

THE DENSITY OF INTEGRAL POINTS ON COMPLETE INTERSECTIONS

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ABSTRACT. In this paper, an upper bound for the number of integral points of bounded height on an affine complete intersection defined over \mathbb{Z} is proven. The proof uses an extension to complete intersections of the method used for hypersurfaces by Heath-Brown [7], the so called “ q -analogue” of van der Corput’s AB process.

1. INTRODUCTION

If X is an affine algebraic set defined by a set of equations

$$f_i(x_1, \dots, x_n) = 0, i = 1, \dots, r$$

with integral coefficients, and if \mathbf{B} is a box in \mathbb{R}^n - that is, a product of closed intervals - then we define the quantity

$$N(X, \mathbf{B}) = \# \{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n; f_i(\mathbf{x}) = 0, \mathbf{x} \in \mathbf{B} \}.$$

If m is a positive integer, and if \mathbf{B} is small enough as to contain at most one representative of each congruence class modulo m , then we define

$$N(X, \mathbf{B}, m) = \# \{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n; f_i(\mathbf{x}) \equiv 0 \pmod{m}, \mathbf{x} \in \mathbf{B} \}.$$

Since $N(X, \mathbf{B}) \leq N(X, \mathbf{B}, m)$ one can obtain upper bounds for $N(X, \mathbf{B})$ by considering $N(X, \mathbf{B}, m)$ for suitably chosen m . If $\mathbf{B} = [-B, B]^n$ for some $B > 0$ we write

$$N(X, B) = N(X, \mathbf{B}) \text{ and } N(X, B, m) = N(X, \mathbf{B}, m).$$

Throughout this paper we shall be concerned with the case when X is a complete intersection, that is, when $\dim X = n - r$, where r is the number of equations defining X in \mathbb{A}^n . Our main concern shall be to find an upper bound for $N(X, B)$. One result in this direction is the following, by Fujiwara [3]: let X be a non-singular hypersurface in \mathbb{A}^n defined by the vanishing of a polynomial f with integer coefficients, of degree at least 2. Then $N(X, B) \ll_{f,n} B^{n-2+2/n}$ for $n \geq 4$. Fujiwara proved this by exhibiting an asymptotic formula for $N(X, B, p)$ for primes p , the proof of which uses the estimates for exponential sums by Deligne [2] as a key tool. Heath-Brown [7] was able to sharpen the exponent to $n - 2 + 2/(n + 1)$ by averaging over primes in an interval. In the same paper he introduced a new technique, the so called q -analogue of van der Corput’s method. He could then prove the bound

$$(1) \quad N(X, B) \ll_{f,n} B^{n-3+15/(n+5)}$$

for a non-singular hypersurface X defined by a polynomial f of degree at least 3 (Theorem 2 in [7]), by considering $N(X, B, pq)$ for two suitable primes p and q .

In this paper we will generalize the method of Heath-Brown to complete intersections of arbitrary codimension. We shall use the following notation: if X is a scheme over \mathbb{Z} we let $X_{\mathbb{Q}} = X \times_{\text{Spec } \mathbb{Z}} \mathbb{Q}$ and $X_q = X_{\mathbb{F}_q} = X \times_{\text{Spec } \mathbb{Z}} \mathbb{F}_q$ for every prime q .

Theorem 1. *Let*

$$X = \text{Spec } \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_r),$$

where the leading forms F_1, \dots, F_r of f_1, \dots, f_r are of degree ≥ 3 , and let

$$Z = \text{Proj } \mathbb{Z}[X_1, \dots, X_n]/(F_1, \dots, F_r).$$

Assume that $Z_{\mathbb{Q}}$ is non-singular of codimension r in $\mathbb{P}_{\mathbb{Q}}^{n-1}$. Then, if $n \geq 4r + 2$, we have for $B \geq 1$

$$N(X, B) \ll_{n,d,\epsilon} B^{n-3r+r^2 \frac{13n-5-3r}{n^2+4nr-n-r-r^2}} (\log B)^{n/2} \left(\sum_{i=1}^r \log \|F_i\| \right)^{2r+1},$$

where $d = \max_i(\deg f_i)$.

Remark. The factor $(\log B)^{n/2}$ can in fact be disposed of, and we sketch in the end of Section 4 how this can be done.

The estimate given by Theorem 1 in the case $r = 1$ is in fact slightly sharper than (1), owing to the use of estimates by Katz [10] on exponential sums modulo q . Theorem 1 is a corollary to the following theorem.

Theorem 2. *Let*

$$X = \text{Spec } \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_r),$$

where $r < n$ and the leading forms F_1, \dots, F_r of f_1, \dots, f_r are of degree ≥ 3 , and let

$$Z = \text{Proj } \mathbb{Z}[X_1, \dots, X_n]/(F_1, \dots, F_r).$$

Let B be a positive number, and let p and q be primes, with $2p < 2B + 1 < q - p$, such that both Z_p and Z_q are non-singular of dimension $n - 1 - r$. Then we have

$$\begin{aligned} N(X, B, pq) &= \frac{(2B+1)^n}{p^r q^r} + O_{n,d} \left(B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} (\log q)^{n/2} \right. \\ &\quad \left. + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} (\log q)^{n/2} + B^{n/2} p^{-r/2} q^{(n-r)/4} (\log q)^{n/2} \right. \\ &\quad \left. + B^{n/2} p^{(n-r)/2} (\log q)^{n/2} + B^n p^{-(n+r-1)/2} q^{-r} + B^{n-1} p^{-r+1} q^{-r} \right), \end{aligned}$$

where $d = \max_i(\deg f_i)$.

