

Laplacians in Odd Symplectic Geometry

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Abstract

We consider odd Laplace operators arising in odd symplectic geometry. Approach based on semidensities (densities of weight $1/2$) is developed. The role of semidensities in the Batalin–Vilkovisky formalism is explained. In particular, we study the relations between semidensities on an odd symplectic supermanifold and differential forms on a purely even Lagrangian submanifold. We establish a criterion of “normality” of a volume form on an odd symplectic supermanifold in terms of the canonical odd Laplacian acting on semidensities.

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1 Symplectic and Poisson structures

A symplectic structure on a manifold M is defined by a non-degenerate closed two-form ω . In a vicinity of an arbitrary point one can consider coordinates (x^1, \dots, x^{2n}) such that $\omega = \sum_{i=1}^n dx^i dx^{i+n}$. Such coordinates are called Darboux coordinates. To a symplectic structure corresponds a non-degenerate Poisson structure $\{ , \}$. In Darboux coordinates $\{x^i, x^j\} = 0$ if $|i - j| \neq n$ and $\{x^i, x^{i+n}\} = -\{x^{i+n}, x^i\} = 1$. The condition of closedness of the two-form ω corresponds to the Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

for the Poisson bracket. If a symplectic or Poisson structure is given, then every function f defines a vector field (the Hamiltonian vector field) \mathbf{D}_f such that $\mathbf{D}_f g = \{f, g\} = -\omega(\mathbf{D}_f, \mathbf{D}_g)$.

A Poisson structure can be defined independently of a symplectic structure (see below). In general it can be degenerate, i.e., there exist non-constant functions f such that $\mathbf{D}_f = 0$. In the case when a Poisson structure is non-degenerate (corresponds to a symplectic structure), the map from T^*M to TM defined by the relation $f \mapsto \mathbf{D}_f$ is an isomorphism.

One can straightforwardly generalize these constructions to the supercase and consider symplectic and Poisson structures (even or odd) on supermanifolds. An even (odd) symplectic structure on a supermanifold is defined by an even (odd) non-degenerate closed two-form. In the same way as the existence of a symplectic structure on an ordinary manifold implies that the manifold is even-dimensional (by the non-degeneracy condition for the form ω), the existence of an even or odd symplectic structure on a supermanifold implies that the dimension of the supermanifold is equal either to $(2p, q)$ for an even structure or to (m, m) for an odd structure. Darboux coordinates exist in both cases. For an even structure, the two-form in Darboux coordinates $z^A = (x^1, \dots, x^{2p}; \theta_1, \dots, \theta_q)$ has the form $\sum_{i=1}^p dx^i dx^{p+i} + \sum_{a=1}^q \varepsilon_a d\theta_a d\theta_a$, where $\varepsilon_a = \pm 1$. For an odd structure, the two-form in Darboux coordinates $z^A = (x^1, \dots, x^m; \theta_1, \dots, \theta_m)$ has the form $\sum_{i=1}^m dx^i d\theta_i$.

The non-degenerate odd Poisson bracket corresponding to an odd symplectic structure has the following appearance in Darboux coordinates: $\{x^i, x^j\} = 0$, $\{\theta_i, \theta_j\} = 0$ for all i, j and $\{x^i, \theta_j\} = -\{\theta_j, x^i\} = \delta_j^i$. Thus for arbitrary two functions f, g

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta_i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial x^i} \right), \quad (1.1)$$

where we denote by $p(f)$ the parity of a function f (e.g., $p(x^i) = 0$, $p(\theta_j) = 1$). Similarly one can write down the formulae for the non-degenerate even Poisson structure corresponding to an even symplectic structure.

A Poisson structure (odd or even) can be defined on a supermanifold independently of a symplectic structure as a bilinear operation on functions (bracket) satisfying the following relations taken as axioms:

$$p(\{f, g\}) = p(f) + p(g) + \varepsilon, \quad (1.2)$$

$$\{f, g\} = -\{g, f\}(-1)^{(p(f)+\varepsilon)(p(g)+\varepsilon)}, \quad (1.3)$$

$$\{f, gh\} = \{f, g\}h + \{f, h\}g(-1)^{p(g)p(h)} \quad (\text{Leibniz rule}), \quad (1.4)$$

$$\{f, \{g, h\}\}(-1)^{(p(f)+\varepsilon)(p(h)+\varepsilon)} + \text{cycl.} = 0 \quad (\text{Jacobi identity}), \quad (1.5)$$

where ε is the parity of the bracket ($\varepsilon = 0$ for an even Poisson structure and $\varepsilon = 1$ for an odd one). The correspondence between functions and Hamiltonian vector fields is defined in the same way as on ordinary manifolds: $\mathbf{D}_f g = \{f, g\}$. Notice a possible parity shift: $p(\mathbf{D}_f) = p(f) + \varepsilon$. Every Hamiltonian vector field \mathbf{D}_f defines an infinitesimal transformation preserving the Poisson structure (and the corresponding symplectic structure in the case of a non-degenerate Poisson bracket).

Notice that even or odd Poisson structures on an arbitrary supermanifold can be obtained as “derived” brackets from the canonical symplectic structure on the cotangent bundle, in the following way.

Let M be a supermanifold and T^*M be its cotangent bundle. By changing parity of coordinates in the fibres of T^*M we arrive at the supermanifold ΠT^*M . If z^A are arbitrary coordinates on the supermanifold M , then we denote by (z^A, p_B) the corresponding coordinates on the supermanifold T^*M and by (z^A, z_{*B}) the corresponding coordinates on ΠT^*M : $p(z^A) = p(p_A) = p(z_{*A}) + 1$. If $(z^{A'})$ are another coordinates on M , $z^A = z^A(z')$, then the coordinates z_{*A} transform in the same way as the coordinates p_A (and as the partial derivatives $\partial/\partial z^A$):

$$p_{A'} = \frac{\partial z^B(z')}{\partial z^{A'}} p_B \quad \text{and} \quad z_{*A'} = \frac{\partial z^B(z')}{\partial z^{A'}} z_{*B}. \quad (1.6)$$

One can consider the canonical non-degenerate even Poisson structure $\{ , \}_0$ (the canonical even symplectic structure) on T^*M defined by the relations $\{z^A, z^B\}_0 = \{p_C, p_D\}_0 = 0$, $\{z^A, p_B\}_0 = \delta_B^A$, and, respectively, the canonical non-degenerate odd Poisson structure $\{ , \}_1$ (the canonical odd symplectic structure) on ΠT^*M defined by the relations $\{z^A, z^B\}_0 = \{z_{*C}, z_{*D}\}_0 = 0$, $\{z^A, z_{*B}\}_0 = \delta_B^A$.

