q}(\zeta_5)}(s)$ denotes the Dedekind zeta-function of the cyclotomic field $K := \mathbb{Q}(\zeta_5)$. It is not difficult to show that $h(K) = 1$, $r_1 = 0$, $r_2 = 2$, $R(K) =$

$\log((3 + \sqrt{5})/2)$, $d(K) = 125$ and $w(K) = 10$ (all of this can be deduced from results proved e.g. in [19]). We thus obtain that

$$C = \frac{4}{5\Gamma(\frac{3}{4})} \left(\frac{\pi^2}{2\sqrt{5} \log(\frac{3+\sqrt{5}}{2})} \right)^{1/4} D.$$

Using the prime number theorem for arithmetic progressions we see that the conditions of Lemma 1 are satisfied and from (7) we deduce that

$$4B_t = -3\gamma - 2\Re \left(\frac{L'(1, \chi_c)}{L(1, \chi_c)} \right) + \frac{L'(1, \chi_5)}{L(1, \chi_5)} - \frac{3}{4} \log 5 + A_{\pm 1} + A_{\pm 2}.$$

with

$$A_{\pm 1} = \sum_{p \equiv 1 \pmod{5}} \log p \left(\frac{-16}{p^4 - 1} + \frac{20}{p^5 - 1} \right) + 4 \sum_{p \equiv 4 \pmod{5}} \frac{\log p}{p^2 - 1} \quad \text{and}$$

$$A_{\pm 2} = \sum_{p \equiv \pm 2 \pmod{5}} \log p \left(\frac{4}{p^2 - 1} - \frac{12}{p^3 - 1} + \frac{12}{p^4 - 1} \right).$$

5 Divisibility of tau by 7

At [1, p. 52] Ramanujan makes a statement of the form (3) with $q = 7$ and $\delta = 1/2$. Put $t_n = 0$ if $7|\tau(n)$ and $t_n = 1$ otherwise. Using that $\tau(n) \equiv n\sigma_3(n) \pmod{7}$ it is easily seen that

$$T(s) = \prod_{p \equiv 3,5,6 \pmod{7}} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 1,2,4 \pmod{7}} \frac{1 - p^{-6s}}{(1 - p^{-s})(1 - p^{-7s})}.$$

A simple computation shows that

$$T(s)^2 = \zeta(s)L(s, \chi_{-7})(1 - 7^{-s}) \prod_{p \equiv 3,5,6 \pmod{7}} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 1,2,4 \pmod{7}} \left(\frac{1 - p^{-6s}}{1 - p^{-7s}} \right)^2.$$

where χ_{-7} denotes the usual Kronecker character of the number field $\mathbb{Q}(\sqrt{-7})$. From this relation we then obtain

$$2B_t = -\gamma - \frac{\log 7}{6} - \frac{L'(1, \chi_{-7})}{L(1, \chi_{-7})} + 2 \sum_{p \equiv 3,5,6 \pmod{7}} \frac{\log p}{p^2 - 1}$$

$$+ \sum_{p \equiv 1,2,4 \pmod{7}} \log p \left(\frac{14}{p^7 - 1} - \frac{12}{p^6 - 1} \right).$$

6 Divisibility of tau and lambda by 3

At [1, p. 64] Ramanujan makes two statements of the form (3) with $q = 3$ and $\delta = 1/2$. Put $t_n = 0$ if $3|\tau(n)$ and $t_n = 1$ otherwise. Using that $\tau(n) \equiv n\sigma_1(n) \pmod{3}$, where $\sigma_1(n)$ denotes the sum of the positive divisors of n , it is easy to see that t_n is multiplicative and that

$$T(s) := \sum_{n=1}^{\infty} \frac{t(n)}{n^s} = \prod_{p \equiv 2 \pmod{3}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 1 \pmod{3}} \frac{1+p^{-s}}{1-p^{-3s}}. \quad (8)$$

