## On the Bilateral Series  $2\psi_2$

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#### Abstract

We obtain a formula which reduces the evaluation of a  $2\psi_2$  series to two  $2\phi_1$  series. In some sense, this identity may be considered as a companion of Slater's formulas. We also find that a two-term  $2\psi_2$  summation formula due to Slater can be derived from a unilateral summation formula of Andrews by bilateral extension and parameter augmentation.

Keywords: basic hypergeometric series, bilateral series, bilateral extension, parameter augmentation, q-Gauss summation.

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# 1 Introduction

It is well known that many bilateral basic hypergeometric identities can be derived from unilateral identities. Using Cauchy's method [5, 15, 20, 21] one may obtain bilateral basic hypergeometric identities from terminating unilateral identities. Starting with nonterminating unilateral basic hypergeometric series, Chen and Fu [8] developed a method to construct semi-finite forms by shifting the summation index by  $m$ . Then the bilateral summations are consequences of the semi-finite forms by letting  $m$  tend to infinity. We call this method bilateral extension. In this paper we use bilateral extensions of a  $3\phi_2$  series and an identity of Andrews [2] to study the bilateral series  $2\psi_2$ :

<span id="page-0-0"></span>
$$
{}_2\psi_2\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, z\right].\tag{1.1}
$$

The above  $2\psi_2$  series is closely related to the question of finding a q-extension of Dougall's bilateral hypergeometric series summation formula [10]:

<span id="page-1-0"></span>
$$
\sum_{k=-\infty}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} = \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)},
$$
\n(1.2)

where Re $(c + d - a - b - 1) > 0$ ,  $(a)_k = a(a + 1) \cdots (a + k - 1)$ ,  $k = 1, 2, \cdots$ ,  $(a)_0 = 1$ and  $(a)_k = (-1)^k/(1-a)_{-k}$  when k is a negative integer.

Bailey  $[6]$  first suggested that there did not exist any q-extension of  $(1.2)$ . Since  $(1.2)$  is an extension of the Gauss  ${}_2F_1$  summation formula, one naturally expects that a  $q$ -analogue of  $(1.2)$  should be concerned with the following series:

<span id="page-1-1"></span>
$$
{}_2\psi_2\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, \frac{cd}{abq}\right].\tag{1.3}
$$

Clearly, when c or d equals q,  $(1.3)$  reduces to the q-Gauss sum [13, Appendix II.8]:

<span id="page-1-4"></span>
$$
{}_2\phi_1\left[\begin{array}{cc} a, & b \\ & c \end{array}; q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}, \qquad |c/ab| < 1. \tag{1.4}
$$

Even for the above series [\(1.3\)](#page-1-1), Gasper [12] pointed out that one could not use analytic continuation to derive an infinite product representation.

On the other hand, many results on the bilateral  $_2\psi_2$  series [\(1.1\)](#page-0-0) have been obtained. In [6], Bailey found several transformation formulas for the  $2\psi_2$  series [\(1.1\)](#page-0-0). Later, Slater obtained a general transformation formula for an  $r\psi_r$  series in [23] based on Sears' transformation on the  $r+s+1\phi_{r+s}$  series in [22] subject to suitable substitutions and the following relation

$$
\sum_{n=0}^{\infty} f(n) = \sum_{n=-\infty}^{-1} f(-n-1)
$$
\n(1.5)

to combine two unilateral series to form a bilateral series.

Gasper and Rahman [13] have shown that based on Slater's transformation formula, one could obtain two expansions of an  $r\psi_r$  series in terms of  $r \phi_{r-1}$  series [13, Eq.  $(5.4.4), (5.4.5)$ . When  $r = 2$ , they become

<span id="page-1-2"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, z\right] = \frac{a(q, qa/b, c/a, d/a, az, q/az, qb, 1/b; q)_{\infty}}{(a/b, qb/a, c, d, q/a, q/b, z, q/z; q)_{\infty}}
$$

$$
\times {}_{2}\phi_{1}\left[\begin{array}{cc} qa/c, & qa/d \\ qa/b & q, abz \end{array}\right] + \text{idem}(a; b) \tag{1.6}
$$

and

<span id="page-1-3"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, z\right] = \frac{q}{c} \frac{(q, c/a, c/b, abz/dq, dq^{2}/abz, q/d; q)_{\infty}}{(c, c/d, q/a, q/b, abz/cd, qcd/abz; q)_{\infty}}
$$

$$
\times {}_{2}\phi_{1}\left[\begin{array}{cc} qa/c, & qb/c \\ q d/c \end{array}; q, z\right] + \text{idem}(c; d), \tag{1.7}
$$

where the symbol "idem $(a, b)$ " after an expression means that the preceding expression is repeated with  $a$  and  $b$  interchanged.

Setting  $d = q$ , [\(1.6\)](#page-1-2) reduces to a three-term transformation formula [13, Appendix III.32] for the  $_2\phi_1$  series:

<span id="page-2-0"></span>
$$
{}_2\phi_1\left[\begin{array}{cc} a, & b \\ & c \end{array}; q, z\right] = \frac{(b, c/a, az, q/az; q)_{\infty}}{(c, b/a, z, q/z; q)_{\infty}} {}_2\phi_1\left[\begin{array}{cc} a, & aq/c \\ & aq/b \end{array}; q, \frac{cq}{abz}\right] + \text{idem}(a; b). \quad (1.8)
$$

However, it should be noted that when c or d equals  $q$ , [\(1.7\)](#page-1-3) does not lead to any nontrivial identity.