The proof of Theorem 2 is carried out in Section 4 and more or less follows [7]. However, in contrast to Heath-Brown, we do not use Poisson summation, but a more direct approach.

We also prove, in Section 3, a generalization (and slight sharpening) of Theorem 3 in [7], a weighted asymptotic formula for the density of \mathbb{F}_q -points on affine complete intersections defined over \mathbb{F}_q . However, for the proof of

Theorem 2, we will use an unweighted version of this result, proven by Salberger in an Appendix to this paper. This is because we desire an unweighted asymptotic formula in Theorem 2.

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2. PRELIMINARY RESULTS FROM ALGEBRAIC GEOMETRY

We recall some facts from algebraic geometry that will provide helpful tools for proving our main results.

Definition. Let X be a scheme. A point $x \in X$ is a *singular point* of X if the local ring $\mathcal{O}_{X,x}$ is not a regular local ring. X is said to be *singular* if it has singular points, and *non-singular* if not. We denote the *singular locus* of X - the set of singular points - by $\text{Sing}X$.

If X is a scheme and x a point on X , then \mathcal{O}_x is the local ring at x , \mathfrak{m}_x its maximal ideal and $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$ the residue field of x . If $X \rightarrow Y$ is a morphism of schemes, $\Omega_{X/Y}$ denotes the sheaf of relative differentials of X over Y , and we abbreviate $\Omega_{X/\text{Spec } R} = \Omega_{X/R}$.

We have the following characterization of singular points on a scheme.

Proposition 1. *Let X be a scheme of finite type over a perfect field k . Suppose that X is equidimensional of dimension n . Then for every point $x \in X$, the following conditions are equivalent:*

- (i) x is a singular point of X ;
- (ii) $\dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x) > n$.

Proof. Since this is a local question, we can assume that $X = \text{Spec } R$ with R equidimensional. Suppose $x = \mathfrak{p} \in \text{Spec } R$. Then we have, by [14, Ex. 14.36],

$$(2) \quad \begin{aligned} n &= \text{ht } \mathfrak{p} + \dim R/\mathfrak{p} \\ &= \dim \mathcal{O}_x + \text{tr.d.} \kappa(x)/k. \end{aligned}$$

By definition, x is singular if and only if

$$\dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 > \dim \mathcal{O}_x.$$

Furthermore, by [6, Ex. II.8.1], we have an exact sequence of $\kappa(x)$ -vector spaces

$$0 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow \Omega_{\mathcal{O}_x/k} \otimes_{\mathcal{O}_x} \kappa(x) \rightarrow \Omega_{\kappa(x)/k} \rightarrow 0.$$

Since $\Omega_{\mathcal{O}_x/k}$ is equal to the stalk $\Omega_{X/k,x}$ of the sheaf of relative differentials, and since $\dim_{\kappa(x)} \Omega_{\kappa(x)/k} = \text{tr.d.} \kappa(x)/k$ by [6, Thm. II.8.6A], this implies that

$$\dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x) = \dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 + \text{tr.d.} \kappa(x)/k.$$

In view of (2) it follows that $x \in \text{Sing}X$ if and only if

$$\dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x) > \dim \mathcal{O}_x + \text{tr.d.} \kappa(x)/k = n.$$

□

Remark 1. By [6, Ex. II.5.8] the function

$$\varphi(x) = \dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x)$$

is upper semicontinuous, so that in the situation described in the proposition, $\text{Sing}X$ is a closed subscheme of X .

Remark 2. The proposition also shows that for X equidimensional and of finite type over a perfect field k , X is non-singular if and only if it is *smooth over k* (see [6, Ch. III.10]).

Remark 3. The particular case where we will use the proposition is for X a complete intersection of positive dimension in projective space over a perfect field. Such X are indeed equidimensional, since firstly, any local complete intersection is Cohen-Macaulay ([6, Prop. 8.23]) and thus locally equidimensional, and secondly, a complete intersection in \mathbb{P}_k^n of dimension ≥ 1 is connected ([6, Ex. III.5.5]).

When working in a projective space \mathbb{P}^n with homogeneous coordinates x_0, \dots, x_n we denote by \mathbb{P}^n the dual projective space with homogeneous coordinates ξ_0, \dots, ξ_n . For a point $\mathbf{a} = (a_0, \dots, a_n)$ in \mathbb{P}^n we will let $H_{\mathbf{a}}$ denote the hyperplane defined in \mathbb{P}^n by the equation $\mathbf{a} \cdot \mathbf{x} = a_0x_0 + \dots + a_nx_n = 0$. We begin by proving the following corollary to Bertini's Theorem. By convention, the dimension of the empty set is defined to be -1 .

Lemma 1. *Let k be an algebraically closed field. Let X be a non-empty complete intersection in \mathbb{P}_k^n . Suppose that*

$$\dim \text{Sing}X = s.$$

Then there is a hyperplane H such that $\dim(X \cap H) = \dim X - 1$ and

$$\dim \text{Sing}(X \cap H) < \max(s, 0).$$

Proof. The case $s = -1$ follows immediately from Bertini's Theorem [9, Cor 6.11(2)]. (X is then smooth over k by Remark 2.) If $s \geq 0$, let $Y = X \setminus \text{Sing}X$, so that Y is smooth. Then, by Bertini's Theorem, there exists a non-empty Zariski open subset U of \mathbb{P}_k^n such that for hyperplanes $H_{\mathbf{a}}$ parametrized by closed k -points \mathbf{a} in U , $Y \cap H_{\mathbf{a}}$ is smooth and thus non-singular by Remark 2. Hence, for $\mathbf{a} \in U(k)$ we have

$$(3) \quad \text{Sing}(X \cap H_{\mathbf{a}}) \subseteq \text{Sing}X \cap H_{\mathbf{a}}.$$

Furthermore, there are non-empty open sets U', U'' such that for all closed k -points \mathbf{a} of U' , no irreducible component of $\text{Sing}X$ of dimension s is contained in $H_{\mathbf{a}}$, and for $\mathbf{a} \in U''(k)$ no irreducible component of X is contained in $H_{\mathbf{a}}$. Then we have, for $\mathbf{a} \in U \cap U' \cap U''(k)$, that $\dim(X \cap H_{\mathbf{a}}) = \dim X - 1$ and $\dim \text{Sing}(X \cap H_{\mathbf{a}}) < s$. \square

Remark 4. For any hyperplane H such that $\dim X \cap H = \dim X - 1$, $\dim \text{Sing}(X \cap H) \geq \dim \text{Sing}X - 1$ (see [10, Lemma 3]).