A simple computation shows that

$$T(s)^2 = \zeta(s)L(s, \chi_{-3})(1-3^{-s}) \prod_{p \equiv 2 \pmod{3}} \frac{1}{1-p^{-2s}} \prod_{p \equiv 1 \pmod{3}} \left(\frac{1-p^{-2s}}{1-p^{-3s}} \right)^2,$$

where χ_{-3} denotes the Kronecker character of the number field $\mathbb{Q}(\sqrt{-3})$. From this relation we then obtain

$$\begin{aligned} 2B_t &= -\gamma - \frac{\log 3}{2} - \frac{L'(1, \chi_{-3})}{L(1, \chi_{-3})} + 2 \sum_{p \equiv 2 \pmod{3}} \frac{\log p}{p^2 - 1} \\ &\quad + \sum_{p \equiv 1 \pmod{3}} \log p \left(\frac{6}{p^3 - 1} - \frac{4}{p^2 - 1} \right). \end{aligned}$$

Let $\lambda(n)$ denote the number of partitions of n as the sum of integers which are not multiples of 9. Put $l_n = 0$ if $3|\lambda(n)$ and $l_n = 1$ otherwise. Ramanujan shows that

$$\sum_{k \leq x} l_k = \sum_{k \leq 3x+1} t_k \quad (9)$$

and then states, [1, (11.8a)], that it can be shown by transcendental methods that

$$\sum_{k=1}^n l_k = C \int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right)$$

and

$$\sum_{k=1}^n t_k = \frac{C}{3} \int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right),$$

where he gives an explicit expression for C . Though these claims are correct for $r \leq 3/2$, *a priori* they cannot be both true for $r > 3/2$, as we then trivially have from (9) that $C_2(l) = C_2(t) - \frac{1}{2} \log 3$, whereas the truth of both claims of Ramanujan would imply that $C_2(l) = C_2(t)$.

7 Divisibility of tau by 691

At [1, p. 66] Ramanujan makes a statement of the form (3) with $q = 691$ and $\delta = 1/690$. He did not write down an explicit first order constant. The truth of this assertion for $r \leq 691/690$ was first established by G.N. Watson [20]. In this

note it will be shown, however, that the statement is false for every $r > 691/690$.

It is not difficult to show, as Ramanujan did, that $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$. Let $\nu(p)$ be the smallest integer > 1 , such that $p^{\nu(p)} \equiv 1 \pmod{691}$. We put $\nu(691) = \infty$. Note that $\sigma_{11}(p^k) \equiv 0 \pmod{691}$ if and only if $k \equiv \nu(p) - 1 \pmod{\nu(p)}$. In case $p = 691$ we interpret this congruence as never being satisfied and $1 - 691^{-\nu(691)s}$ as being 1. We thus can write

$$T(s) = \prod_p \frac{1 - p^{-(\nu(p)-1)s}}{(1 - p^{-s})(1 - p^{-\nu(p)s})}.$$

Around $s = 1$ this function is quite close to $\zeta(s)$ (we have $T(s) = \sum_{n=1}^{11053} n^{-s} - \sum_{m=1}^8 (m1381)^{-s} - 5527^{-s} - 8291^{-s} + \sum_{n=11054}^{\infty} t_n/n^s$) and hence we expect B_t to be close to $-\gamma$, cf. Example 1, which is indeed the case by Table 1.