The first result of this paper is to give a new formula for the  $2\psi_2$  series [\(1.1\)](#page-0-0) in terms of two  $_2\phi_1$  series which is different from Slater's formulas [\(1.6\)](#page-1-2) and [\(1.7\)](#page-1-3). It reduces to a different three-term transformation formula  $(2.4)$  when  $c = q$  compared with the threeterm transformation formula [\(1.8\)](#page-2-0) deduced by Slater's transformation. Moreover, this identity may be considered as a companion of Slater's formulas [\(1.6\)](#page-1-2) and [\(1.7\)](#page-1-3). Note that Slater's formulas do not seem to imply the special cases that can be deduced from our formula except for Ramanujan's  $_1\psi_1$  summation formula [13, Appendix II.29]. As a consequence, our formula yields a two-term closed product form for the  $2\psi_2$  series:

<span id="page-2-1"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc}b,&c\\aq/b,&aq/c\end{array};q,-\frac{aq}{bc}\right]=\frac{(-b,aq/bc,-q/b,b/a,q;q)_{\infty}(aq^{2}/c^{2};q^{2})_{\infty}}{(aq/c,-1,q/c,q/b,-aq/bc;q)_{\infty}(b^{2}/a;q^{2})_{\infty}}+\frac{(aq/bc,b,-aq/b,-b/a,q;q)_{\infty}(aq^{2}/c^{2};q^{2})_{\infty}}{(aq/b,aq/c,-1,-aq/bc,q/c;q)_{\infty}(b^{2}/a;q^{2})_{\infty}}.
$$
(1.9)

For comparison, we recall the known formula for the well-poised  $_2\psi_2$  series [13, Appendix II.30]:

<span id="page-2-2"></span>
$$
{}_2\psi_2\left[\begin{array}{cc}b,&c\\aq/b,&aq/c\end{array};q,-\frac{aq}{bc}\right] = \frac{(aq/bc;q)_\infty(aq^2/b^2,aq^2/c^2,q^2,aq,q/a;q^2)_\infty}{(aq/b,aq/c,q/b,q/c,-aq/bc;q)_\infty}.\tag{1.10}
$$

Let us turn our attention back to Dougall's formula. As pointed out by Askey  $|4|$ , Bailey seemed to have been partly right concerning his opinion towards the q-extension of Dougall's formula. According to Askey  $[4]$ , in certain sense the following q-extension of Cauchy's beta integral was similar to a  $q$ -extension of Dougall's formula:

<span id="page-2-3"></span>
$$
\int_{-\infty}^{\infty} \frac{(ct, -dt; q)_{\infty}}{(at, -bt; q)_{\infty}} d_q t = 2 \frac{(1 - q)(c/a, d/b, -c/b, -d/a, ab, q/ab; q)_{\infty} (q^2; q^2)^2_{\infty}}{(cd/abq, q; q)_{\infty} (a^2, q^2/a^2, b^2, q^2/b^2; q^2)_{\infty}}.
$$
 (1.11)

In fact, this integral can be recast as a two-term summation formula for the  $2\psi_2$  series [\(1.3\)](#page-1-1):

$$
\frac{(c, -d; q)_{\infty}}{(a, -b; q)_{\infty}} {}_{2}\psi_{2} \begin{bmatrix} a, & -b \\ c, & -d \end{bmatrix}; q, q \end{bmatrix} + \frac{(-c, d; q)_{\infty}}{(-a, b; q)_{\infty}} {}_{2}\psi_{2} \begin{bmatrix} -a, & b \\ -c, & d \end{bmatrix}; q, q \end{bmatrix}
$$

$$
= 2 \frac{(1 - q)(c/a, d/b, -c/b, -d/a, ab, q/ab; q)_{\infty} (q^{2}; q^{2})_{\infty}^{2}}{(cd/abq, q; q)_{\infty} (a^{2}, q^{2}/a^{2}, b^{2}, q^{2}/b^{2}; q^{2})_{\infty}}.
$$
(1.12)

As observed by Ismail and Rahman [14], the above two-term summation formula is a special of a transformation formula due to Slater [23]. When  $r = 2$ , by substitutions and the q-Gauss sum [\(1.4\)](#page-1-4), Slater's general transformation on the  $r\psi_r$  series reduces to the following two-term summation formula:

<span id="page-3-0"></span>
$$
\frac{(c/ef, qef/c, q, q/a, q/b, c/a, c/b; q)_{\infty}}{(e, f, q/e, q/f, c/ab; q)_{\infty}} = \frac{q}{e} \frac{(c/qf, q^2f/c, e/a, e/b, qc/e, q^2/e; q)_{\infty}}{(e, q/e, e/f, qf/e; q)_{\infty}}
$$

$$
\times_2 \psi_2 \begin{bmatrix} e/c, & e/q \\ e/a, & e/b \end{bmatrix} + \text{idem}(e; f). \tag{1.13}
$$