Now consider Hamiltonians on T^*M or on ΠT^*M that are quadratic in coordinates of the fibres. An arbitrary odd quadratic Hamiltonian on T^*M (an arbitrary even quadratic Hamiltonian on ΠT^*M):

$$\mathfrak{S}(z, p) = \mathfrak{S}^{AB} p_A p_B \quad (p(\mathfrak{S}) = 1) \quad \text{or} \quad \mathfrak{S}(z, z_*) = \mathfrak{S}^{AB} z_{*A} z_{*B} \quad (p(\mathfrak{S}) = 0), \quad (1.7)$$

satisfying the condition that the canonical Poisson bracket of this Hamiltonian with itself vanishes:

$$\{\mathfrak{S}, \mathfrak{S}\}_0 = 0 \quad \text{or} \quad \{\mathfrak{S}, \mathfrak{S}\}_1 = 0 \quad (1.8)$$

defines an odd Poisson structure (an even Poisson structure) on M by the formula

$$\{f, g\}_{\varepsilon+1}^{\mathfrak{S}} = \{f, \{\mathfrak{S}, g\}_{\varepsilon}\}_{\varepsilon}. \quad (1.9)$$

The Hamiltonian \mathfrak{S} which generates an odd (even) Poisson structure on M via the canonical even (odd) Poisson structure on T^*M (ΠT^*M) can be called the master Hamiltonian. The bracket (1.9) is a “derived bracket”. The Jacobi identity for it is equivalent to the vanishing of the canonical Poisson bracket for the master Hamiltonian. One can see that an arbitrary Poisson structure on a supermanifold can be obtained as a derived bracket.

What happens if we change the parity of the master Hamiltonian in (1.9)? The answer is the following. If \mathfrak{S} is an even quadratic Hamiltonian on T^*M (an odd quadratic Hamiltonian on ΠT^*M), then the condition of vanishing of the canonical even Poisson bracket $\{ , \}_0$ (the canonical odd Poisson bracket $\{ , \}_1$) becomes empty (it is obeyed automatically) and the relation (1.9) defines an even Riemannian metric (an odd Riemannian metric) on M .

Formally, odd symplectic (and odd Poisson) geometry is a generalization of symplectic (Poisson) geometry to the supercase. However, there are unexpected analogies between the constructions in odd symplectic geometry and in Riemannian geometry (see [10] and later below). The construction of derived brackets could explain close relations between odd Poisson structures in supermathematics and the Riemannian geometry (see [10]).

The construction of the derived bracket (without the name) and the elaboration of the unified viewpoint for different geometries in terms of derived brackets are due to T. Voronov [17]. Derived brackets (under this name) were independently introduced and studied in [11]. It has to be noted that in the physical literature the relations of the type (1.9) for brackets of different parity were considered in [12] and [13], where they were used for obtaining derived brackets on Lagrangian surfaces. This approach was considered later in [5] and essentially developed in [4], where constructions involving generalized “higher order” even and odd Poisson brackets appeared.

In what follows we consider second order differential operators on an odd symplectic supermanifold and study their geometric properties. Some of our constructions can be automatically considered in the case of a general odd Poisson structure.

2 Odd Laplacians on functions

2.1 Definition and properties

In ordinary symplectic geometry the symplectomorphisms of M^{2n} (the diffeomorphisms preserving the two-form ω) preserve the volume form $\rho_\omega = \omega^n$ (Liouville’s theorem). What is the situation in the supercase? In spite of the fact that differential forms are not objects of integration on supermanifolds (see details in [16]), the Liouville theorem still holds in even symplectic ge-

ometry. One can see that the coordinate volume form $\rho = \mathcal{D}z$ in Darboux coordinates defines a global volume form that is preserved under symplectomorphisms. The situation is drastically different for an odd symplectic structure. Let $z^A = (x^1, \dots, x^n, \theta_1, \dots, \theta_n)$ be Darboux coordinates. Consider, for example, the transformation $x^1 \mapsto 2x^1$, $\theta_1 \mapsto \frac{1}{2}\theta_1$ to another Darboux coordinates. The Berezinian of this transformation is equal to 4, hence the coordinate volume $\mathcal{D}z$ form is not preserved. One can prove (see below) that on an odd symplectic supermanifold there is no volume form invariant w.r.t. all symplectomorphisms.

Let ρ be an arbitrary volume form on an odd symplectic supermanifold. Consider the linear differential operator Δ_ρ on functions such that its action on a function f is equal (up to a coefficient) to the divergence $\text{div}_\rho \mathbf{D}_f$ of the Hamiltonian vector field \mathbf{D}_f w.r.t. to the volume form ρ :

$$\Delta_\rho f := \frac{1}{2} (-1)^{p(f)} \text{div}_\rho \mathbf{D}_f = \frac{1}{2} (-1)^{p(f)} \mathcal{L}_{\mathbf{D}_f} \log \rho = \frac{1}{2} (-1)^{p(f)} \frac{\mathcal{L}_{\mathbf{D}_f} \rho}{\rho}. \quad (2.1)$$

Let $(x^1, \dots, x^n; \theta_1, \dots, \theta_n)$ be Darboux coordinates and let $\rho = \rho(x, \theta) \mathcal{D}(x, \theta)$ in these coordinates. It follows from (1.1) that

$$\mathbf{D}_f = \frac{\partial f}{\partial x^i} \frac{\partial}{\partial \theta_i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial x^i}.$$

Hence

$$\Delta_\rho f = \Delta_0 f + \frac{1}{2} \{\log \rho, f\}, \quad (2.2)$$

where

$$\Delta_0 f = \frac{\partial^2 f}{\partial x^i \partial \theta_i}. \quad (2.3)$$

We come to the *odd Laplacian on functions*, a second order differential operator depending on the volume form.