The next lemma is an "effective" version of Bertini's Theorem. For a more explicit result of the same type, see [1].

Lemma 2. *Let n, r, d_1, \dots, d_r be natural numbers, and let F_1, \dots, F_r be forms in X_0, \dots, X_n with integer coefficients, and with $\deg F_i = d_i$. Let $V = \text{Proj } \mathbb{Z}[X_0, \dots, X_n]/(F_1, \dots, F_r)$, and suppose that $V_{\mathbb{Q}}$ has dimension $n - r \geq 0$. Then for every prime q such that V_q has dimension $n - r$, there is a non-zero form $\Phi_q \in \mathbb{F}_q[\xi_0, \dots, \xi_n]$ with degree bounded in terms of n and d_1, \dots, d_r only, such that for every point $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{P}_{\mathbb{F}_q}^n$ satisfying $\Phi_q(a_0, \dots, a_n) \neq 0$ we have*

- (i) $\dim \text{Sing}(V_q \cap H_{\mathbf{a}}) = \max(-1, \dim \text{Sing} V_q - 1)$
- (ii) $\dim V_q \cap H_{\mathbf{a}} = \dim V_q - 1$.

In particular, for each $q \geq q_0 = q_0(n, d_1, \dots, d_r)$ there is an $\mathbf{a} \in \mathbb{P}_{\mathbb{F}_q}^n$ with the properties (i) and (ii).

Proof. We let \mathbb{P}_i , for each $i = 1, \dots, r$, be the projective space over \mathbb{Z} parametrizing all hypersurfaces in $\mathbb{P}_{\mathbb{Z}}^n$ of degree d_i (as a Hilbert scheme), and work in the large multiprojective space $\mathbf{P} = \mathbb{P}_1 \times \dots \times \mathbb{P}_r$. For a k -point in \mathbf{P} representing a tuple (F_1, \dots, F_r) we write $V(F_1, \dots, F_r)$ for the intersection of the corresponding r hypersurfaces in \mathbb{P}_k^n . Let $W \subseteq \mathbf{P} \times \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_{\mathbb{Z}}^n$ be defined as the closed set of points $P \in \mathbf{P} \times \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_{\mathbb{Z}}^n$ representing $(F_1, \dots, F_r, \mathbf{a}, \mathbf{x})$ that satisfy

$$\mathbf{x} \in V(F_1, \dots, F_r) \cap H_{\mathbf{a}}.$$

Let

$$\pi : W \rightarrow \mathbf{P}' := \mathbf{P} \times \mathbb{P}_{\mathbb{Z}}^n$$

be the projection. The function $\varphi(P) := \dim_{\kappa(P)} \Omega_{W/\mathbf{P}', P}$ is upper semicontinuous (see Remark 1), so the set

$$S = \{P \in W; \varphi(P) \geq n - r\}$$

is closed. Now, let $\tilde{\pi} : S \rightarrow \mathbf{P}'$ be the restriction of π to S , and let for every $s \in \{-1, 0, 1, \dots, n\}$

$$A_s = \{Q \in \mathbf{P}'; \dim \tilde{\pi}^{-1}(Q) \geq s\}.$$

By Chevalley's Semicontinuity Theorem [5, Cor 13.1.5], A_s is closed in \mathbf{P}' , as is the set

$$D = \{Q \in \mathbf{P}'; \dim \pi^{-1}(Q) \geq n - r\}.$$

For each $s \in \{-1, 0, \dots, n\}$, let $T_s = D \cup A_s$. Then T_s is closed as well, so there exist multihomogeneous forms H_1^s, \dots, H_t^s over \mathbb{Z} that define T_s .

For a closed k -point $P \in W$ representing $(F_1, \dots, F_r, \mathbf{a}, \mathbf{x})$ we have an isomorphism of stalks $\Omega_{W/\mathbf{P}', P} \cong \Omega_{Y/k, \mathbf{x}}$, where

$$Y = V(F_1, \dots, F_r) \cap H_{\mathbf{a}} \subseteq \mathbb{P}_k^n.$$

Thus, for each tuple $(F_1, \dots, F_r, \mathbf{a})$ such that both $V = V(F_1, \dots, F_r)$ and $V \cap H_{\mathbf{a}}$ are complete intersections of codimension r and $r + 1$, respectively, the fiber $\tilde{\pi}^{-1}(F_1, \dots, F_r, \mathbf{a})$ is precisely $\text{Sing}(V \cap H_{\mathbf{a}})$ by Proposition 1. For every other point $(F_1, \dots, F_r, \mathbf{a})$ we have $\tilde{\pi}^{-1}(F_1, \dots, F_r, \mathbf{a}) = \mathbb{P}_k^n$. We conclude that T_s , for each s , is the set of tuples $(F_1, \dots, F_r, \mathbf{a})$ such that $V(F_1, \dots, F_r) \cap H_{\mathbf{a}}$ either has codimension $\leq r$ or has a singular locus of dimension at least s . In particular, if we have a closed k -point $Q \in \mathbf{P}$ representing (F_1, \dots, F_r) such that $V = V(F_1, \dots, F_r)$ satisfies

$$(4) \quad \dim V = n - r, \quad \dim \text{Sing} V = s,$$

and if $\pi_s : T_s \rightarrow \mathbf{P}$ is the projection, then the fiber $\pi_s^{-1}(Q)$ is the closed set of points $\mathbf{a} \in \mathbb{P}_k^n$ such that either $\dim \text{Sing}(V \cap H_{\mathbf{a}}) \geq \dim \text{Sing} V$ or $\dim(V \cap H_{\mathbf{a}}) = \dim V$.