The second result of this paper is concerned with the above two-term summation formula [\(1.13\)](#page-3-0) for  $2\psi_2$ . Andrews [2] established a three-term transformation formula which is the key to proving many of Ramanujan's identities for partial θ-functions. In view of the symmetry in this formula, he obtained a generalization of Ramanujan's  $_1\psi_1$ summation:

<span id="page-3-1"></span>
$$
d\sum_{n=0}^{\infty} \frac{(q/bc, acdf;q)_n}{(ad, df;q)_{n+1}} (bd)^n - c \sum_{n=0}^{\infty} \frac{(q/bd, acdf;q)_n}{(ac, cf;q)_{n+1}} (bc)^n
$$
  
= 
$$
d\frac{(q, qd/c, c/d, abcd, acdf, bcdf;q)_{\infty}}{(ac, ad, bc, bd, cf, df;q)_{\infty}}, \qquad |bc|, |bd| < 1.
$$
 (1.14)

Using the approach of parameter augmentation developed by Chen and Liu [9], we find that the two-term summation formula [\(1.13\)](#page-3-0) for  $_2\psi_2$  series is a consequence of the above identity [\(1.14\)](#page-3-1) of Andrews by bilateral extension and parameter augmentation.

As is customary, we employ the notation and terminology of basic hypergeometric series in [13]. For  $|q| < 1$ , the q-shifted factorial is defined by

$$
(a;q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k)
$$
 and  $(a;q)_n = \frac{(a;q)_{\infty}}{(aq^n;q)_{\infty}}$ , for  $n \in \mathbb{Z}$ .

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$
(a_1, a_2,..., a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,
$$

where n is an integer or infinity. In particular, for a nonnegative integer  $k$ , we have

$$
(a;q)_{-k} = \frac{1}{(aq^{-k};q)_k}.
$$
\n(1.15)

The (unilateral) basic hypergeometric series  $r\phi_s$  is defined by

$$
{}_r\phi_s \left[ \begin{array}{cccc} a_1, & a_2, & \dots, & a_r \\ b_1, & b_2, & \dots, & b_s \end{array} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_k}{(q, b_1, b_2, \dots, b_s; q)_k} \left[ (-1)^k q^{\binom{k}{2}} \right]^{1+s-r} z^k, \quad (1.16)
$$

while the bilateral basic hypergeometric series  $r\psi_s$  is defined by

$$
{}_r\psi_s\left[\begin{array}{cccc} a_1, & a_2, & \dots, & a_r \\ b_1, & b_2, & \dots, & b_s \end{array}; q, z\right] = \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_k}{(b_1, b_2, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{s-r} z^k. \tag{1.17}
$$

# 2 An Expansion Formula for the  $2\psi_2$  Series

In this section, we derive a representation for the  $2\psi_2$  series [\(1.1\)](#page-0-0) in terms of two  $2\phi_1$  series. This formula can be considered as a companion of Slater's formulas [\(1.6\)](#page-1-2) and [\(1.7\)](#page-1-3). We also present some consequences including a two-term infinite product representation for the sum of a well-poised  $_2\psi_2$  series [\(1.9\)](#page-2-1).

## <span id="page-4-2"></span>Theorem 2.1 We have

<span id="page-4-3"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ c, & d \end{array};q,z\right] = \frac{(c/b, abz/d, dq/abz, q/d, q;q)_{\infty}}{(c, az/d, q/a, q/b, cd/abz;q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} cd/abz, & d/a \\ dq/az \end{array};q, \frac{bq}{d}\right] -\frac{(cq/d, b, d/a, az/q, q^{2}/az, q/d, q;q)_{\infty}}{(d/q, c, bq/d, az/d, dq/az, q^{2}/d, q/a;q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} aq/d, & bq/d \\ qq/d \end{array};q,z\right], (2.1)
$$

where  $|cd/ab| < |z| < 1$  and  $|bq/d| < 1$ .

*Proof.* We start with a three-term transformation of  $_3\phi_2$  series [13, Appendix III.33]:

$$
{}_{3}\phi_{2}\left[\begin{array}{cc}a, & b, & c \\ & d, & e \end{array}; q, \frac{de}{abc}\right] = \frac{(e/b, e/c, cq/a, q/d; q)_{\infty}}{(e, cq/d, q/a, e/bc; q)_{\infty}} {}_{3}\phi_{2}\left[\begin{array}{cc}c, & d/a, & cq/e \\ & cq/a, & bcq/e \end{array}; q, \frac{bq}{d}\right] -\frac{(q/d, eq/d, b, c, d/a, de/bcq, bcq^{2}/de; q)_{\infty}}{(d/q, e, bq/d, cq/d, q/a, e/bc, bcq/e; q)_{\infty}} {}_{3}\phi_{2}\left[\begin{array}{cc}aq/d, & bq/d, & cq/d \\ & q^{2}/d, & eq/d \end{array}; q, \frac{de}{abc}\right],
$$

where  $|bq/d|, |de/abc| < 1$ .