Notice that (2.1) defines the operator Δ_ρ in terms of the Poisson bracket. This expression defines a linear operator on functions for an arbitrary Poisson structure, even or odd. It is an operator of the second order for an odd Poisson structure and an operator of the first order for an even Poisson structure. In the even case this operator of the first order is a Poisson vector field (the divergence of the Poisson bivector) specifying the so-called Weinstein's modular class of an even Poisson manifold [18].

If ρ and ρ' are two volume forms, $\rho' = g\rho$, then

$$\text{div}_{\rho'} f - \text{div}_\rho f = \frac{\mathcal{L}_{\mathbf{D}_f} g}{g} = \{f, \log g\}. \quad (2.4)$$

In particular, the existence of a “canonical” volume form ρ_ω preserved under all symplectomorphisms would imply that the operator (2.1) is a first order differential operator given by the r.h.s. of (2.4), because (2.1) would evidently vanish for $\rho = \rho_\omega$. On the other hand, by (2.2) and (2.3), Δ_ρ is a second order differential operator. Thus we have proved that *on an odd symplectic supermanifold there is no canonical volume form*.

This should be compared with the even Poisson situation where Weinstein’s modular class (see above) is the obstruction to the existence of a volume form invariant under all Hamiltonian flows, and it vanishes in the (even) symplectic case. We see that the situation with the odd bracket (symplectic or not) is more complicated.

Now consider the properties of the odd Laplacian Δ_ρ .

One can see that for an arbitrary odd Poisson supermanifold the Leibniz rule for the second derivatives takes the following form for the odd Laplacian:

$$\Delta_\rho(f \cdot g) = \Delta_\rho f \cdot g + (-1)^{p(f)}\{f, g\} + (-1)^{p(f)}f \cdot \Delta_\rho g. \quad (2.5)$$

In other words, the operator Δ_ρ generates the Poisson structure.

We have already mentioned that (2.5) has a straightforward analogue in Riemannian geometry: $\Delta(fg) = g\Delta f + \langle df, dg \rangle + f\Delta g$, where $\langle \cdot, \cdot \rangle$ is the scalar product given by the Riemannian metric and Δ is the Beltrami–Laplace operator corresponding to the metric (see details in [10]).

Another very important property of the odd Laplacian (on an arbitrary odd Poisson supermanifold) is that it preserves the Poisson bracket:

$$\Delta_\rho\{f, g\} = \{\Delta_\rho f, g\} + (-1)^{p(f)+1}\{f, \Delta_\rho g\}. \quad (2.6)$$

Now let us return to the canonical odd symplectic structure on ΠT^*M (see the first section). We consider the case when M is a usual n -dimensional manifold. The base manifold M is a Lagrangian $(n,0)$ -dimensional surface in the (n,n) -dimensional odd symplectic supermanifold ΠT^*M . This example can be considered as the basic example of an odd symplectic supermanifold ¹.

Functions on ΠT^*M encode multivector fields on M :

$$\begin{aligned} f(x, \theta) &= f(x) + f^i(x)\theta_i + f^{ik}(x)\theta_i\theta_k + \dots \\ &\leftrightarrow T = f(x) + f^i(x)\partial_i + f^{ik}(x)\partial_i \wedge \partial_k + \dots \end{aligned} \quad (2.7)$$

¹For a usual symplectic manifold E and a Lagrangian surface L in it there exists a symplectomorphism between the cotangent bundle T^*L and, in general, only a tubular neighborhood of L in E . The triviality of topology in odd directions allows to identify the cotangent bundle ΠT^*L to a purely even Lagrangian surface in an odd symplectic supermanifold E with the whole supermanifold E if L coincides with the underlying manifold of E (see [9] for details).

The odd Poisson bracket of functions on ΠT^*M corresponds to the Schouten bracket (“skew-symmetric concomitant”) of multivector fields.

To every diffeomorphism of M naturally corresponds the induced symplectomorphism of ΠT^*M , but in general one can consider symplectomorphisms that do not correspond to diffeomorphisms of the base and destroy the cotangent bundle structure (see the next section).

Let us now analyze the meaning of an odd Laplacian (2.1) on ΠT^*M . Let (x^i) be arbitrary coordinates on M and (x^i, θ_j) the corresponding Darboux coordinates on ΠT^*M . Let $\sigma = \mathcal{D}(x)$ and $\rho = \mathcal{D}(x, \theta)$ be the coordinate volume forms on M and ΠT^*M respectively. Then the odd Laplacian Δ_ρ on ΠT^*M is given by the formula (2.3). It is obvious from (2.3) that in this case the action of the operator Δ_ρ on functions on ΠT^*M corresponds to the divergence of multivector fields on M with respect to the coordinate volume form σ . Every volume form on M has a local appearance as a coordinate volume form (in some local coordinate system). On the other hand, it follows from (1.6) that the determinant of an arbitrary coordinate transformation $x \mapsto x' = x'(x)$ on M is equal to the square root of the Berezinian of the corresponding coordinate transformation $(x, \theta) \mapsto (x', \theta')$:

$$\text{Ber} \frac{\partial(x', \theta')}{\partial(x, \theta)} = \text{Ber} \begin{pmatrix} \frac{\partial x^i}{\partial x^i} & * \\ 0 & \frac{\partial x^i}{\partial x^{i'}} \end{pmatrix} = \left(\det \left(\frac{\partial x^i}{\partial x^{i'}} \right) \right)^2. \quad (2.8)$$

Hence we come to important conclusions: (a) to every volume form σ on M corresponds a volume form $\rho = \sigma^2$ on ΠT^*M ; (b) $\text{div}_\sigma T = \Delta_{\sigma^2} f$, where we identify multivector fields on M with functions on ΠT^*M by (2.7); (c) for a volume form $\rho = \sigma^2$ on ΠT^*M holds the condition

$$\Delta_\rho^2 = 0, \quad (2.9)$$

because the square of the divergence operator on multivector fields equals zero.