Notice that each local factor of $T(s)$ has the term $1 - p^{-s}$ in it, unless $p \equiv -1 \pmod{691}$, in which case the local factor is $(1 - p^{-2s})^{-1}$. By multiplying $T(s)$ with $\prod_{p \equiv -1 \pmod{691}} (1 - p^{-s})^{-1}$ we can then write it as $\zeta(s)h(s)$ with $h(s)$ a regular function for $\text{Re}(s) > 1/2$. On noting that the primes p with $p \equiv -1 \pmod{691}$ are exactly the primes that split completely in $\mathbb{Q}(\zeta_{691})$ but not in $\mathbb{Q}(\cos(\frac{2\pi}{691}))$, the product $\prod_{p \equiv -1 \pmod{691}} (1 - p^{-s})^{-1}$ can be expressed in terms of the Dedekind zetafunctions of the latter two fields and some regular function for $\text{Re}(s) > 1/2$, which can be explicitly determined using the splitting behaviour of a prime p in these fields. Using the factorisation of these Dedekind zetafunctions in terms of L-series, we then obtain the following identity, with χ_c the character uniquely determined by $\chi_c(\bar{3}) = \exp(2\pi i/690)$;

$$\begin{aligned} T(s)^{690} &= \zeta(s)^{689} (1 - 691^{-s})^{-1} L(s, \chi_c) \prod_{j=1}^{344} \frac{L(s, \chi_c^{2j+1})}{L(s, \chi_c^{2j})} \\ &\quad \prod_{p \equiv -1 \pmod{691}} (1 - p^{-2s})^{-345} \prod_{p \equiv 1 \pmod{691}} \left(\frac{1 - p^{-690s}}{1 - p^{-691s}} \right)^{690} \\ &\quad \prod_{\substack{p \neq 691 \\ 2|\nu(p), \nu(p) \geq 4}} \left(\frac{1 + p^{-\nu(p)\frac{s}{2}}}{1 - p^{-\nu(p)\frac{s}{2}}} \right)^{\frac{690}{\nu(p)}} \prod_{\substack{p \\ 2 < \nu(p) < 691}} \left(\frac{1 - p^{-(\nu(p)-1)s}}{1 - p^{-\nu(p)s}} \right)^{690}, \end{aligned}$$

the truth of which is most easily established by checking that the local factors on both sides agree for every prime p . As before a formula for B_t can now be easily written down, but for reasons of space we leave this to the interested reader. It turns out that the contribution of the last four products in the formula for $T(s)^{690}$ to B_t is less than 10^{-5} in absolute value. We thus have

$$B_t \approx \frac{\log 691}{690^2} - \frac{689}{690} \gamma - \frac{1}{690} \sum_{j=0}^{344} \frac{L'(1, \chi_c^{2j+1})}{L(1, \chi_c^{2j+1})} + \frac{1}{690} \sum_{j=1}^{344} \frac{L'(1, \chi_c^{2j})}{L(1, \chi_c^{2j})}. \quad (10)$$

with an error of at most 10^{-5} .

8 Divisibility of tau by 23

At [1, p. 80] Ramanujan makes a statement of the form (3) with $q = 23$ and $\delta = 1/2$. Using a trick of Wilton [24], $T(s)$ can be found in this case more easily

than by Ramanujan's approach. Using Euler's identity we note that

$$\begin{aligned} \sum_{n=1}^{\infty} \tau(n)x^n &= x\{(1-x)(1-x^2)\cdots\}^{23}(1-x)(1-x^2)\cdots \\ &\equiv xk(x^{23}) \sum_{m=-\infty}^{\infty} (-1)^m x^{\frac{m(3m+1)}{2}} \pmod{23}, \end{aligned}$$

where $k(x) = \prod_{n=1}^{\infty} (1-x^n)$. Now if we also apply Euler's identity to $k(x^{23})$, then we obtain

$$\sum_{n=1}^{\infty} \tau(n)x^n \equiv x \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{m+r} x^{\frac{m(3m+1)}{2} + 23\frac{r(r+1)}{2}} \pmod{23}.$$

From the latter identity Wilton's congruences are easily deduced;

$$\tau(p) \equiv \begin{cases} 1 \pmod{23} & \text{if } p = 23; \\ 0 \pmod{23} & \text{if } (p/23) = -1; \\ 2 \pmod{23} & \text{if } p = U^2 + 23V^2 \text{ with } U \neq 0; \\ -1 \pmod{23} & \text{for other } p \neq 23. \end{cases}$$