Shifting the index of summation on the left hand side of the above identity by  $m$ such that the new sum runs from  $-m$  to infinity, and then replacing a, b, d, e by  $aq^{-m}$ ,  $bq^{-m}$ ,  $dq^{-m}$ ,  $eq^{-m}$ , respectively, we get

<span id="page-4-0"></span>
$$
\sum_{k=-m}^{\infty} \frac{(a, b, cq^m; q)_k}{(q^{m+1}, d, e; q)_k} \left(\frac{de}{abc}\right)^k = \frac{(cq/e, q/d, q; q)_m}{(c, q/a, q/b; q)_m} \frac{(e/b, e/c, cq^{1+m}/a, q^{1+m}/d; q)_{\infty}}{(e, cq^{1+m}/d, q^{1+m}/a, e/bc; q)_{\infty}}
$$
\n
$$
\times_3 \phi_2 \left[\begin{array}{cc} c, & d/a, & cq^{1+m}/e \\ cq^{1+m}/a, & bcq/e \end{array}; q, \frac{bq}{d}\right] - \frac{(bcq^2/de, q/d, q; q)_m}{(q^2/d, q/a, c; q)_m} \frac{(q^{1+m}/d, eq/d, b; q)_{\infty}}{(d/q, e, bq/d; q)_{\infty}}
$$
\n
$$
\times \frac{(c, d/a, de/bcq, bcq^{2+m}/de; q)_{\infty}}{(cq^{1+m}/d, q^{1+m}/a, e/bc, bcq/e; q)_{\infty}} 3\phi_2 \left[\begin{array}{cc} aq/d, & bq/d, & cq^{1+m}/d \\ q^{2+m}/d, & eq/d \end{array}; q, \frac{de}{abc}\right], (2.2)
$$

where  $|bq/d|, |de/abc| < 1$ .

Setting  $m \to \infty$  in [\(2.2\)](#page-4-0) and assuming  $|c| < 1$ , Tannery's theorem [7] enables us to interchange the limit and the summation. This gives

<span id="page-4-1"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ d, & e \end{array}; q, \frac{de}{abc}\right] = \frac{(cq/e, q/d, q, e/b, e/c; q)_{\infty}}{(c, q/a, q/b, e, e/bc; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} c, & d/a \\ & bcq/e \end{array}; q, \frac{bq}{d}\right]
$$

$$
-\frac{(bcq^{2}/de, q/d, q, eq/d, b, d/a, de/bcq; q)_{\infty}}{(q^{2}/d, q/a, d/q, e, bq/d, e/bc, bcq/e; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} aq/d, & bq/d \\ & eq/d \end{array}; q, \frac{de}{abc}\right], (2.3)
$$

where  $|bq/d|, |c|, |de/abc| < 1$ .

By the substitutions  $c \to de/abz$  and  $e \to c$  in [\(2.3\)](#page-4-1), we get the desired formula.

Note that Theorem [2.1](#page-4-2) may be considered as a bilateral extension of the following three-term transformation formula [13, Appendix III.31]

<span id="page-5-0"></span>
$$
{}_{2}\phi_{1}\left[\begin{array}{cc} a, & b \\ & d \end{array}; q, z\right] = \frac{(abz/d, q/d; q)_{\infty}}{(az/d, q/a; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} d/a, & dq/abz \\ & dq/az \end{array}; q, \frac{bq}{d}\right] -\frac{(b, d/a, az/q, q^{2}/az, q/d; q)_{\infty}}{(d/q, bq/d, az/d, dq/az, q/a; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} aq/d, & bq/d \\ & q^{2}/d \end{array}; q, z\right], (2.4)
$$

where  $|bq/d|, |z| < 1$ . It is clear that  $(2.4)$  is a special case of  $(2.1)$  for  $c = q$ .

Since Slater's formula [\(1.7\)](#page-1-3) and our formula [\(2.1\)](#page-4-3) deal with the same series, we are naturally led to an identity on  $_2\phi_1$  series. The right hand sides of [\(1.7\)](#page-1-3) and [\(2.1\)](#page-4-3) give rise to the following identity by replacing a, b, c, z by  $d/b$ ,  $dz/q$ ,  $adz/c$ ,  $bq/c$ , respectively,

<span id="page-5-1"></span>
$$
{}_{2}\phi_{1}\left[\begin{array}{c} a, & b \\ & c \end{array}; q, z \right] = \frac{(abz/c, q/c; q)_{\infty}}{(az/c, q/a; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{c} cq/abz, & c/a \\ & cq/az \end{array}; q, \frac{bq}{c} \right]
$$

$$
+ \left(\frac{q(1-a)(b, q/z, d/aq, aq^{2}/d, cq/adz, adz/c, q/c; q)_{\infty}}{d(d, c/az, 1/a, aq/c, dz/c, cq/dz, q/d; q)_{\infty}} + \frac{(azq/c, dz/q, b, d/c, cq/d, q^{2}/dz, a; q)_{\infty}}{(d/q, z, c, q^{2}/d, aq/c, dz/c, cq/dz; q)_{\infty}} \right) {}_{2}\phi_{1}\left[\begin{array}{c} q/b, & z \\ & azq/c \end{array}; q, \frac{bq}{c} \right]. \tag{2.5}
$$

It is worth noting that the parameter d occurs only in the factors of the second term on the right hand side of [\(2.5\)](#page-5-1). Hence the sum of the two products in the parentheses does not depend on d. This fact does not seem to be obvious by direct verification. Setting  $d = aq$ , it follows that