These relations make a bridge between odd symplectic geometry and classical vector calculus. They are closely related with the geometric meaning of the Batalin–Vilkovisky formalism (see [8]).

What can we say about Δ_ρ^2 in the general case?

For an arbitrary odd Poisson supermanifold the operator Δ_ρ^2 is a Poisson vector field: $\Delta_\rho^2(fg) = (\Delta_\rho^2 f)g + f(\Delta_\rho^2 g)$ and Δ_ρ^2 preserves the Poisson bracket. This follows from relations (2.5), (2.6). Under a change of volume form $\rho \mapsto \rho' = g\rho$ this Poisson vector field changes by a Hamiltonian vector field:

$$\Delta_{\rho'}^2 - \Delta_\rho^2 = \mathbf{D}_{H(\rho', \rho)} \quad \text{where} \quad H(\rho', \rho) = \frac{1}{\sqrt{g}} \Delta_\rho \sqrt{g}. \quad (2.10)$$

This relation leads to a non-trivial groupoid structure [10].

For an odd symplectic supermanifold one can always pick a volume form ρ such that $\Delta_\rho^2 = 0$, namely, as we have shown above, one can identify the symplectic supermanifold with ΠT^*L for an purely even Lagrangian surface L by a suitable symplectomorphism and then choose $\rho = \sigma^2$, yielding (2.9). Hence, it follows from (2.10) that for odd Laplacians on a symplectic supermanifold the operator Δ_ρ^2 is always a Hamiltonian vector field.

Definition 2.1. *A volume form ρ on an odd symplectic supermanifold E is called normal if in a vicinity of an arbitrary point there exist Darboux coordinates (x, θ) such that ρ is the coordinate volume form in these Darboux coordinates: $\rho = \mathcal{D}(x, \theta)$.*

The volume form $\rho = \sigma^2$ on ΠT^*M in (2.9) is a normal volume form by definition. If a volume form ρ is normal, then $\Delta_\rho^2 = 0$, by the definition of the odd Laplacian (2.2). Does the Hamiltonian field Δ_ρ^2 has to be equal to zero for every volume form ρ ? As it follows from (2.10), the answer is, generally, no. Does the condition $\Delta_\rho^2 = 0$ implies that ρ is a normal volume form? We will give a detailed analysis in the next sections, where we will study the canonical odd Laplacian acting on semidensities. Now two words about where odd Laplacians come from.

2.2 Where an odd Laplacian comes from

Odd Laplacians have appeared in mathematical physics for the first time around 1981 in the pioneer works by Batalin and Vilkovisky [1, 2, 3] for the purpose of constructing a Lagrangian version of the BRST quantization (the “BV-formalism”). Batalin and Vilkovisky introduced an odd Laplacian acting on functions on an odd symplectic supermanifold as $\Delta_0 = \frac{\partial^2}{\partial x^i \partial \theta_i}$, where (x^i, θ_i) are some Darboux coordinates. (The invariant definition (2.1) of an odd Laplacian Δ_ρ depending on a volume form ρ was given later in [6].) The following very important properties of this operator were fixed in their works (see [3]). If (x', θ') are another Darboux coordinates, and Δ_0 and Δ'_0 denote the odd Laplacians (2.3) in the Darboux coordinates (x, θ) and (x', θ') respectively, then

$$\Delta_0 = \Delta'_0 + \frac{1}{2} \left\{ \log \text{Ber} \frac{\partial(x', \theta')}{\partial(x, \theta)}, \right\} \quad (2.11)$$

and

$$\Delta_0 \left(\text{Ber} \frac{\partial(x', \theta')}{\partial(x, \theta)} \right)^{1/2} = 0 \quad (\text{“Batalin–Vilkovisky identity”}). \quad (2.12)$$

The property (2.11) of the operator Δ_0 is closely related with the invariant expression (2.2) for Δ_ρ . The second property (2.12), which was stated in [3]

(in a non-explicit form it also appeared in [15]), is highly non-trivial. This identity is deeply related with the existence of canonical odd Laplacian on semidensities (see the next section).

The operator Δ_0 was introduced in [1] for the purpose of formulating the so-called Batalin–Vilkovisky quantum master equation, which is the equation

$$\Delta_0 \sqrt{f(x, \theta)} = 0 \quad (2.13)$$

on the function $f(x, \theta) = \exp \frac{i\mathcal{S}(x, \theta)}{\hbar}$, where $\rho = \mathcal{D}(x, \theta)$ is a coordinate volume form in the space of fields and antifields ((x, θ) are Darboux coordinates). The master action $\mathcal{S}(x, \theta)$ defines a measure element on Lagrangian surfaces corresponding to gauge choice. This measure is gauge invariant if the master equation is satisfied (see [1, 2, 3]). Using the identity $\Delta_0 \exp g = (\Delta_0 g + 1/2\{g, g\}) \exp g$, we can rewrite the quantum master equation (2.13) as $-4\hbar\Delta_0\mathcal{S} + \{\mathcal{S}, \mathcal{S}\} = 0$ and taking $\hbar \rightarrow 0$ we arrive at the Batalin–Vilkovisky classical master equation: $\{\mathcal{S}, \mathcal{S}\} = 0$. The geometrical meaning of the master equation was studied in [7, 8] and most notably by A. S. Schwarz in [14]. In particular, the following result was obtained. Suppose ρ is a normal volume form. Then there are implications:

$$\text{the volume form } \rho' = f(x, \theta)\rho \text{ is normal} \Rightarrow \Delta_\rho \sqrt{f} = 0 \Rightarrow \Delta_\rho^2 = 0. \quad (2.14)$$

In this statement the master equation is not formulated invariantly, but it stands between two invariant conditions. The exact statement about the relations between the three conditions in (2.14) will be formulated in the next section in the language of semidensities.

3 Canonical odd Laplacian on semidensities

3.1 Definition of the canonical Laplacian

A density of weight t on a supermanifold is a function of local coordinates such that under a change of variables it is multiplied by the t -th power of the Berezinian of transformation. We will consider semidensities (densities of weight $t = \frac{1}{2}$) on an odd symplectic supermanifold.

