Analogues of Weyl's Formula for Reduced Enveloping Algebras

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Abstract

In this note we study simple modules for a reduced enveloping algebra $U_\chi(\mathfrak{g})$ in the critical case when $\chi \in \mathfrak{g}^*$ is "nilpotent". Some dimension formulas computed by Jantzen suggest modified versions of Weyl's dimension formula, based on certain reflecting hyperplanes for the affine Weyl group which might be associated to Kazhdan–Lusztig cells.

1 Introduction

In the last decade or so there has been significant progress in understanding the non-restricted representations of the Lie algebra of a reductive group over a field of prime characteristic. Friedlander and Parshall extended the earlier foundations laid by Kac and Weisfeiler, while Premet proved the Kac–Weisfeiler conjecture on the minimum p-power dividing dimensions. More recently the work of Jantzen has reinforced ideas of Lusztig which arise in the framework of affine Hecke algebras and Springer fibers in the flag variety.

In spite of the progress made, serious obstacles remain to a definitive treatment of the representations. Here we attempt to interpret Jantzen's explicit dimension calculations in terms of analogues of Weyl's classical formula, imitating Kazhdan–Lusztig theory for the case of restricted representations.

2 Reduced enveloping algebras

First we recall briefly some essential background and notation, referring for details to the survey [5] and the lectures by Jantzen [8], whose notation we mainly follow. For Lusztig's perspective on these questions, see [17].

2.1

Let G be a simply connected, semisimple algebraic group over an algebraically closed field of characteristic $p > 0$, with Lie algebra $\mathfrak{g}$. Following work of Kac and Weisfeiler, the simple modules for the universal enveloping algebra $U(\mathfrak{g})$ partition into modules for quotients $U_\chi(\mathfrak{g})$ of $U(\mathfrak{g})$ (reduced enveloping algebras) associated with linear functionals $\chi \in \mathfrak{g}^*$. All χ in a coadjoint G-orbit yield isomorphic algebras. If $\chi = 0$, $U_\chi(\mathfrak{g})$ is the restricted enveloping algebra, whose representations include those derived from representations of G.

It has been known since early work of Jacobson and Zassenhaus that the maximum possible dimension of a simple module for $\mathfrak{g}$ or $U(\mathfrak{g})$ is p^N ($N =$ number of positive roots). The Steinberg module in the restricted case is an example where this dimension is achieved.

2.2

The "nilpotent" χ (including $\chi = 0$) play the main role. These correspond to nilpotent elements of $\mathfrak{g}$ when $\mathfrak{g}$ can be identified in a G-equivariant way with $\mathfrak{g}^*$, and form finitely many G-orbits (corresponding naturally to the characteristic 0 orbits when p is good). Probably the most important question about the representation theory of $U_\chi(\mathfrak{g})$ is this:

Question. How does the geometry of the G-orbit $G\chi$ influence the category of $U_\chi(\mathfrak{g})$-modules?

The orbit geometry involves a number of important ideas which have played a major role in characteristic 0 representation theory: Springer's resolution of the nilpotent variety, the flag variety and Springer fibers, affine Weyl groups and Hecke algebras, Kazhdan–Lusztig theory. It seems clear from recent work of Lusztig that many of these same ideas should recur in prime characteristic. In particular, the affine Weyl group W_p relative to p (defined in terms of the Langlands dual of G) has for a long time been known to play a major role in organizing the representation theory of G.

2.3

The category of $U_\chi(\mathfrak{g})$-modules can be enriched by adding a natural action of the centralizer group $C_G(\chi)$. When this group contains at least a 1-dimensional torus T_0, Jantzen is able to obtain graded versions of the Lie algebra actions and exploit translation functors much as in the restricted case.

The best-behaved case occurs when χ has *standard Levi form* in the sense of Friedlander–Parshall [4]: for some choice of Borel subalgebra $\mathfrak{b}$, we have $\chi(\mathfrak{b}) = 0$ while χ vanishes on all negative root vectors $x_{-\alpha}$ except for a set I of simple roots α. (This always happens in type A.) Then the simple $U_\chi(\mathfrak{g})$-modules are parametrized uniformly by linked weights $w \cdot \lambda$ with w running over coset representatives for the subgroup W_I of the Weyl group W generated by corresponding reflections.