<span id="page-5-2"></span>
$$
{}_2\phi_1\left[\begin{array}{cc} a, & b \\ c & \end{array}; q, z\right] = \frac{(abz/c, q/c; q)_{\infty}}{(az/c, q/a; q)_{\infty}} {}_2\phi_1\left[\begin{array}{cc} cq/abz, & c/a \\ cq/az & \end{array}; q, \frac{bq}{c} \right] \\
+\frac{(az, b, c/a, q/az; q)_{\infty}}{(z, c, q/a, c/az; q)_{\infty}} {}_2\phi_1\left[\begin{array}{cc} q/b, & z \\ azq/c & \end{array}; q, \frac{bq}{c} \right].\n\tag{2.6}
$$

From Heine's transformation [13, Appendix III.1]

<span id="page-5-3"></span>
$$
{}_2\phi_1\left[\begin{array}{cc} a, & b \\ & c \end{array}; q, z\right] = \frac{(b, az; q)_{\infty}}{(c, z; q)_{\infty}} {}_2\phi_1\left[\begin{array}{cc} c/b, & z \\ & az \end{array}; q, b\right],\tag{2.7}
$$

<span id="page-5-4"></span>it is easily seen that [\(2.6\)](#page-5-2) is equivalent to [\(2.4\)](#page-5-0) by the substitution  $c \to d$ .

Corollary 2.2 We have

<span id="page-5-5"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, z\right] = \frac{(abz/d, c/b, dq/abc, q/d, q; q)_{\infty}}{(c, az/d, q/a, q/b, cd/abz; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} cd/abz, & d/a \\ dq/az \end{array}; q, \frac{bq}{d}\right] + \frac{(d/a, b, az, q/az, q; q)_{\infty}}{(d, c, d/az, z, q/a; q)_{\infty}} {}_{2}\phi_{1}\left[\begin{array}{cc} c/b, & z \\ azq/d \end{array}; q, \frac{bq}{d}\right],
$$
(2.8)

where  $|cd/ab| < |z| < 1$  and  $|bq/d| < 1$ .

Proof. By Heine's transformation [\(2.7\)](#page-5-3), the second term on the right hand side of  $(2.1)$  equals

$$
-\frac{(cq/d, b, d/a, az/q, q^2/az, q/d, q; q)_{\infty}}{(d/q, c, bq/d, az/d, dq/az, q^2/d, q/a; q)_{\infty}} 2\phi_1 \begin{bmatrix} aq/d, & bq/d \\ & cq/d \end{bmatrix}; q, z \end{bmatrix}
$$
  
= 
$$
-\frac{(b, d/a, az/q, q^2/az, q/d, q, azq/d; q)_{\infty}}{(d/q, c, az/d, dq/az, q^2/d, q/a, z; q)_{\infty}} 2\phi_1 \begin{bmatrix} c/b, & z \\ & azq/d \end{bmatrix}; q, \frac{bq}{d} \end{bmatrix}
$$
  
= 
$$
-\frac{(d/a, b, az, q/az, q; q)_{\infty} (1 - az/q)(1 - q/d)(1 - d/az)}{(d, c, d/az, z, q/a; q)_{\infty} (1 - d/q)(1 - az/d)(1 - q/az)}
$$
  

$$
\times 2\phi_1 \begin{bmatrix} c/b, & z \\ & azq/d \end{bmatrix}; q, \frac{bq}{d} \end{bmatrix}
$$
  
= 
$$
\frac{(d/a, b, az, q/az, q; q)_{\infty}}{(d, c, d/az, z, q/a; q)_{\infty}} 2\phi_1 \begin{bmatrix} c/b, & z \\ & azq/d \end{bmatrix}; q, \frac{bq}{d} \end{bmatrix}.
$$
 (2.9)

Remark 2.3 Corollary [2.2](#page-5-4) can also be obtained from the following three-term transformation formula [13, Appendix III.34]

$$
{}_{3}\phi_{2}\left[\begin{array}{cc}a, & b, & c \\ & d, & e \end{array}; q, \frac{de}{abc}\right] = \frac{(e/b, e/c; q)_{\infty}}{(e, e/bc; q)_{\infty}} {}_{3}\phi_{2}\left[\begin{array}{cc}d/a, & b, & c \\ & d, & bcq/e \end{array}; q, q\right] + \frac{(d/a, b, c, de/bc; q)_{\infty}}{(d, e, bc/e, de/abc; q)_{\infty}} {}_{3}\phi_{2}\left[\begin{array}{cc}e/b, & e/c, & de/abc \\ & de/bc, & eq/bc \end{array}; q, q\right].
$$

Shifting the summation index by m on the left hand side and replacing  $a, c, d, e$  by  $aq^{-m}$ ,  $cq^{-m}$ ,  $dq^{-m}$ ,  $eq^{-m}$ , respectively, we are led to [\(2.8\)](#page-5-5) by taking the limit  $m \to \infty$ and making suitable substitutions.

As a consequence of Corollary [2.2,](#page-5-4) we may deduce the following expansion of a  $_2\psi_2$ series in terms of a  $_2\phi_1$  series [11, Eq. (3.13.1.7)]. Setting  $z = q/a$  in [\(2.8\)](#page-5-5), the second summation on the right hand side vanishes. It follows from [\(2.7\)](#page-5-3) that

$$
{}_2\psi_2\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, \frac{q}{a}\right] = \frac{(c/b, d/b, bq/a, q; q)_{\infty}}{(c, d, q/a, q/b; q)_{\infty}} {}_2\phi_1\left[\begin{array}{cc} bq/c, & bq/d \\ bq/a \end{array}; q, \frac{cd}{bq}\right],\tag{2.10}
$$

which was originally derived from a double sum transformation formula of Slater, see [11, Section 3.13].