First of all, let us consider again the supermanifold ΠT^*M for a usual manifold M , with the canonical symplectic structure. Recall that functions on ΠT^*M encode multivector fields on M (see (2.7)). Our claim [9] is that *semidensities on ΠT^*M encode differential forms on M .*

Indeed, let (x^1, \dots, x^n) be arbitrary local coordinates on the manifold M . Let $(x^1, \dots, x^n; \theta_1, \dots, \theta_n)$ be the corresponding Darboux coordinates on ΠT^*M . In the same way as multivector fields on M can be identified with

functions on ΠT^*M , differential forms on M can be identified with functions on the supermanifold $\Pi T M$ obtained from the tangent bundle $T M$ by changing parity of coordinates in the fibres. If $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ are the coordinates on $\Pi T M$ corresponding to coordinates (x^1, \dots, x^n) on M ($p(\xi^k) = 1$), then a differential form $\omega_0(x) + dx^i \omega_i(x) + \dots$ can be identified with the function $\omega(x, \xi) = \omega_0(x) + \xi^i \omega_i(x) + \dots$ on $\Pi T M$.

Recall that given a volume form $\sigma = \sigma(x) \mathcal{D}x = \sigma(x) dx^1 \dots dx^n$ on M the Hodge operator transforms a multivector field on M corresponding to the function $f(x, \theta)$ on ΠT^*M to the differential form on M corresponding to the function $\omega(x, \xi)$ on $\Pi T M$, where

$$\omega(x, \xi) = \int \exp(i\xi^i \theta_i) f(x, \theta) \sigma(x) \Delta \theta$$

(a ‘‘Fourier transform’’). (In a conventional language it is a contraction of a top order form with a multivector, but the language of integrals is more flexible.) It follows that without a volume form, the Hodge operator acts on multivector densities of weight $t = 1$ on M transforming them into differential forms on M , and vice versa.

On the other hand, from (2.8) it follows that under canonical transformations on ΠT^*M induced by changes of coordinates of M a density on M transforms as a semidensity on ΠT^*M .

Hence we come to a 1-1-correspondence between differential forms on M (functions on $\Pi T M$) and semidensities on ΠT^*M , as follows²:

$$\omega(x, \xi) \mapsto \mathbf{s} = s(x, \theta) \sqrt{\Delta(x, \theta)} = \left(\int \exp(-i\xi\theta) \omega(x, \xi) \Delta \xi \right) \sqrt{\mathcal{D}(x, \theta)} \quad (3.1)$$

(This map for the first time appeared in [14] in a non-explicit way.) For example, if M is a two-dimensional manifold, then $f(x) \mapsto f(x) \theta_1 \theta_2 \sqrt{\Delta(x, \theta)}$, $\omega_1 dx^1 + \omega_2 dx^2 \mapsto (\omega_1 \theta_2 - \omega_2 \theta_1) \sqrt{\Delta(x, \theta)}$, $\omega dx^1 dx^2 \mapsto -\omega \sqrt{\Delta(x, \theta)}$

The relation (3.1) between forms on M and semidensities on ΠT^*M suggests that there exists a linear operator on semidensities on odd symplectic supermanifolds corresponding to the exterior differential.

Definition 3.1 ([9]). *Let \mathbf{s} be a semidensity on an odd symplectic supermanifold E . We assign to it a semidensity $\Delta \mathbf{s}$ by the following formula: if $\mathbf{s} = s(x, \theta) \sqrt{\mathcal{D}(x, \theta)}$ in some Darboux coordinates, then in the same coordinates*

$$\Delta \mathbf{s} := (\Delta_0 s(x, \theta)) \sqrt{\mathcal{D}(x, \theta)} = \frac{\partial^2 s}{\partial x^i \partial \theta_i} \sqrt{\mathcal{D}(x, \theta)}. \quad (3.2)$$

²In these considerations we assume that the manifold M is orientable and an orientation is chosen.

We call this operator the canonical Laplacian on semidensities. (See [9] for details.)

In the case of $E = \Pi T^*M$ one can see from (3.1) that this definition gives exactly the de Rham exterior differential:

$$\omega \mapsto \mathbf{s}_\omega \quad \Rightarrow \quad \Delta \mathbf{s}_\omega = \mathbf{s}_{d\omega}. \quad (3.3)$$

Of course, this relation is not a proof that the operator given by (3.2) is well-defined for a general odd symplectic supermanifold E , because though one might consider E as ΠT^*M for some manifold M the identification (3.1) fails under symplectomorphisms which are not induced by diffeomorphisms of M .

To prove that the canonical operator is well-defined by formula (3.2) one has to prove that the r.h.s. of (3.2) indeed defines a semidensity, i.e., under an arbitrary transformation from Darboux coordinates (x, θ) to another Darboux coordinates (x', θ') we have

$$(\Delta_0 s) \cdot \left(\text{Ber} \frac{\partial(x, \theta)}{\partial(x', \theta')} \right)^{1/2} = \Delta'_0 \left(s \cdot \left(\text{Ber} \frac{\partial(x, \theta)}{\partial(x', \theta')} \right)^{1/2} \right), \quad (3.4)$$

where we denote by Δ_0 and Δ'_0 the ‘‘coordinate’’ odd Laplacians (2.3) in the Darboux coordinates (x, θ) and (x', θ') , respectively.

Notice (see [9] for details) that every transformation from Darboux coordinates (x, θ) to new Darboux coordinates (x', θ') can be represented as the composition of transformations of the following types:

transformations corresponding to $x' = x'(x)$:

$$x' = x'(x), \quad \theta_{i'} = \frac{\partial x^i}{\partial x^{i'}} \theta_i, \quad (3.5)$$

transformations identical on the surface $\theta = 0$:

$$x'(x, \theta)|_{\theta=0} = x, \quad \theta'(x, \theta)|_{\theta=0} = \theta, \quad (3.6)$$

transformations identical on even coordinates:

$$x' = x, \quad \theta_{i'} = \theta_i + \alpha_i \quad \text{such that} \quad \partial_i \alpha_j - \partial_j \alpha_i = 0. \quad (3.7)$$

It is sufficient to check condition (3.4) for transformations (3.5), (3.6) and (3.7) separately. For transformations (3.5) it follows from the identity (2.8). The Berezinian of transformation (3.7) equals 1, hence (3.4) is satisfied. One can show that transformation (3.6) is induced by a Hamiltonian vector field (see [9]), hence and it is sufficient to check it infinitesimally.