Now let F_1, \dots, F_r be forms as in the hypothesis, and let q be a prime such that (4) is satisfied for $Q \in \mathbf{P}$ representing the tuple of $(\text{mod } q)$ -reductions $((F_1)_q, \dots, (F_r)_q)$. Then $\pi_s^{-1}(Q)$ is defined in \mathbb{P}_k^n , where $k = \kappa(Q) = \mathbb{F}_q$, by the specializations $H_i^s|_Q$ of the multihomogeneous forms H_i^s . Applying Lemma 1 we get that $\pi_s^{-1}(Q) \times \text{Spec } \bar{k}$ is a proper closed subset of $\mathbb{P}_{\bar{k}}^n$ (where \bar{k} is an algebraic closure of k). Therefore one of the forms $H_i^s|_Q \in k[\xi_0, \dots, \xi_n]$ must be non-zero, so the form

$$\Phi_q(\xi_0, \dots, \xi_n) = H_i^s|_Q(\xi_0, \dots, \xi_n)$$

has the desired properties.

The last assertion of the lemma follows from the easy observation that a polynomial of degree at most q cannot vanish at every point of $\mathbb{P}_{\mathbb{F}_q}^n$. \square

The following lemma explores the new geometry arising from the Weyl differencing in Section 4. For a polynomial $f(X_1, \dots, X_n)$ we denote by ∇f the gradient $\left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right)^t$ and by $\nabla^2 f$ the Hessian matrix $\left(\frac{\partial^2 f}{\partial X_i \partial X_j}\right)_{1 \leq i, j \leq n}$.

Lemma 3. *Let G_1, \dots, G_r be homogeneous polynomials in $\mathbb{Z}[X_1, \dots, X_n]$ of degrees d_1, \dots, d_r , and let*

$$V = \text{Proj } \mathbb{Z}[X_1, \dots, X_n]/(G_1, \dots, G_r).$$

Let q be a prime such that $q \nmid d_i$ for all $i = 1, \dots, r$ and suppose that V_q is a non-singular complete intersection of codimension r in $\mathbb{P}_{\mathbb{F}_q}^{n-1}$.

(i) Let

$$S = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{P}_{\mathbb{F}_q}^{n-1} \times \mathbb{P}_{\mathbb{F}_q}^{n-1}; \mathbf{y} \cdot \nabla G_i(\mathbf{x}) = 0, i = 1, \dots, r, \right. \\ \left. \text{rank}(\mathbf{y} \cdot \nabla^2 G_i(\mathbf{x}))_{1 \leq i \leq r} < r \right\}.$$

Then $\dim S \leq n - 2$.

(ii) For $\mathbf{y} \in \mathbb{P}_{\mathbb{F}_q}^{n-1}$, let

$$S_{\mathbf{y}} = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}_q}^{n-1}; \mathbf{y} \cdot \nabla G_i(\mathbf{x}) = 0, i = 1, \dots, r, \right. \\ \left. \text{rank}(\mathbf{y} \cdot \nabla^2 G_i(\mathbf{x}))_{1 \leq i \leq r} < r, \right\}.$$

For $s = -1, 0, 1, \dots, n - 1$, let $T_s = \left\{ \mathbf{y} \in \mathbb{P}_{\mathbb{F}_q}^{n-1}; \dim S_{\mathbf{y}} \geq s \right\}$. Then T_s is Zariski closed and $\dim T_s \leq n - s - 2$.

(iii) For each s , let $T_s^{(1)}, T_s^{(2)}, \dots$ be the irreducible components of T_s . Then

$$\sum_j \deg(T_s^{(j)}) = O_{n,r,d_1,\dots,d_r}(1).$$

To prove Lemma 3 we shall need the following lemma.

Lemma 4. *Let k be a field, and let V be a closed subscheme of $\mathbb{P}_k^n \times \mathbb{P}_k^n$. Let $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal, $\Delta = \{(\mathbf{x}, \mathbf{x}); \mathbf{x} \in \mathbb{P}_k^n\}$. If $\dim V \geq n$, then $V \cap \Delta \neq \emptyset$.*

Proof. Consider the rational map

$$f : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

given by

$$(X_0 : \dots : X_{2n+1}) \mapsto ((X_0 : \dots : X_n), (X_{n+1} : \dots : X_{2n+1})).$$

Its domain of definition is the Zariski open set $U := \mathbb{P}^{2n+1} \setminus (L \cup M)$, where $L = \{X_0 = \dots = X_n = 0\}$ and $M = \{X_{n+1} = \dots = X_{2n+1} = 0\}$. Moreover, let $\hat{\Delta}$ be the variety in \mathbb{P}^{2n+1} defined by $X_0 = X_{n+1}, \dots, X_n = X_{2n+1}$. Then f is an isomorphism between $\hat{\Delta}$ and Δ . Let \hat{V} be the Zariski closure in \mathbb{P}^{2n+1} of $f^{-1}(V)$. Then

$$\dim \hat{V} = \dim V + 1 \geq n + 1,$$

so that

$$\text{codim} \hat{\Delta} + \text{codim} \hat{V} \leq 2n + 1.$$

Thus, by the Projective Dimension Theorem [11, Ex. 3.3.4], $\hat{\Delta} \cap \hat{V}$ is nonempty. But a point P in this intersection automatically lies in U , since $\hat{\Delta} \cap (L \cup M)$ is empty, and we get a point $f(P)$ in $\Delta \cap V$. \square