Corollary 2.4 We have

<span id="page-6-0"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc}b,&c\\aq/b,&aq/c\end{array};q,-\frac{aq}{bc}\right]=\frac{(-b,aq/bc,-q/b,b/a,q;q)_{\infty}(aq^{2}/c^{2};q^{2})_{\infty}}{(aq/c,-1,q/c,q/b,-aq/bc;q)_{\infty}(b^{2}/a;q^{2})_{\infty}}+\frac{(aq/bc,b,-aq/b,-b/a,q;q)_{\infty}(aq^{2}/c^{2};q^{2})_{\infty}}{(aq/b,aq/c,-1,-aq/bc,q/c;q)_{\infty}(b^{2}/a;q^{2})_{\infty}},
$$
\n(2.11)

where  $|aq/bc| < 1$ .

*Proof.* Setting  $c = cq/a$ ,  $d = cq/b$ , and  $z = -cq/ab$  in [\(2.8\)](#page-5-5), we find that the summations on the right hand side of the identity are both equal to

$$
\sum_{k=0}^{\infty} \frac{(c^2 q^2 / a^2 b^2; q^2)_k}{(q^2; q^2)_k} \left(\frac{b^2}{c}\right)^k, \tag{2.12}
$$

which can be summed by the Cauchy  $q$ -binomial theorem [13, Appendix II.3]

<span id="page-7-1"></span>
$$
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_{\infty}}{(z;q)_{\infty}}, \qquad |z| < 1.
$$
 (2.13)

Thus the following relation holds

$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ cq/a, & cq/b \end{array} ;q,-\frac{cq}{ab} \right] \;\; = \;\; \frac{(-b,cq/ab,-q/b,b/c,q;q)_{\infty}(cq^{2}/a^{2};q^{2})_{\infty}}{(cq/a,-1,q/a,q/b,-cq/ab;q)_{\infty}(b^{2}/c;q^{2})_{\infty}} \\ + \frac{(cq/ab,b,-cq/b,-b/c,q;q)_{\infty}(cq^{2}/a^{2};q^{2})_{\infty}}{(cq/b,cq/a,-1,-cq/ab,q/a;q)_{\infty}(b^{2}/c;q^{2})_{\infty}}.
$$

The proof is thus completed by interchanging  $a$  and  $c$ .

Combining  $(2.11)$  and  $(1.10)$ , we are led to the following identity

<span id="page-7-0"></span>
$$
(-b, -q/b, b/a, aq/b; q)_{\infty} + (b, q/b, -b/a, -aq/b; q)_{\infty}
$$
  
= 
$$
\frac{2(aq, q/a, b^2/a, aq^2/b^2; q^2)_{\infty}}{(q; q^2)_{\infty}^2}.
$$
 (2.14)

П

To restate the above identity in a symmetric form, we replace a by  $b/a$  in [\(2.14\)](#page-7-0).

### Theorem 2.5 We have

$$
(a, -b, q/a, -q/b; q)_{\infty} + (-a, b, -q/a, q/b; q)_{\infty} = \frac{2(ab, q^2/ab, aq/b, bq/a; q^2)_{\infty}}{(q; q^2)_{\infty}^2}.
$$
 (2.15)

More identities on sums of infinite products have been found by Bailey [5] and Slater [24–26].

While no attempt will be made to derive a closed product formula for the series  $(1.3)$ , we obtain a formula involving a product and a summation which has the advantage that it reduces to the q-Gauss summation [\(1.4\)](#page-1-4) when  $c = q$  or  $d = q$ . Combining Corollary [2.2](#page-5-4) and Cauchy's  $q$ -binomial theorem  $(2.13)$ , we deduce

#### Corollary 2.6

$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, \frac{cd}{abq}\right] = \frac{(c/b, c/q, q^{2}/c, q/d; q)_{\infty}}{(c, c/bq, q/a, q/b; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(d/a; q)_{k}}{(bq^{2}/c; q)_{k}} \left(\frac{bq}{d}\right)^{k} + \frac{(c/a, d/a, b, cd/bq, bq^{2}/cd, q; q)_{\infty}}{(c, d, bq/c, bq/d, q/a, cd/abq; q)_{\infty}},
$$
(2.16)

where  $|bq/d|, |cd/abq| < 1$ .

# 3 A Two-term Summation Formula for  $2\psi_2$

In this section, we show that a two-term summation formula for the  $2\psi_2$  series [\(1.13\)](#page-3-0) due to Slater can be derived from an identity of Andrews [\(1.14\)](#page-3-1) by bilateral extension and parameter augmentation.

We recall that the *q*-difference operator, or Euler derivative, is defined as

$$
D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}.\tag{3.1}
$$

The q-shift operator  $\eta$  in the literature [1, 19] is defined as follows:

$$
\eta\{f(a)\}=f(aq)
$$
 and  $\eta^{-1}\{f(a)\}=f(aq^{-1}),$  (3.2)

which was introduced by Rogers in [16–18].