Infinitesimal transformations are generated by odd functions (Hamiltonians) via the corresponding Hamiltonian vector fields. To an odd Hamiltonian $Q(z)$ corresponds the infinitesimal canonical transformation $\tilde{z}^A = z^A + \varepsilon\{Q, z^A\}$ generated by the vector field \mathbf{D}_Q . The action of it on a semidensity \mathbf{s} can be expressed by a “differential” $\delta_Q(s\sqrt{\Delta z}) = \Delta_0 Q \cdot s\sqrt{\Delta z} - \{Q, s\}\sqrt{\Delta z}$, because $\delta s = -\varepsilon\{Q, s\}$ and $\delta\Delta z = \varepsilon\delta\text{Ber}(\partial z/\partial\tilde{z})\Delta z = \varepsilon 2\Delta_0 Q\Delta z$ for the infinitesimal transformation generated by Q . Using $\Delta_0^2 = 0$ and equation (2.5), we come to the commutation relation $\Delta_0\delta_Q = \delta_Q\Delta_0$. Thus condition (3.4) is satisfied for infinitesimal transformations.

3.2 Properties of the canonical Laplacian

The canonical Laplacian obviously obeys the condition $\Delta^2 = 0$.

Let ρ be an arbitrary volume form on an odd symplectic supermanifold. Then it is easy to check using (2.2), (2.5) and (2.6) that the canonical Laplacian on semidensities Δ obeys the following condition:

$$\Delta(f\sqrt{\rho}) = (\Delta_\rho f)\sqrt{\rho} + (-1)^{p(f)}f\Delta\sqrt{\rho}, \quad (3.8)$$

where Δ_ρ is the Laplacian (2.2) on functions. Using (2.4) one can rewrite this relation in the following way, for the semidensity $\mathbf{s} = \sqrt{\rho}$:

$$[\Delta, f]\mathbf{s} \equiv (\Delta \circ f - (-1)^{p(f)}f \circ \Delta)\mathbf{s} = \mathcal{L}_{\mathbf{D}_f}\mathbf{s}, \quad (3.9)$$

where $\mathcal{L}_{\mathbf{D}_f}$ is the Lie derivative along the Hamiltonian vector field \mathbf{D}_f . Notice that (3.9) holds for an arbitrary semidensity \mathbf{s} , not only for an even non-degenerate semidensity $\mathbf{s} = \sqrt{\rho}$ corresponding to a volume form ρ . This relation is very important for the study of Laplacians on semidensities on arbitrary odd Poisson supermanifolds (see [10]).

If ρ is an arbitrary volume form, then by applying the canonical Laplacian to the semidensity $\mathbf{s} = \sqrt{\rho}$ we can obtain a “derived” function $H = \Delta\sqrt{\rho}/\sqrt{\rho}$. It turns out that the Hamiltonian vector field \mathbf{D}_H corresponding to H is nothing but the vector field Δ_ρ^2 :

$$\Delta_\rho^2 f = \left\{ \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}, f \right\}. \quad (3.10)$$

(Compare this formula with (2.10).)

It is evident that if the form ρ is normal, then by definition

$$\Delta\sqrt{\rho} = 0. \quad (3.11)$$

This is just an invariant expression for the Batalin–Vilkovisky identity (2.12)

Now let us return to the relation between differential forms on M and semidensities on ΠT^*M , and to its generalization for arbitrary odd symplectic supermanifolds.

We call a semidensity \mathbf{s} *closed* or *exact* if $\Delta \mathbf{s} = 0$ or $\mathbf{s} = \Delta \mathbf{r}$ respectively. The condition $\Delta^2 = 0$ for the canonical Laplacian corresponds to $d^2 = 0$ for the exterior differential.

Equations (3.1), (3.3) and (3.9) allow the translation of formulae of vector calculus on M into formulae for semidensities on ΠT^*M (see [9]). For example, under the map (3.1) the “interior multiplication” of a differential form ω by a multivector field T transforms into the usual product of the semidensity \mathbf{s}_ω with the function corresponding to the multivector field. Hence, equation (3.9) corresponds to the formula for the Lie derivative of a differential form along a multivector field (a generalization of Cartan’s homotopy formula).

Consider in more details the following two constructions which do not appear naturally in classical calculus of forms. (Below we use familiar formal properties of the Fourier transform.)

a) If $a = a_i(x)dx^i$ is a 1-form on M and a semidensity $\mathbf{s} = s(x, \theta)\sqrt{\mathcal{D}(x, \theta)}$ on ΠT^*M corresponds to another form ω , then one can see that the semidensity $a_i \frac{\partial \mathbf{s}}{\partial \theta_i} \sqrt{\mathcal{D}(x, \theta)}$ corresponds to the form $a \wedge \omega$. (Compare with the familiar relation between the differentiation and multiplication by a coordinate for the classical Fourier transform.) Consider the following generalization. Let $a = a_i dx^i$ be a one-form on M with odd coefficients (we have to allow “external odd parameters” for this). For an arbitrary semidensity $\mathbf{s} = s(x, \theta)\sqrt{\mathcal{D}(x, \theta)}$ consider a new semidensity \mathbf{s}' , which we denote by $a \lceil \mathbf{s}$, given by the formula $\mathbf{s}' = a \lceil \mathbf{s} := s(x, \theta_i + a_i)\sqrt{\mathcal{D}(x, \theta)}$. It is a well-defined operation, because the coefficients a_i have the same transformation law as the variables θ_i . (Notice that the Berezinian of the transformation $(x^i, \theta_i) \mapsto (x^i, \theta_i + a_i)$ equals 1.) Respectively, if the semidensity \mathbf{s} corresponds to a differential form $\omega = \sum \omega_k$, then we denote by $a \lceil \omega$ the differential form such that the semidensity $a \lceil \mathbf{s}$ corresponds to $a \lceil \omega$. One can see that

$$a \lceil \omega = \sum_{p=0}^k \frac{1}{p!} \underbrace{a \wedge \cdots \wedge a}_p \wedge \omega_{k-p}, \quad (k = 0, \dots, n).$$