In general the parametrization by weights is much less well understood. It may depend in part on the choice of a Borel subalgebra on which χ vanishes: such a Borel subalgebra lies on one or more irreducible components of the Springer fiber. It is also possible that the component group $C_G(\chi)/C_G(\chi)^\circ$ and its characters will play a significant role, as they do in Springer theory. Recent work of Brown and Gordon [3] confirms, at any rate, that the blocks of $U_\chi(\mathfrak{g})$ (when χ is nilpotent) are in natural bijection with linkage classes of restricted weights. Here we consider only the most generic situation, involving simple modules in blocks parametrized by p-regular weights. This requires $p \geq h$ (the Coxeter number).

2.4

So far the most striking general fact about $U_\chi(\mathfrak{g})$-modules is the theorem of Premet [20], valid for arbitrary χ (under mild restrictions on $\mathfrak{g}$ and p):

If d is half the dimension of the coadjoint orbit $G\chi$, then the dimension of every $U_\chi(\mathfrak{g})$-module is divisible by p^d.

This had been conjectured much earlier by Kac and Weisfeiler. In particular, when χ is regular, all simple modules have the maximum possible dimension p^N (N = number of positive roots). Premet's theorem suggests a natural question:

With d as above, does there always exist a simple $U_\chi(\mathfrak{g})$-module $L_\chi(\lambda)$ of the smallest possible dimension p^d?

The answer is yes in the cases investigated so far, but for no obvious conceptual reason unless χ lies in a Richardson orbit (permitting an easy construction by parabolic induction from a trivial module). Our proposed interpretation of dimension formulas stems partly from trying to understand this question better.

3 The restricted case

We recall briefly the standard framework [6] for the study of simple G-modules, which include all simple $U_\chi(\mathfrak{g})$-modules when $\chi = 0$.

For each dominant weight λ there is a Weyl module $V(\lambda)$, whose formal character and dimension are given by Weyl's formulas. In particular,

$$\dim V(\lambda) = \frac{\prod_{\alpha>0}\langle \lambda+\rho, \alpha^\vee\rangle}{\prod_{\alpha>0}\langle \rho, \alpha^\vee\rangle}.$$

Each Weyl module has a unique simple quotient $L(\lambda)$. Those for which λ is *restricted* (the coordinates of λ relative to fundamental weights lying between 0 and $p-1$) are precisely the p^r simple $U_0(\mathfrak{g})$-modules, where r is the rank. Knowing just these modules would allow one to recover all $L(\lambda)$ as twisted tensor products, by Steinberg's Tensor Product Theorem [22]. But so far the broader study of Weyl modules for G has yielded the most concrete results.

Knowledge of the formal characters and dimensions of all $L(\lambda)$ is equivalent to knowledge of the composition factor multiplicities of all $V(\lambda)$. When $p < h$ (the Coxeter number), there is no specific program for finding these multiplicities, but for $p \geq h$ the answer is expected to be given by Lusztig's conjecture. (This is known to be true for "sufficiently large" p, from the work of Andersen–Jantzen–Soergel [1].)

Lusztig's approach depends on the fact that composition factors of $V(\lambda)$ must have highest weights linked to λ under the standard dot action of the affine Weyl group W_p relative to p. Write dominant weights as $w \cdot \lambda$, where λ lies in the lowest alcove of the dominant Weyl chamber. One can in principle express the character of $L(w \cdot \lambda)$ as an alternating sum (with multiplicities) of the known Weyl characters for various weights $w' \cdot \lambda \leq w \cdot \lambda$. The multiplicities are in turn predicted to be the values of Kazhdan–Lusztig polynomials for pairs in W_p related to (w', w), after evaluation at 1. This procedure is inherently recursive and even in low ranks cannot usually be expected to produce simple closed formulas for characters or dimensions of simple modules.

Note how use of the lowest dominant alcove as a starting point locates in a natural way the unique weight $\lambda = 0$ for which $L(\lambda)$ has the smallest possible dimension $p^0 = 1$. This weight is as close as possible to all hyperplanes bounding the alcove below, i.e., minimizes the numerator of Weyl's formula. But cancellation by the denominator is needed to produce 1.

4 Special cases

At the opposite extreme from $\chi = 0$, in the case where the coadjoint orbit of χ is *regular* (with $d = N$), one has $\dim L_\chi(\lambda) = p^N$ for all λ. Much less is known between these extremes.