Proof of Lemma 3. (i) Assume that $\dim S \geq n - 1$. According to Lemma 4, we then must have $S \cap \Delta \neq \emptyset$. Thus, suppose $(\mathbf{x}, \mathbf{x}) \in S \cap \Delta$. By the definition of S , we then have

$$\begin{cases} \mathbf{x} \cdot \nabla G_i(\mathbf{x}) = 0, & i = 1, \dots, r \\ \text{rank}(\mathbf{x} \cdot \nabla^2 G_i(\mathbf{x}))_{1 \leq i \leq r} < r. \end{cases}$$

But $\mathbf{x} \cdot \nabla^2 G_i(\mathbf{x}) = \nabla(\mathbf{x} \cdot \nabla G_i(\mathbf{x}))$, so by Euler's identity we have (since q does not divide any of the degrees of the G_i)

$$\begin{cases} G_i(\mathbf{x}) = 0, & i = 1, \dots, r \\ \text{rank}(\nabla G_i(\mathbf{x}))_{1 \leq i \leq r} < r. \end{cases}$$

Therefore, by the Jacobian Criterion, \mathbf{x} is a singular point of V , in contradiction with the hypothesis.

(ii) Let $\pi : S \rightarrow \mathbb{P}^{n-1}$ be the projection onto the second coordinate, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{y}$. Then $S_{\mathbf{y}} = \pi^{-1}(\mathbf{y}) \times \{\mathbf{y}\}$. The fact that T_s is closed follows from Chevalley's semicontinuity theorem [5, Cor 13.1.5]. Now let $S_s = S \cap (\mathbb{P}^{n-1} \times T_s)$ for each $s = -1, \dots, n-1$. Since S_s is the disjoint union of fibres

$$S_s = \bigcup_{\mathbf{y} \in T_s} \pi^{-1}(\mathbf{y}),$$

we have, by (i)

$$\dim T_s + s \leq \dim S_s \leq \dim S \leq n - 2,$$

whence $\dim T_s \leq n - s - 2$.

(iii) As in Lemma 2, we shall let \mathbb{P}_i be the projective spaces parametrizing hypersurfaces of degree d_i in $\mathbb{P}_{\mathbb{Z}}^n$, and put $\mathbf{P} = \mathbb{P}_1 \times \dots \times \mathbb{P}_r$. Now, let

$$\mathcal{S} = \left\{ (G_1, \dots, G_r, \mathbf{x}, \mathbf{y}) \in \mathbf{P} \times \mathbb{P}_{\mathbb{Z}}^{n-1} \times \mathbb{P}_{\mathbb{Z}}^{n-1}; \mathbf{y} \cdot \nabla G_i(\mathbf{x}) = 0, i = 1, \dots, r, \right. \\ \left. \text{rank}(\mathbf{y} \cdot \nabla^2 G_i(\mathbf{x}))_{1 \leq i \leq r} < r, \right\}.$$

Let $\tilde{\pi} : \mathcal{S} \rightarrow \mathbf{P} \times \mathbb{P}_{\mathbb{Z}}^{n-1}$ be the projection $(G_1, \dots, G_r, \mathbf{x}, \mathbf{y}) \mapsto (G_1, \dots, G_r, \mathbf{y})$, and define for each s

$$\mathcal{T}_s = \left\{ \mathcal{P} = (G_1, \dots, G_r, \mathbf{y}); \dim \tilde{\pi}^{-1}(\mathcal{P}) \geq s \right\}.$$

Then \mathcal{T}_s is closed by Chevalley's theorem, so it is defined in $\mathbf{P} \times \mathbb{P}_{\mathbb{Z}}^{n-1}$ by multihomogeneous polynomials H_1, \dots, H_t where $t = O_{n,r,d_1, \dots, d_r}(1)$. Now we fix polynomials G_1, \dots, G_r and a prime q . The set T_s is then defined in $\mathbb{P}_{\mathbb{F}_q}^{n-1}$ by $H_1|_{G_1, \dots, G_r}, \dots, H_t|_{G_1, \dots, G_r}$. Now by Bézout's Theorem [4, Ex. 8.4.6] we have

$$\sum_j \deg(T_s^{(j)}) \leq \prod_i \deg(H_i) \ll_{n,r,d_1, \dots, d_r} 1.$$

□

3. POINTS ON COMPLETE INTERSECTIONS OVER \mathbb{F}_q

The following result is well-known and trivial, but we include a proof for the sake of completeness.