Using that $\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1})$ for $k \geq 1$, we can now easily compute $\tau(p^k)$ modulo 23. Let \mathcal{S}_1 denote the set of primes p such that p is a quadratic non-residue mod 23. Let \mathcal{S}_3 denote the set of primes p which can be written as $U^2 + 23V^2$ with $U \neq 0$. Let \mathcal{S}_2 be the set of remaining primes $p \neq 23$. (The sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ have natural densities of respectively, $1/2, 1/3$ and $1/6$, which can be shown using that the class number of $\mathbb{Q}(\sqrt{-23})$ equals 3.) We now find that

$$T(s) = \frac{1}{1-23^{-s}} \prod_{p \in \mathcal{S}_1} \frac{1}{1-p^{-2s}} \prod_{p \in \mathcal{S}_2} \frac{1+p^{-s}}{1-p^{-3s}} \prod_{p \in \mathcal{S}_3} \frac{1-p^{-2s}}{(1-p^{-s})(1-p^{-23s})}.$$

The factor at 23 is not present in Ramanujan's formula (17.6), although it should be there according to his argument. This leads then to an incorrect formula for the first order constant (at the bottom of [1, p. 80]). One should replace the factor $66^{1/2}23^{-3/4}$ by $23^{1/4}\sqrt{3/22}$. It is also clear that where Ramanujan writes 'all primes of the form $23a^2 + b^2$ ', he excludes the prime 23.

We have

$$T(s)^2 = \frac{\zeta(s)L(s, \chi_{-23})}{1-23^{-s}} \prod_{p \in \mathcal{S}_1} \frac{1}{1-p^{-2s}} \prod_{p \in \mathcal{S}_2} \left(\frac{1-p^{-2s}}{1-p^{-3s}} \right)^2 \prod_{p \in \mathcal{S}_3} \left(\frac{1-p^{-2s}}{1-p^{-23s}} \right)^2.$$

From this we easily deduce that

$$\begin{aligned} B_t &= -\frac{\gamma}{2} - \frac{L'(1, \chi_{-23})}{2L(1, \chi_{-23})} + \frac{\log 23}{44} \\ &+ \sum_{p^{11} \equiv -1 \pmod{23}} \frac{\log p}{p^2 - 1} + \sum_{p^{11} \equiv 1 \pmod{23}} \log p \left(\frac{3}{p^3 - 1} - \frac{2}{p^2 - 1} \right) \\ &+ \sum_{\substack{p=U^2+23V^2 \\ p>23}} \log p \left(\frac{2}{p^2 - 1} - \frac{3}{p^3 - 1} + \frac{23}{p^{23} - 1} - \frac{22}{p^{22} - 1} \right). \end{aligned}$$

Remark. To the reader familiar with Cox's beautiful book [3], we suggest as an exercise showing that $p \in \mathcal{S}_3$ if and only if $\left(\frac{p}{23}\right) = 1$ and the congruence $x^3 \equiv x + 1 \pmod{p}$ has an integer solution.

9 Numerical evaluation of the second order constants

The expressions obtained for the various B_t involve both prime sums and values of L and L' at $s = 1$. The prime sums we evaluate termwise and estimate the tail using that, for $k > 1$ and $x \geq 7481$,

$$\sum_{p>x} \frac{\log p}{p^k - 1} \leq \frac{x}{x^k - 1} (-0.98 + 1.017 \frac{k}{k-1}),$$

which follows easily on using the estimate $0.98x \leq \theta(x) \leq 1.017x$ for $x \geq 7481$ [13].