In a series of recent papers, Jantzen has studied a number of special cases when χ is nilpotent and of small codimension in the nilpotent variety. He obtains explicit dimension formulas for simple modules, as well as many details about projective modules, Ext groups, etc. Unlike the case $\chi = 0$, it is feasible here to work out closed formulas for dimensions.

(a) Type B_2, with χ in the minimal (nonzero) nilpotent orbit was first treated by *ad hoc* methods in [7] and then more systematically in [11]. We take a closer look at this in the following section.

(b) The case when χ lies in the subregular orbit ($d = N - 1$) was initially treated in [9] for the two simple types A_n, B_n where χ can be chosen in standard Levi form. A more comprehensive treatment was then given in [10]. The results are more complete for simply-laced types. When there are two root lengths, the number of simple modules in a typical block is less certain (leading to uncertainty about some of the dimensions), but everything is expected to agree with Lusztig's predictions.

(c) Unpublished work by Jantzen (assisted by B. Jessen for type G_2) deals with a number of other cases, including the nilpotent orbits of G_2 for which $d = 3, 4$ (while $N = 6$) and the "middle" orbit of A_3 (with $d = 4, N = 6$). He also works out families of examples involving standard Levi form: $C_n (n \geq 3)$ with I of type C_{n-1} and $D_n (n \geq 4)$ with I of type D_{n-1}. In each case $d = N - 2$. The results are somewhat less complete in types G_2 $(d = 3)$ and C_n, just as in the subregular case.

It is a striking fact that, in all of these cases, the dimension formulas for simple modules have the same quotient format as Weyl's formula. There is a constant denominator, together with a numerator written as the product of N factors: p repeated d times (in accordance with Premet's Theorem), as well as $N - d$ other factors. Each of these factors involves an affine expression in the coordinates of a ρ-shifted weight based in one reference alcove. One or more weights will minimize the numerator, giving a dimension equal to p^d after dividing by the denominator.

The main drawback to these formulas is that there is a separate one for each simple module (or small family of simple modules) in a typical block. Moreover, there is no obvious way to predict the formulas in advance, apart from the occurrence of p^d.

5 Example: B_2

To explain more concretely our approach to dimension formulas, we look at type B_2 (say $p \geq 5$). Denote the simple roots by α_1 (long) and α_2 (short), with corresponding fundamental weights ϖ_1 and ϖ_2.

5.1

Consider the case when χ lies in the minimal nilpotent orbit (with $N = 4, d = 2$). Here χ has standard Levi form, relative to the subset $I = \{\alpha_1\}$. There is a one-dimensional torus T_0 in $C_G(\chi)$ which acts naturally on $U_\chi(\mathfrak{g})$-modules. A generic block has four simple modules $L_\chi(\lambda)$, each labelled by two "highest" weights λ linked by the subgroup of W generated by the simple reflection s_1. The dimensions of the $L_\chi(\lambda)$ were first worked out by Jantzen in [7]; he later developed a streamlined version based on the systematic use of translation functors in [11, §5]. As required by Premet's Theorem, p^2 divides all dimensions.

To parametrize the simple modules by weights, Jantzen starts in the conventional lowest alcove of the dominant Weyl chamber, fixing a p-regular weight λ. In order to simplify formulas, he builds in the ρ-shift by writing $\lambda + \rho = r\varpi_1 + s\varpi_2 = (r, s)$. Thus $r, s > 0$ while $2r + s < p$. The dimensions of simple modules corresponding to linked weights in the four restricted alcoves are:

$$\frac{s(p-2r-s)}{2}p^2,\ \frac{2pr}{2}p^2,\ \frac{(2p-s)(p-2r-s)}{2}p^2\ \frac{s(p+2r+s)}{2}p^2.$$

Notice that there are two choices of (r, s) which yield a simple module of smallest possible dimension p^2: $((p-3)/2, 1)$ and $((p-3)/2, 2)$. These weights (which parametrize a single module) lie in the second restricted alcove, which suggests that we might instead view that alcove as "lowest" and use a weight there to rewrite Jantzen's formulas. The alcove in question is labelled A in Figure 1.