Lemma 5. *Let $X = \text{Spec } \mathbb{F}_q[X_1, \dots, X_n]/(f_1, \dots, f_\rho)$ be a closed subscheme of $\mathbb{A}_{\mathbb{F}_q}^n$, and let $d = \max_i(\deg f_i)$. Let $B \geq 1$. Then, for any box $\mathbf{B} = [a_1 - b_1, a_1 + b_1] \times \dots \times [a_n - b_n, a_n + b_n]$, with $|b_i| \leq B$, containing at most one representative of each congruence class modulo q , we have*

$$N(X, \mathbf{B}, q) \ll_{n,\rho,d} B^{\dim X}.$$

Proof. We identify $\mathbb{A}_{\mathbb{F}_q}^n$ with the open subset $\{X_0 \neq 0\}$ of $\mathbb{P}_{\mathbb{F}_q}^n$ and consider the scheme-theoretic closure Y of X in $\mathbb{P}_{\mathbb{F}_q}^n$ defined by the homogenizations F_1, \dots, F_ρ of f_1, \dots, f_ρ . Then the sum D_X of the degrees of the irreducible components of Y is at most d^ρ by Bézout's Theorem [4, Ex. 8.4.6]. Thus it suffices to show that $N(X, \mathbf{B}, q) \ll_{n,D_X} B^{\dim X}$ for every closed subscheme X . We prove this by induction over $\nu = \dim X$. If $\nu = 0$, then $\#X(\mathbb{F}_q) \leq D_X$, so we are done. Thus, suppose that $\nu \geq 1$. Since X has at most D_X irreducible components, it is enough to prove that $N(X', \mathbf{B}, q) \ll_{n,D_X} B^\nu$ for an arbitrary irreducible component X' of X . For some $i \in \{1, \dots, n\}$, all the hyperplanes $H_a : x_i = a$, where a ranges over \mathbb{F}_q , intersect X' properly. Since $D_{X \cap H_a} \leq D_X$, the induction hypothesis yields that $N(X' \cap H_a, \mathbf{B}, q) \ll_{n,D_X} B^{\nu-1}$ for each $a \in \mathbb{F}_q$. Since we only need to consider at most $2B$ values of a , we get

$$N(X', \mathbf{B}, q) = \sum_a N(X' \cap H_a, \mathbf{B}, q) \leq 2B \cdot O_{n,D_X}(B^{\nu-1}) \ll_{n,D_X} B^\nu,$$

as desired. □

Delignes work on the Weil Conjectures [2] yields a sharp asymptotic formula for the number of \mathbb{F}_q -points on a non-singular projective complete intersection. In the paper by Hooley [8] (with an appendix by Katz) an extension

to the singular case is proven. The following lemma is an affine reformulation of Hooley's result.

Lemma 6. *Let Y be a closed subscheme of $\mathbb{P}_{\mathbb{F}_q}^n$ that is a complete intersection of codimension $r \leq n$ and multidegree (d_1, \dots, d_r) . Let $Z = Y \cap \{x_0 = 0\}$ and suppose that $\dim Z = \dim Y - 1$. Put $X = Y \setminus Z$ and $s = \dim \text{Sing} Z$. Then we have*

$$\#X(\mathbb{F}_q) = q^{n-r} + O_{n,d_1,\dots,d_r}(q^{(n-r+2+s)/2}).$$

Proof. In case $n = r$ the lemma is a trivial consequence of Bézout's Theorem. We may thus assume that $n > r$. By [8, Appendix, Thm. 1] we have

$$\#Z(\mathbb{F}_q) = 1 + q + \dots + q^{n-r-1} + O(q^{(n-r+s)/2}).$$

However, $s \geq \dim \text{Sing} Y - 1$ by Remark 4, so by the same theorem we get

$$\#Y(\mathbb{F}_q) = 1 + q + \dots + q^{n-r} + O(q^{(n-r+2+s)/2}).$$

Subtracting these two equations, we get

$$\#X(\mathbb{F}_q) = q^{n-r} + O(q^{(n-r+2+s)/2}),$$

as stated. \square

The following result is a generalization of Theorem 3 in [7]. However, even in the case of a hypersurface we get a slightly sharper estimate. The reason for this is the use of estimates by Katz [10] for "singular" exponential sums. A similar application of those results are found in a paper by Luo [12].

Notation. For an element $\mathbf{x} = (x_1, \dots, x_n)$ in \mathbb{Z}^n we let $\mathbf{x}_q = (x_1 + q\mathbb{Z}, \dots, x_n + q\mathbb{Z}) \in \mathbb{F}_q^n$.

Theorem 3. *Let $W : \mathbb{R}^n \rightarrow \mathbb{R}$ be an infinitely differentiable function, supported in a cube of side $2L$. Let q be a prime and B a real number with $1 \leq B \ll_L q$. Let*

$$X = \text{Spec } \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_r),$$

where the leading forms F_1, \dots, F_r of f_1, \dots, f_r are of degree at least 2, and let

$$Z_q = \text{Proj } \mathbb{Z}[X_1, \dots, X_n]/(q, F_1, \dots, F_r).$$

Assume that $\dim Z_q = n - 1 - r$. Let $s = \dim \text{Sing} Z_q$ and $d = \max_i(\deg F_i)$. Define a weighted counting function

$$N_W(X, B, q) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \mathbf{x}_q \in X_q}} W\left(\frac{1}{B}\mathbf{x}\right).$$

Then we have

$$(5) \quad \begin{aligned} N_W(X, B, q) &= q^{-r} N_W(\mathbb{A}^n, B, q) \\ &\quad + O_{n,d,L}\left(D_{2n} B^{s+1} q^{(n-r-s-2)/2} (B + q^{1/2})\right), \end{aligned}$$

where, for each natural number k , D_k is the maximum over \mathbb{R}^n of all partial derivatives of W of order k .

Proof. We begin with some preparatory considerations, to justify the use of Lemma 6 later in the proof. Let

$$Y_q = \text{Proj } \mathbb{Z}[X_0, \dots, X_n]/(q, G_1, \dots, G_r),$$

where $G_i(X_0, \dots, X_n) = X_0^{d_i} f_i(X_1/X_0, \dots, X_n/X_0)$ for $i = 1, \dots, n$. Then $Z_q = Y_q \cap \{X_0 = 0\}$ and $X_q = Y_q \setminus Z_q$. Moreover, since $\dim Z_q = n - 1 - r$ we must have $\dim Y_q = n - r$.