In [19], Roman combined  $q$ -differential operator and the  $q$ -shift operator to built an operator which was denoted by  $\theta$  in [9]:

$$
\theta = \eta^{-1} D_q. \tag{3.3}
$$

In [9], Chen and Liu introduced the operator:

$$
E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n},\tag{3.4}
$$

and proved the following basic relations:

<span id="page-8-0"></span>
$$
E(b\theta) \{ (at;q)_{\infty} \} = (at, bt;q)_{\infty}, \qquad (3.5)
$$

$$
E(b\theta) \{ (as, at; q)_{\infty} \} = \frac{(as, at, bs, bt; q)_{\infty}}{(abst/q; q)_{\infty}}, \qquad |abst/q| < 1.
$$
 (3.6)

The procedure to apply the operator  $E(b\theta)$  in order to derive a new identity is called parameter augmentation.

The following theorem is equivalent to Slater's formula [\(1.13\)](#page-3-0), as pointed out by Ismail and Rahman [14]. We proceed to demonstrate how to derive it from the identity [\(1.14\)](#page-3-1) of Andrews by bilateral extension and parameter augmentation.

Theorem 3.1 We have

<span id="page-8-1"></span>
$$
{}_{2}\psi_{2}\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, \frac{cd}{abq}\right] - \frac{\alpha}{q} \frac{(q/c, q/d, \alpha/a, \alpha/b; q)_{\infty}}{(q/a, q/b, \alpha/c, \alpha/d; q)_{\infty}} {}_{2}\psi_{2}\left[\begin{array}{cc} aq/\alpha, & bq/\alpha \\ cq/\alpha, & dq/\alpha \end{array}; q, \frac{cd}{abq}\right]
$$

$$
= \frac{(\alpha, q/\alpha, cd/\alpha q, \alpha q^{2}/cd, q, c/a, c/b, d/a, d/b; q)_{\infty}}{(c/\alpha, \alpha q/c, d/\alpha, \alpha q/d, c, d, q/a, q/b, cd/abq; q)_{\infty}}, \qquad (3.7)
$$

where  $|cd/abq| < 1$ .

*Proof.* Shifting the index of summation by m and then replacing a, b, f by  $aq^{-m}$ ,  $bq^m$ ,  $fq^{-m}$  in [\(1.14\)](#page-3-1), respectively, we obtain

<span id="page-9-0"></span>
$$
\frac{d(q^{1-m}/bc, acdfq^{-2m};q)_m (bdq^m)^m}{(1-adq^{-m})(1-dfq^{-m})(adq^{1-m}, dfq^{1-m};q)_m} \sum_{k=-m}^{\infty} \frac{(q/bc, acdfq^{-m};q)_k (bdq^m)^k}{(adq, dfq; q)_k} (bdq^m)^k
$$

$$
-\frac{c(q^{1-m}/bd, acdfq^{-2m};q)_m (bcq^m)^m}{(1-acq^{-m})(1-cfq^{-m})(acq^{1-m}, cfq^{1-m};q)_m} \sum_{k=-m}^{\infty} \frac{(q/bd, acdfq^{-m};q)_k}{(acq, cfq;q)_k} (bcq^m)^k
$$

$$
=\frac{d(q, qd/c, c/d, abcd, acdfq^{-2m}, bcdf; q)_{\infty}}{(acq^{-m}, adq^{-m}, bcq^m, bdq^m, cfq^{-m}, dfq^{-m};q)_{\infty}}.
$$
(3.8)

Letting  $m \to \infty$  in [\(3.8\)](#page-9-0) and employing Tannery's theorem, we get

<span id="page-9-1"></span>
$$
\frac{c(bc;q)_{\infty}}{(1/ad,1/df;q)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(q/bc;q)_k}{(adq,dfq;q)_k} (-abcd^2f)^k q^{k \choose 2} \n- \frac{d(bd;q)_{\infty}}{(1/ac,1/cf;q)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(q/bd;q)_k}{(acq,cfq;q)_k} (-abc^2df)^k q^{k \choose 2} \n= \frac{acd^2f(q,qd/c,c/d,abcd,acdf,bcdf,q/acdf;q)_{\infty}}{(ac,ad,cf,df,q/ac,q/ad,q/cf,q/df;q)_{\infty}}.
$$
\n(3.9)

Now, [\(3.9\)](#page-9-1) can be written as

<span id="page-9-2"></span>
$$
\frac{c}{(1/ad, 1/df; q)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(bcq^{-k}; q)_{\infty}}{(adq, dfq; q)_k} (ad^2fq)^k q^{2\binom{k}{2}}
$$

$$
-\frac{d}{(1/ac, 1/cf; q)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(bdq^{-k}; q)_{\infty}}{(acq, cfq; q)_k} (ac^2fq)^k q^{2\binom{k}{2}}
$$

$$
= \frac{acd^2f(q, qd/c, c/d, abcd, acdf, bcdf, q/cdf; q)_{\infty}}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df; q)_{\infty}}.
$$
(3.10)

Next, applying  $E(g\theta)$  to both sides of [\(3.10\)](#page-9-2) with respect to the parameter b gives