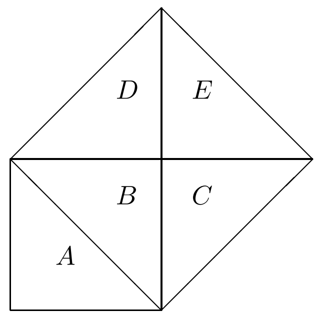

Figure 1: Some alcoves for type B_2

The four dimensions above correspond to linked weights in the respective alcoves A, B, C, D. The simple module of dimension p^2 corresponds to two weights in the lower left corner of alcove A, as close to both vertical and horizontal walls as possible. Moreover, the dimension formula $s(p-2r-s)p^2/2$ for alcove A corresponds in a transparent way to the defining equations $s=0$ and $2r+s=p$ of these two hyperplanes (if we retain Jantzen's standard coordinates). The two special weights minimize the numerator in this dimension formula, giving $2p^2$; the denominator is then needed to cancel the 2. In this way, we can begin to imitate the interpretation of Weyl's formula in Section 3.

5.2

To develop further the analogy with the restricted case, we have to rewrite the dimension formula for alcove A in terms of new (ρ-shifted) coordinates (r, s) of a weight in this alcove. Set

$$\delta(r,s) := s(2r+s-p)p^2/2.$$

Now the idea is to apply this formula to the (ρ-shifted) coordinates (r, s) of an arbitrary weight, in the spirit of Weyl's formula. In particular the formula yields 0 when applied to a weight in the indicated orthogonal hyperplanes bounding alcove A below.

Write briefly $\delta_A, \delta_B, \ldots$ for the formal dimensions obtained by applying δ to linked weights in alcoves $A, B, \ldots$ Denoting the corresponding simple $U_\chi(\mathfrak{g})$-modules by $L_A, L_B, \ldots$, we find the following pattern:

$$\begin{aligned}
\dim L_A &= \delta_A \\
\dim L_B &= \delta_B - \delta_A \\
\dim L_C &= \delta_C - \delta_B + \delta_A \\
\dim L_D &= \delta_D - \delta_B + \delta_A
\end{aligned}$$

This in turn raises the question of the possible existence of modules $V_\chi(\lambda)$ (analogous to Weyl modules in the case $\chi = 0$) having dimensions given by the function δ. These should exist for weights lying in an appropriate collection of alcoves (here infinite) and should be modules for $U_\chi(\mathfrak{g})$ as well as for $C_G(\chi)$.

The alternating sum formulas above are certainly suggestive of a general pattern, though we usually must expect (as for $\chi = 0$) coefficients of absolute value > 1 coming from Kazhdan–Lusztig theory. In our example, alcove E should carry the simple module L_A, but the most likely alternating sum formula will produce a multiple of its dimension such as $3\delta_A$. This suggests associating to a weight in alcove E a $U_\chi(\mathfrak{g})$-module together with a nontrivial representation of an SL_2-type subgroup of $C_G(\chi)$ (whose trivial representation would occur for alcove A). Such a pairing, somewhat analogous to the Springer Correspondence, would be compatible with Lusztig's cell conjectures [12, §10].

In any case, the main thrust of our formulation is the derivation of diverse-looking dimension formulas from a single formula based on a special choice of affine hyperplanes. This much can be conjectured in general, but the explanation for such regularity remains speculative.

6 Kazhdan–Lusztig cells and hyperplanes

6.1

How can one identify suitable affine hyperplanes which might support a version of Weyl's dimension formula for arbitrary nilpotent χ, in the spirit of the above discussion of the minimal nilpotent orbit for type B_2? An answer is suggested by Lusztig's bijection [12] between nilpotent orbits (in good characteristic) and two-sided cells in the affine Weyl group (for the Langlands dual of G). As shown by Lusztig and Xi [18], each two-sided cell in turn meets the dominant Weyl chamber in a "canonical" left cell. In characteristic p we identify the affine Weyl group in question with W_p, allowing us to view the cells as unions of p-alcoves.

For example, the minimal nilpotent orbit of B_2 corresponds to the canonical left cell whose lower portion (beginning with alcoves $A, B, \dots$) is the strip along one wall pictured in Figure 2. To rewrite the previous discussion of dimensions in terms of weights lying in these translated alcoves, we just have to redefine the function δ by

$$\delta(r, s) := (s - p)(2r + s - 2p)p^2/2.$$

This leads to the same dimension formulas as in Sec. 5.

Empirical study of Jantzen's formulas in a variety of cases shows a strong correlation with the hyperplanes bounding below the canonical left cell for χ (but with hyperplanes bounding the dominant region omitted). This is the rationale for our reformulation of his B_2 results above.