We shall follow the approach of Heath-Brown [7] and use induction with respect to s , starting with the case when Z_q is non-singular, that is, when $s = -1$. In case $n - r \geq 2$ we shall use Katz' results. We begin, however, with two trivial cases. Suppose firstly that $n - r = 1$. Then

$$N_W(X, B, q) \ll_{n,L} D_0 N(X, B, q) \ll_{n,d} D_0 B$$

by Lemma 5, and

$$q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,L} D_0 q^{-n+1} B^n \ll_{n,L} D_0 B,$$

so

$$N_W(X, B, q) - q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,d,L} D_{2n}(B + q^{1/2})$$

as required for (5). Next, suppose that $n - r = 0$. Also in this case the formula (5) holds, since $N_W(X, B, q) \ll_{n,d,L} D_0$ and $q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,L} D_0 q^{-n} B^n \ll_{n,L} D_0$, whereas the error term required for (5) is $D_{2n}(Bq^{-1/2} + 1)$.

From now on, we assume that $n - r \geq 2$. By the Poisson Summation Formula we have

$$\begin{aligned} N_W(X, B, q) &= \sum_{\mathbf{z} \in X_q} \sum_{\mathbf{u} \in \mathbb{Z}^n} W\left(\frac{1}{B}(\mathbf{z} + q\mathbf{u})\right) \\ &= \sum_{\mathbf{z} \in X_q} \left(\frac{B}{q}\right)^n \sum_{\mathbf{a} \in \mathbb{Z}^n} e_q(\mathbf{a} \cdot \mathbf{z}) \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \\ &= \left(\frac{B}{q}\right)^n \sum_{\mathbf{a} \in \mathbb{Z}^n} \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \Sigma_q(\mathbf{a}), \end{aligned}$$

where

$$\Sigma_q(\mathbf{a}) = \sum_{\mathbf{z} \in X_q} e_q(\mathbf{a} \cdot \mathbf{z}),$$

a sum which we shall now investigate. In case $\mathbf{a} \equiv \mathbf{0} \pmod{q}$, we can use Lemma 6 to conclude that we have

$$\Sigma_q(\mathbf{a}) = \#X_q(\mathbb{F}_q) = q^{n-r} + O_{n,d}(q^{(n-r+1)/2}).$$

Next we consider $\Sigma_q(\mathbf{a})$ for $\mathbf{a} \not\equiv \mathbf{0} \pmod{q}$. Since Z_q is a projective complete intersection of dimension at least 1, it is geometrically connected. Being non-singular, it is thus geometrically integral. The hypothesis that $\deg F_i \geq 2$ for all i now implies that for each $\mathbf{a} \in \mathbb{F}_q^n \setminus \{\mathbf{0}\}$ we have $\dim(Z_q \cap H_{\mathbf{a}}) = n - r - 2$, where $H_{\mathbf{a}}$ is the hyperplane defined by $\mathbf{a} \cdot \mathbf{x} = 0$. Then, by Theorems 23 and 24 in [10], we have

$$\Sigma_q(\mathbf{a}) \ll q^{(n-r+1+\delta(\mathbf{a}))/2},$$

where $\delta(\mathbf{a}) = \dim \text{Sing}(Z_q \cap H_{\mathbf{a}})$. Thus we get

$$(6) \quad N_W(X, B, q) = \left(\frac{B}{q}\right)^n \left(\sum_{q|\mathbf{a}} \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \left(q^{n-r} + O_{n,d}\left(q^{(n-r+1)/2}\right) \right) \right) + O\left(\left(\frac{B}{q}\right)^n \sum_{\mathbf{a} \in \mathbb{Z}^n} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r+1+\delta(\mathbf{a}))/2} \right).$$

The first term here equals

$$(7) \quad \left(\frac{B}{q}\right)^n q^{n-r} \sum_{\mathbf{v} \in \mathbb{Z}^n} \hat{W}(B\mathbf{v}) + O_{n,d} \left(\left(\frac{B}{q}\right)^n q^{(n-r+1)/2} \sum_{\mathbf{v} \in \mathbb{Z}^n} \hat{W}(B\mathbf{v}) \right) = q^{-r} N_W(\mathbb{A}^n, B, q) + O_{n,d,L} \left(B^n q^{-(n+r-1)/2} \right),$$

by the Poisson formula in the reverse direction and since $N_W(\mathbb{A}^n, B, q) = O_{n,d,L}(B^n)$. In order to estimate the second term in (6) we write

$$\sum_{\mathbf{a} \in \mathbb{Z}^n} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r+1+\delta(\mathbf{a}))/2} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{|\mathbf{a}| \leq q/2} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r+1+\delta(\mathbf{a}))/2} \text{ and} \\ \Sigma_2 = \sum_{|\mathbf{a}| > q/2} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r+1+\delta(\mathbf{a}))/2}.$$