<span id="page-9-5"></span>
$$
\frac{c}{(1/ad, 1/df; q)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(ad^2f q)^k q^{2\binom{k}{2}}}{(adq, dfq; q)_k} E(g\theta) \left\{ (bcq^{-k}; q)_{\infty} \right\}
$$

$$
-\frac{d}{(1/ac, 1/cf; q)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(ac^2f q)^k q^{2\binom{k}{2}}}{(acq, cfq; q)_k} E(g\theta) \left\{ (bdq^{-k}; q)_{\infty} \right\}
$$

$$
= \frac{acd^2f(q, qd/c, c/d, acdf, q/cdf; q)_{\infty}}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df; q)_{\infty}} E(g\theta) \left\{ abcd, bcdf; q)_{\infty} \right\}. (3.11)
$$

From [\(3.5\)](#page-8-0) and [\(3.6\)](#page-8-0), it is evident that

<span id="page-9-3"></span>
$$
E(g\theta) \left\{ (bcq^{-k};q)_{\infty} \right\} = (bcq^{-k}, cqq^{-k};q)_{\infty},
$$
\n(3.12)

<span id="page-9-4"></span>
$$
E(g\theta) \left\{ (bdq^{-k}; q)_{\infty} \right\} = (bdq^{-k}, dgq^{-k}; q)_{\infty}, \tag{3.13}
$$

and

<span id="page-10-0"></span>
$$
E(g\theta) \{abcd, bcdf; q)_{\infty}\} = \frac{(abcd, acdg, bcdf, cdfg; q)_{\infty}}{(abc^2d^2fg/q; q)_{\infty}}.
$$
 (3.14)

Substituting  $(3.12)$ ,  $(3.13)$ , and  $(3.14)$  into  $(3.11)$ , we see that

$$
\frac{c(bc, cg;q)_{\infty}}{(1/ad, 1/df;q)_{\infty}}^{2}\psi_{2}\left[\begin{array}{c}q/bc, q/cg\\adq, dfq\end{array}; q, \frac{abc^{2}d^{2}fg}{q}\right] - \frac{d(bd, dg;q)_{\infty}}{(1/ac, 1/cf;q)_{\infty}}^{2}\psi_{2}\left[\begin{array}{c}q/bd, q/dg\\acq, cfq\end{array}; q, \frac{abc^{2}d^{2}fg}{q}\right] = \frac{acd^{2}f(q, qd/c, c/d, abcd, acdf, acdg, bcdf, q/cdf, cdfg;q)_{\infty}}{(ac, ad, cf, df, q/ac, q/ad, q/cf, q/df, abc^{2}d^{2}fg/q;q)_{\infty}},
$$
(3.15)

where  $|abc^2d^2fg/q| < 1$ .

Finally, the proof is completed by replacing a, b, c, d, f, g by  $c/fq$ , e,  $q/ae$ , f,  $d/fq$ ,  $ae/b$ , respectively, and then setting  $ae f = \alpha$ .

Substitute a, b, c, d,  $\alpha$  with  $qa/e$ ,  $qb/e$ ,  $qc/e$ ,  $q^2/e$ ,  $fq/e$  in [\(3.7\)](#page-8-1), respectively, we may recover the original formula [\(1.13\)](#page-3-0) due to Slater.

If we set  $d = q$  in [\(3.7\)](#page-8-1), then the second term on the left hand side vanishes, and so we get the  $q$ -Gauss summation  $(1.4)$  as a special of  $(3.7)$ .

To conclude this paper, we represent [\(3.7\)](#page-8-1) in an equivalent form and give the explicit substitutions to reach Askey's  $q$ -extension of Cauchy's beta integral [\(1.11\)](#page-2-3). By the relation

$$
{}_2\psi_2\left[\begin{array}{cc} a, & b \\ c, & d \end{array}; q, z\right] = {}_2\psi_2\left[\begin{array}{cc} q/c, & q/d \\ q/a, & q/b \end{array}; q, \frac{cd}{abz}\right],\tag{3.16}
$$

we may rewrite [\(3.7\)](#page-8-1) as

<span id="page-10-1"></span>
$$
\frac{(q/a, q/b; q)_{\infty}}{(q/c, q/d; q)_{\infty}} {}_{2}\psi_{2} \left[ \begin{array}{c} q/c, q/d \\ q/a, q/b \end{array} ; q, q \right] - \frac{\alpha}{q} \frac{(\alpha/a, \alpha/b; q)_{\infty}}{(\alpha/c, \alpha/d; q)_{\infty}} {}_{2}\psi_{2} \left[ \begin{array}{c} \alpha/c, \alpha/d \\ \alpha/a, \alpha/b \end{array} ; q, q \right]
$$

$$
= \frac{(\alpha, q/\alpha, cd/\alpha q, \alpha q^{2}/cd, q, c/a, c/b, d/a, d/b; q)_{\infty}}{(c/\alpha, \alpha q/c, d/\alpha, \alpha q/d, c, d, q/c, q/d, cd/\alpha bq; q)_{\infty}}, \qquad (3.17)
$$

where  $|cd/abq| < 1$ . Replacing a, b, c, d,  $\alpha$  by  $q/c$ ,  $-q/d$ ,  $q/a$ ,  $-q/b$ , q, respectively, then [\(3.17\)](#page-10-1) takes the form of Askey's q-extension of Cauchy's beta integral.

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