We obtain an action of the abelian supergroup of differential one-forms “with odd values” (i.e., $\Pi \Omega^1(M)$) in the spaces of semidensities and differential forms (see [9]).

b) Let $\omega = \sum \omega_k$ and $\omega' = \sum \omega'_k$ be differential forms on M^n such that their top-degree components ω_n and ω'_n are non-zero, and let the semidensities \mathbf{s} and \mathbf{s}' correspond to ω and ω' respectively. Then we can define a new form $\tilde{\omega} := \omega * \omega'$ such that the corresponding semidensity is equal to $\sqrt{\mathbf{s} \cdot \mathbf{s}'}$. The

condition $\omega_n \neq 0$, $\omega'_n \neq 0$ for the top-degree components makes the square root operation uniquely defined.

The 1-1 correspondence between forms on M and semidensities on ΠT^*M is defined using the cotangent bundle structure on the odd symplectic supermanifold ΠT^*M . Bearing in mind that every odd symplectic manifold E with an underlying manifold M is symplectomorphic to ΠT^*M (see [9] and the footnote in subsection 2.1), let us analyze the relation between semidensities on E and differential forms on M . The map (3.1) is not invariant under arbitrary symplectomorphisms of the total symplectic supermanifold $E = \Pi T^*M$. In other words, if L is an arbitrary $(n,0)$ -dimensional Lagrangian surface in E , then the correspondence between semidensities on E and differential forms on L depends on an *identifying symplectomorphism*, i.e. a symplectomorphism $\varphi : \Pi T^*L \rightarrow E$ such that $\varphi|_L = \text{id}$.

Consider the following symplectomorphisms of an odd symplectic supermanifold $E = \Pi T^*M$:

$$\text{symplectomorphisms induced by diffeomorphisms of } M \quad (3.12)$$

(these symplectomorphisms preserve the cotangent bundle structure), symplectomorphisms “adjusted to M ”, i.e. identical on M :

$$\varphi: \quad \varphi^*\omega = \omega, \quad \varphi|_M = \text{id} \quad (3.13)$$

(they destroy the cotangent bundle structure except for $\varphi = \text{id}$), symplectomorphisms corresponding to closed one-forms on M :

$$\varphi^*x^i = x^i, \quad \varphi^*\theta_i = \theta_i + \alpha_i(x), \quad (\partial_i\alpha_j - \partial_j\alpha_i = 0) \quad (3.14)$$

where (x^i, θ_i) are coordinates on ΠT^*M corresponding to some coordinates (x^i) on M and $\alpha = \alpha_i(x)dx^i$ is a closed one-form on M with odd values (these symplectomorphisms move the Lagrangian surface M ; notice that an arbitrary $(n,0)$ -dimensional Lagrangian surface L is given by the equations $\theta_i - \alpha_i(x) = 0$ where $\alpha_i(x)dx^i$ is a closed odd-valued one-form). It might be worth noting that symplectomorphisms of ΠT^*M form a supergroup.

One can prove that an arbitrary symplectomorphism of ΠT^*M can be represented as the composition of symplectomorphisms (3.12), (3.13) and (3.14) (see [9]). (Compare this statement with the statement that an arbitrary transformation of Darboux coordinates can be represented as the composition of transformations (3.5), (3.6) and (3.7)).

Notice that every adjusted symplectomorphism (3.13) has the following appearance:

$$\begin{aligned} x^i &\rightarrow x^i + f^i(x, \theta), & \text{where } f^i(x, \theta) &= O(\theta) \\ \theta_i &\rightarrow \theta_i + g_i(x, \theta), & \text{where } g_i(x, \theta) &= O(\theta^2) \end{aligned} \quad (3.15)$$

where (x^i, θ_i) are the Darboux coordinates on ΠT^*M corresponding to coordinates x^i on M . One can show that there exists a Hamiltonian $Q(x, \theta) = Q^{ik}(x, \theta)\theta_i\theta_k$ that generates this transformation, i.e., (3.15) can be included in a 1-parameter family of transformations defined by the differential equation $\dot{z} = \{Q, z\}$ (see [9] for details).

Comparing (3.15) with (3.1) we arrive at an important conclusion:

Proposition 3.1. *The top-degree component of the form corresponding to a semidensity on ΠT^*M does not change under any symplectomorphism adjusted to M . In other words, a semidensity on an odd symplectic supermanifold E defines a volume form (density) for all $(n,0)$ -dimensional Lagrangian surfaces³.*

Now consider a *closed* semidensity on ΠT^*M . To it corresponds a closed differential form on M . It follows from (3.9) and (3.13) that the action of an adjusted symplectomorphism (3.15) changes the semidensity and the corresponding form by an exact semidensity and an exact form respectively. Hence, we arrive at another important conclusion:

Proposition 3.2. *To a closed semidensity on ΠT^*M corresponds a cohomology class of differential forms on M independently of the bundle structure. If two closed semidensities \mathbf{s} and \mathbf{s}' coincide on M and differ by an exact semidensity, then there exists an adjusted symplectomorphism $\varphi : \Pi T^*M \rightarrow \Pi T^*M$ such that $\varphi^*\mathbf{s} = \mathbf{s}'$.*

Propositions 3.1 and 3.2 were stated and proved in [14] and in [9], but in the work [14] semidensities do not appear explicitly.

Based on the concept of semidensities, the properties of the canonical Laplacian and the above Propositions we will now analyze the statements (2.14) concerning the Batalin–Vilkovisky formalism.