The L and L' values above can be evaluated using generalized Euler constants for arithmetical progressions. We define

$$\gamma_k(r, m) := \lim_{x \rightarrow \infty} \left\{ \sum_{\substack{0 < n \leq x \\ n \equiv r \pmod{m}}} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{m(k+1)} \right\}.$$

Note that $\gamma_0(0, 1) = \gamma$, Euler's constant. Let χ be a non-principal character modulo m . It is not difficult to show [7] that for $k \geq 0$ we have

$$L^{(k)}(1, \chi) = (-1)^k \sum_{r=1}^m \chi(r) \gamma_k(r, m).$$

Using Proposition 12 of [4], the Euler constants $\gamma_k(r, m)$ can be computed with any degree of precision and thus the same holds true for $L^{(k)}(1, \chi)$.

For $q = 5$ we find, using Dilcher's Table 1 [4, S21], that $L'(1, \chi_5)/L(1, \chi_5) = 0.82767947 \dots$ and $L'(1, \chi_c)/L(1, \chi_c) = 0.15786453 \dots - i0.08833613 \dots$. For $q = 7$ we find using Dirichlet's formula that $L(1, \chi_{-7}) = \pi/\sqrt{7}$ and, using Dilcher's Table 1, that $L'(1, \chi_{-7}) = 0.01856598 \dots$. The quotient $L'(1, \chi_{-3})/L(1, \chi_{-3})$ is evaluated with many decimal accuracy in, e.g., [9]. We have $L(1, \chi_{-23}) = 3\pi/\sqrt{23}$ and, on implementing Proposition 12 of [4] in Maple, we find $L'(1, \chi_{-23}) = -0.82955295 \dots$. Similarly we find that the sum involving the odd, respectively even characters in (10) equal $1.9018228 \dots$, respectively $5.10942407 \dots$ (note that *a priori* these sums must be real).

For $q = 3$ we can use the relationship $-\zeta'(2)/\zeta(2) = \sum_p \log p/(p^2 - 1)$ to rewrite B_t as

$$2B_t = 6 \sum_{p \equiv 2 \pmod{3}} \frac{\log p}{p^2 - 1} + 4 \frac{\zeta'(2)}{\zeta(2)} - \frac{L'(1, \chi_{-3})}{L(1, \chi_{-3})} - \gamma + 6 \sum_{p \equiv 1 \pmod{3}} \frac{\log p}{p^3 - 1}.$$

Since all but the last term were either computed with high accuracy in [9] or are easily computable with high accuracy in Maple, we now obtain that $B_t = -0.5349219 \dots$.

Let us for a function f satisfying the conditions of Theorem 1 define $H_f(x) := \sum_{n \leq x} \Lambda_f(n)/n - \tau \log x$. The function Λ_f is most easily computed by computing minus the logarithm of the generating series of f . The numbers $H_f(10^5)$ and

$H_f(10^6)$ ought to be approximations of B_f . The function b is the indicator function of the set of integers that can be written as a sum of two squares. This is, as was already known to Fermat, a multiplicative function. The final column in Table 1 gives Ramanujan’s predicted value for C_2 .

Table 1: Numerical values

	f	$H_f(10^5)$	$H_f(10^6)$	B_f	$C_2(f)$	R’s C_2
$B(x)$	b	+0.163...	+0.162...	+0.1638...	0.5819...	1/2
$q = 5$	t	-0.401...	-0.400...	-0.3995...	0.1501...	1/4
$q = 7$	t	-0.232...	-0.232...	-0.2316...	0.3841...	1/2
$q = 3$	t	-0.532...	-0.534...	-0.5349...	0.2325...	1/2
$q = 691$	t	-0.571...	-0.571...	-0.5717...	0.0006...	1/690
$q = 23$	t	-0.217...	-0.217...	-0.2166...	0.6083...	1/2

Remark. The computations were not carried out far enough to determine the fifth digit in $-0.2166\dots$; it is either a 6 or a 7.

Now we are in the position to prove the following result.