In the B_2 example, there are two orthogonal hyperplanes, corresponding to an $A_1 \times A_1$ root system. In other cases studied one gets more complicated root systems, taking in each case the natural hyperplanes corresponding to the

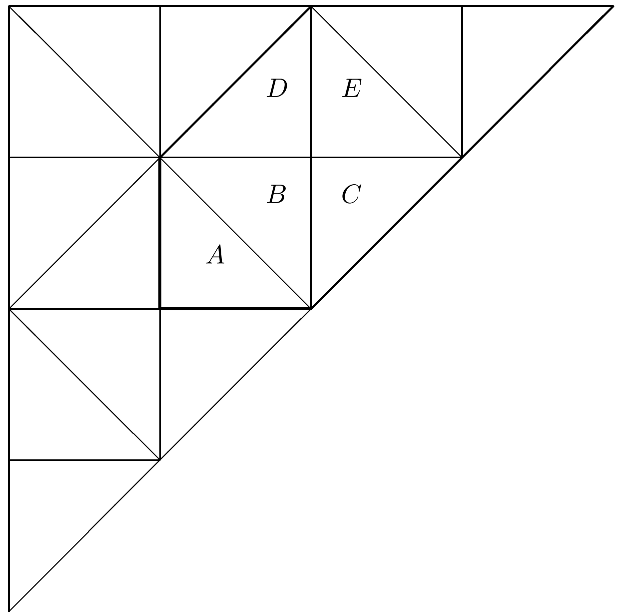

Figure 2: Lower part of a canonical left cell for type B_2

associated positive roots as a framework for the basic dimension formula. Of course, when $\chi = 0$ we are just reverting to Weyl's formula in this way. Such hyperplane systems must come from various proper subsets of the extended Dynkin diagram.

6.2

This type of interpretation agrees well with the location of weights which yield $\dim L_\chi(\lambda) = p^d$, as in the B_2 example. However, this small example is oversimplified in some respects. There are several complicating factors in the attempt to correlate $U_\chi(\mathfrak{g})$-modules with cells:

(a) It is not easy to describe geometrically the lower boundary behavior of canonical left cells, though the work of Shi [21] in type A (and further work by him and his associates in other cases) provides a lot of combinatorial data. It is clear that one cannot in general expect to find a unique lowest alcove in a canonical left cell. For example, the minimal orbit for type A_3 (where $N = 6$ and $d = 3$) yields a cell with two symmetrically placed configurations of lower hyperplanes of A_2 type. Here one expects three factors corresponding to positive roots of A_2 in the conjectural dimension formulas.

(b) One cannot always point to an obvious hyperplane configuration of the right size. An extreme case to keep in mind is the minimal nilpotent orbit of E_8, where $N = 120$ and $d = 29$. The 91 expected factors in a dimension formula might well arise from a combination of the 28 positive roots in an

A_7 root system and another 63 positive roots in an E_7 root system (both found in the extended Dynkin diagram). It is unclear how to predict such patterns in general, though they may be related to a duality for nilpotent orbits studied by Sommers. Note that for type A one has a simple version of duality (based on transpose partitions) which might suggest a natural choice of hyperplanes.

(c) When the component group $C_G(\chi)/C_G(\chi)^\circ$ is nontrivial, it may permute a number of nonisomorphic simple modules having the same dimension [10]. This already shows up in subregular cases for B_2 or G_2, in a way that looks consistent with Lusztig's conjectures in [12, §10]: each intersection of a left cell with its inverse should correspond to an orbit of the component group in the set of simple modules belonging to a typical block of $U_\chi(\mathfrak{g})$.

6.3

How can one construct modules $V_\chi(\lambda)$ for $U_\chi(\mathfrak{g})$ and $C_G(\chi)$ which carry dimension formulas of the shape we have described? One approach has been initiated by Mirković and Rumynin [19], but many technical problems remain. The natural starting point is the Springer fiber associated with χ, whose dimension is $N - d$.

Ultimately all of this should connect naturally with the ideas of Lusztig [13, 14, 15, 16, 17] involving Springer fibers, equivariant K-theory, affine Hecke algebras, cells, etc. Significant progress has recently been made by Bezrukavnikov, Mirković, and Rumynin [2].

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