3.3 Master equation on semidensities

The claim is that *the Batalin–Vilkovisky master equation (2.13) is an equation on the semidensity $\mathbf{s} = \sqrt{f(x, \theta)}\boldsymbol{\rho}$. A solution of the Batalin–Vilkovisky quantum master equation is a closed semidensity: $\Delta\mathbf{s} = 0$.*

Suppose E is an odd symplectic supermanifold with the compact connected orientable underlying manifold M . E can be identified with ΠT^*L for every closed $(n,0)$ -dimensional Lagrangian submanifold L (see [9] for details) and any two identifications differ by an adjusted symplectomorphism. An arbitrary $(n,0)$ -dimensional closed Lagrangian surface L is given by a closed one-form on M (see (3.14)).

³A relation between semidensities on E and densities on Lagrangian surfaces can be defined for arbitrary Lagrangian surfaces (see [14] and [7]).

Let us now rewrite the implications (2.14) in terms of semidensities:

$$\rho \text{ is a normal volume form} \Rightarrow \Delta\sqrt{\rho} = 0 \Rightarrow \Delta_\rho^2 = 0. \quad (3.16)$$

The first implication follows from the definition of the canonical operator Δ . The second implication follows from equation (3.10). Let us analyze to what extent these conditions are equivalent. Let ρ be a volume form such that $\Delta_\rho^2 = 0$. By (3.10), then it follows that $\Delta\sqrt{\rho} = \nu\sqrt{\rho}$, where ν is an odd constant. (If external odd parameters are not allowed, then, of course, $\nu = 0$. Our analysis takes into consideration possible ‘‘odd moduli’’.) This odd constant is the obstruction to the condition $\Delta\sqrt{\rho} = 0$, i.e. to the closedness of the semidensity $\sqrt{\rho}$.

Suppose $\nu = 0$. Then the master equation $\Delta\sqrt{\rho} = 0$ is satisfied. Consider in this case an arbitrary closed $(n,0)$ -dimensional Lagrangian surface L and an arbitrary identifying symplectomorphism $\varphi: E \rightarrow \Pi T^*L$. Under the map (3.1) to every closed semidensity $\sqrt{\rho}$ on E corresponds a closed differential form $\omega = \omega_0 + \omega_1 + \dots + \omega_n$ on L , where the top degree form ω_n defines a volume form on L . The closed 0-form ω_0 is a constant. It is easy to see from (3.1) and (3.14) that the value of this constant (up to a sign) does not depend on the choice of the Lagrangian surface and on the choice of the identifying symplectomorphism. The top degree form, clearly, depends on the Lagrangian surface but does not depend on the identifying symplectomorphism (by Proposition 3.1). On the other hand, all other closed forms ω_k ($1 < k < n$) can be eliminated by a suitable choice of the identifying symplectomorphism.

We have arrived, finally, to the following theorem:

Theorem 3.1. *Let $E^{n,n}$ be an odd symplectic supermanifold with the closed orientable compact underlying manifold M . Let ρ be a volume form on $E^{n,n}$ such that $\Delta_\rho^2 = 0$. To this volume form corresponds an odd constant ν . If this odd constant is equal to zero, then the volume form defines a closed semidensity $\mathbf{s} = \sqrt{\rho}$, a solution of the Batalin–Vilkovisky quantum master equation. To a closed semidensity corresponds a constant c defined by the zero cohomology class of the differential form corresponding to the semidensity \mathbf{s} . If this constant is equal to zero, then the volume form ρ is normal.*

4 Discussion

The existence of Darboux coordinates that can locally make flat every surface in an odd symplectic supermanifold together with the absence of an invariant volume form make odd symplectic geometry a poor candidate for finding local

invariants if no extra structure is provided. Hence the existence of the canonical odd Laplacian (3.2) looks mysterious. We shall to explain this fact briefly . (See details in [10])

Consider an arbitrary n -th order linear operator acting on functions or densities of some weight t on an arbitrary manifold (or supermanifold). One can consider its principal symbol, i.e., the coefficients at the highest order derivatives. It is a contravariant tensor field of rank n . In the case of an n -th order operator \hat{A} acting on densities of weight t , the adjoint operator \hat{A}^\dagger acting on densities of weight $1 - t$ can be defined by the equation $\int \mathbf{s}_1 \cdot (\hat{A}\mathbf{s}_2) = \int (\hat{A}^\dagger\mathbf{s}_1) \cdot \mathbf{s}_2$, where \mathbf{s}_2 is an arbitrary density of weight t and \mathbf{s}_1 is an arbitrary density of weight $1 - t$. Hence in the case of an operator \hat{A} acting on semidensities ($t = 1/2$) the operators \hat{A} and \hat{A}^\dagger act on the same space. Assuming that the coefficients are real, the operators \hat{A} and \hat{A}^\dagger have the same principal symbol. One can consider the principal symbol of $\hat{A} - \hat{A}^\dagger(-1)^n$, which is a tensor field of rank $n - 1$. It is the so-called *subprincipal symbol* of the operator \hat{A} .

Let us consider the canonical operator Δ on an odd symplectic supermanifold in arbitrary coordinates (not necessary Darboux coordinates). The highest order coefficients make the principal symbol, which is here the tensor of rank 2 defining the odd symplectic structure. (More precisely it is the tensor \mathfrak{S}^{AB} that defines the master Hamiltonian (1.8).) It is easy to see that if \hat{A} is an arbitrary linear differential operator of the second order on semidensities having the principal symbol defined by the odd symplectic structure, then the subprincipal symbol of this operator is equal to $[\hat{A}, f] - \mathcal{L}_{\mathbf{D}_f}$. Hence it follows from (3.9) that the subprincipal symbol of the canonical Laplacian Δ is equal to zero. The coefficients at the first derivatives are fixed by this condition. In fact, these two conditions on the principal symbol and subprincipal symbol are equivalent to the equation (3.9). An arbitrary linear operator Δ' on semidensities obeying the condition (3.9) is equal to $\Delta + C$, where C is a scalar (a zero-order operator). The condition $\Delta'\sqrt{\rho} = 0$ for an arbitrary normal volume form fixes this scalar $C = 0$ according to (3.11): hence, *on an odd symplectic supermanifold there is no distinguished volume form, but there is a distinguished class of normal volume forms.*

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