Theorem 3 *All assertions made by Ramanujan in his ‘unpublished’ manuscript on the partition and tau functions [1] of the format (3) are false.*

Proof. Let q be a prime from Table 1. Assume (3) holds true for $r > 1 + \delta$, with δ as in the last column of Table 1. Then the second order coefficient equals δ , which does not match the value of $C_2(t)$ given in Table 1. For the function λ from Section 6 we have $C_2(l) = 0.2325\dots - \frac{1}{2} \log 3 \neq 0.5$. \square

10 On a 1928 paper of Geraldine Stanley

The purpose of Stanley’s paper [17] is to show that two assertions due to Ramanujan are false. The first assertion was already mentioned in the introduction. Stanley’s analysis of this case contains, unfortunately, several misprints and errors, which are corrected in [16].

The second assertion concerns the $5|\tau(n)$ case. In one of the footnotes we read: ”In discussing this question I have used a manuscript of Prof. Hardy, who at one time intended to complete Ramanujan’s work”. Hardy made some headway with this and then apparently later asked Stanley to fill in the further details. The purpose of this section is to correct Geraldine Stanley’s analysis of this case and point out typo’s. With respect to the analysis of the first assertion Shanks [16, p. 75] has written: ”In fact, there are several errors, and these nullify the proof that Ramanujan’s second term is wrong”. Indeed, we will see that the same wording applies to her analysis of the $5|\tau(n)$ case. For information on Geraldine Stanley, the reader is referred to [11].

In the definition of $\psi_1(s)$, for p_2^{-s} read p_1^{-s} . For p_2^{-s} read p_1^{-5s} . A few lines down read $(1, i, -i, -1)$ instead of $(1, i, -i, 1)$. For $(1 - 5^{-s})^{5/4}$ in the formula for $h(s)$ read $(1 - 5^{-s})^{3/4}$. In the denominator of $h(s)$, the factor $(1 - p_3^{-4s})^{3/4}$ has to be added in the denominator (the so corrected $h(s)$ is thus invariant under

permuting p_2 and p_3). In the formula for A at the bottom of p. 236, the factor $(4/5)^{3/2}$ has to be changed to $4/5$. This formula shows that the omission of the factor $(1 - p_3^{-4s})^{3/4}$ in the formula for $h(s)$ was not a mathematical mistake. Also the formula for a_1/A given at p. 237 shows that this factor was in the original formula for $h(s)$. The exponent $5/4$ instead of $3/4$ is consistently worked with in the remainder of the paper though. This leads then to $(4/5)^{3/2}$ instead of $4/5$ in the formula for A and to $5(\log 5)/16$ instead of $3(\log 5)/16$ in the formula for a_1/A .

From the numerical point of view the formula for A is quite awkward since the values $L_j(1)$, $2 \leq j \leq 4$ have been expanded as Euler products and have been regrouped. (On not doing this a numerically more convenient expression for A can be obtained, cf. Section 4.) Thus we have to read

$$\prod \left(\frac{1 - p_4^{-1}}{(1 - p^{-1})^{1/4}} \right) \text{ as } \lim_{x \rightarrow \infty} \frac{\prod_{p_4 \leq x} (1 - p_4^{-1})}{\prod_{p \leq x} (1 - p^{-1})^{1/4}}.$$

Indeed, by the Mertens' theorem for arithmetic progressions [23] this limit exists.

The factor $1/(s+1)^{3/4}$ at the middle of p. 237 has to be replaced by $1/(s-1)^{3/4}$. The formula for b_1 at p. 237 is off by a minus sign (cf. [21, pp. 720-723]). It should read $b_1 = -a_1 \Gamma(\frac{5}{4})/\pi\sqrt{2}$. Consequently the term $(a_1 - A)$ in the first formula for $T(x)$ becomes $(-a_1 - A)$ and in the formula for a_1/A , a_1/A has to be replaced by $-a_1/A$. This sign error nullifies the last two sentences of the paper. (Note that apart from the sign error, $5(\log 5)/16$ has to be replaced by $3(\log 5)/16$ and $L_4(1)$ by $4L_4(1)$.)

References

- [1] B.C. Berndt and K. Ono, Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary, *The Andrews Festschrift* (Maratea, 1998), (Eds.) D. Foata and G.N. Han, 2001, 39–110.
- [2] B.C. Berndt and R.A. Rankin, *Ramanujan. Letters and commentary*, History of Mathematics **9**, AMS, London, 1995.
- [3] D.A. Cox, *Primes of the form $x^2 + ny^2$. Fermat, class field theory and complex multiplication*, Wiley, New York, 1989.
- [4] K. Dilcher, Generalized Euler constants for arithmetical progressions, *Math. Comp.* **59** (1999), 259–282.
- [5] H.M. Edwards, *Fermat's Last Theorem*, Graduate Texts in Mathematics **50**, Springer-Verlag, New York, 1977.
- [6] G.H. Hardy, *Ramanujan: twelve lectures on subjects suggested by his life and work*, Chelsea Publishing Company, New York, 1959.
- [7] J. Knopfmacher, Generalized Euler constants, *Proc. Edinburgh Math. Soc.* **21** (1978), 25–32.

- [8] E. Landau, Über die zu einem algebraischen Zahlkörper gehörige Zetafunktion und die Ausdehnung der Tschebyscheffschen Primzahlentheorie auf das Problem der Verteilung der Primideale, *J. Reine Angew. Math.* **125** (1902), 64-188.
- [9] P. Moree, Chebyshev's bias for composite numbers with restricted prime divisors, arXiv:math.NT/0112100, to appear in *Math. Comp.*.
- [10] P. Moree and J. Cazanar, On a claim of Ramanujan in his first letter to Hardy, *Exposition. Math.* **17** (1999), 289–311.
- [11] R. A. Rankin, Obituary: Gertrude Katherine Stanley, *Bull. London Math. Soc.* **14** (1982), 554–555.
- [12] R. A. Rankin, Ramanujan's manuscripts and notebooks, *Bull. London Math. Soc.* **14** (1982), 81-97.
- [13] J.B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois Journal Math.* **6** (1962), 64-94.
- [14] J.-P. Serre, Divisibilité de certaines fonctions arithmétiques, *Enseignement Math.* **22** (1976), 227-260.
- [15] J.-P. Serre, Quelques applications du théorème de densité de Chebotarev, *Publ. Math. I. H. E. S.* **54** (1981), 123-201.
- [16] D. Shanks, The second-order term in the asymptotic expansion of $B(x)$, *Math. Comp.* **18** (1964), 75–86.
- [17] G.K. Stanley, Two assertions made by Ramanujan, *Journal London Math. Soc.* **3** (1928), 232-237 (1928), Corrigenda, *ibid.* **4** (1929), 32.
- [18] H.P.F. Swinnerton-Dyer, Congruence properties of $\tau(n)$, in *Ramanujan revisited* (Urbana-Champaign, Ill., 1987), 289–311, Academic Press, Boston, MA, 1988.
- [19] L.C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics **83**. Springer-Verlag, New York, 1982.
- [20] G.N. Watson, Über Ramanujansche Kongruenzeigenschaften der Zerfallungsanzahlen. I, *Math. Z.* **39** (1935), 712–731.
- [21] G.N. Watson, Ramanujans Vermutung über Zerfallungsanzahlen, *J. Reine Angew. Math.* **179** (1938), 97–128.
- [22] G.N. Watson, A table of Ramanujan's function $\tau(n)$, *Proc. London Math. Soc.* **51** (1949), 1-13.
- [23] K.S. Williams, Mertens' theorem for arithmetic progressions, *J. Number Theory* **6** (1974), 353-359.
- [24] J.R. Wilton, Congruence properties of Ramanujan's function $\tau(n)$, *Proc. London Math. Soc.* **31** (1930), 1-10.