It follows from a result of Zak (see [8, Appendix, Thm. 2]) that $\delta(\mathbf{a}) = -1$ or 0 for all \mathbf{a} . By Lemma 2, all \mathbf{a} for which $\delta(\mathbf{a}) = 0$ satisfy $\Phi(\mathbf{a}) \equiv 0 \pmod{q}$ for a non-zero polynomial $\Phi(\xi_1, \dots, \xi_n)$ with integer coefficients, whose degree is $O_{n,d}(1)$. Thus, let us split Σ_1 into two sums

$$\Sigma_1 = \sum_{\substack{|\mathbf{a}| \leq q/2 \\ \Phi(\mathbf{a}) \equiv 0(q)}} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r+1)/2} + \sum_{\substack{|\mathbf{a}| \leq q/2 \\ \Phi(\mathbf{a}) \not\equiv 0(q)}} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r)/2}$$

and denote the first by Σ_{11} and the second by Σ_{12} . We observe that, since the infinitely differentiable function W has compact support, we have an estimate $|\hat{W}(\mathbf{t})| \ll_{n,L} D_k |\mathbf{t}|^{-k}$ for $|\mathbf{t}| \geq 1$ and any $k \geq 0$, and moreover $D_k \ll_{n,L} D_{k+1}$ for every k . In particular, for any $t \in \mathbb{R}^n$ we have the estimate

$$(8) \quad \left| \hat{W}(\mathbf{t}) \right| \ll_{n,L} D_k \min(1, |\mathbf{t}|^{-k}), \quad k \geq 0$$

Thus we get

$$\sum_{\substack{|\mathbf{a}| \leq q/2 \\ \Phi(\mathbf{a}) \equiv 0(q)}} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| \ll_{n,L} D_{2n} \sum_{\substack{|\mathbf{a}| \leq q/2 \\ \Phi(\mathbf{a}) \equiv 0(q)}} \min\left(1, \left|\frac{B}{q}\mathbf{a}\right|^{-2n}\right).$$

Without loss of generality we can assume that ξ_n occurs in the polynomial $\Phi(\xi_1, \dots, \xi_n)$. Then, for each fixed determination of a_1, \dots, a_n , there are $O_{n,d}(1)$ values for which $\Phi(a_1, \dots, a_n) \equiv 0 \pmod{q}$, and we get

$$\begin{aligned} \sum_{\substack{|\mathbf{a}| \leq q/2 \\ \Phi(\mathbf{a}) \equiv 0(q)}} \min\left(1, \left|\frac{B}{q}\mathbf{a}\right|^{-2n}\right) &= \sum_{|a_1| \leq q/2} \cdots \sum_{|a_{n-1}| \leq q/2} \sum_{\substack{|a_n| \leq q/2 \\ \Phi(\mathbf{a}) \equiv 0(q)}} \min\left(1, \left|\frac{B}{q}\mathbf{a}\right|^{-2n}\right) \\ &\ll_{n,d} \prod_{i=1}^{n-1} \sum_{|a_i| \leq q/2} \min\left(1, \left|\frac{B}{q}a_i\right|^{-2}\right). \end{aligned}$$

Now, for each $i = 1, \dots, n-1$ we have

$$\sum_{|a_i| \leq q/2} \min\left(1, \left|\frac{B}{q}a_i\right|^{-2}\right) = \sum_{|a_i| \leq q/B} 1 + \sum_{q/B < |a_i| \leq q/2} \left|\frac{B}{q}a_i\right|^{-2} \ll \frac{q}{B},$$

and we conclude that

$$\Sigma_{11} \ll_{n,d,L} D_{2n} \left(\frac{q}{B}\right)^{n-1} q^{(n-r+1)/2}.$$

Moreover, using (8) and the fact that

$$(9) \quad \sum_{\substack{\mathbf{u} \in \mathbb{Z}^n \\ |\mathbf{u}| > U}} |\mathbf{u}|^{-(n+1)} \ll_n U^{-1}$$

we have

$$\begin{aligned} \Sigma_{12} &\leq \sum_{|\mathbf{a}| \leq q/2} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r)/2} \\ &\leq q^{(n-r)/2} \left(\sum_{|\mathbf{a}| \leq q/B} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| + \sum_{q/B < |\mathbf{a}| \leq q/2} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| \right) \\ &\ll_{n,L} D_{n+1} \left(\frac{q}{B}\right)^n q^{(n-r)/2}. \end{aligned}$$

We arrive at the estimate

$$(10) \quad \Sigma_1 \ll_{n,d,L} D_{2n} \left(\frac{q}{B}\right)^n q^{(n-r-1)/2} (B + q^{1/2}).$$

It turns out that Σ_2 does not contribute to the error term. Indeed, using (8) and (9) again we have

$$\Sigma_2 \leq \sum_{|\mathbf{a}| > q/2} \left| \hat{W}\left(\frac{B}{q}\mathbf{a}\right) \right| q^{(n-r+1)/2} \ll_{n,L} D_{n+1} \left(\frac{q}{B}\right)^n q^{(n-r-1)/2},$$

which is dominated by the bound (10) for Σ_1 . Thus, inserting (7) and (10) into the formula (6) yields

$$N_W(X, B, q) = q^{-r} N_W(\mathbb{A}^n, B, q) + O_{n,d,L} \left(D_{2n} q^{(n-r-1)/2} (B + q^{1/2}) \right),$$

as required for the case $s = -1$.

Suppose now that Z_q is singular, so that $s \geq 0$. Following Heath-Brown [7] we will count points on hyperplane sections. We begin with remarking that it is enough to prove the theorem for q greater than some constant

$q_0 = q_0(n, d)$. Indeed, if $q \ll_{n,d} 1$, then $B \ll_{n,d,L} 1$, so that trivially we have $N_W(X, B, q) - q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,d,L} 1$. Thus, using Lemma 2, we can assume that it is possible to find a primitive integer vector \mathbf{b} , with $\mathbf{b} \ll_{n,d} 1$, such that $\dim(Z_q \cap H_{\mathbf{b}}) = n - r - 2$ and $\dim \text{Sing}((Z_q \cap H_{\mathbf{b}})_q) = s - 1$, where $H_{\mathbf{b}}$ is the hyperplane in \mathbb{P}^{n-1} defined by $\mathbf{b} \cdot \mathbf{x} = 0$. We can find a unimodular integer matrix M , all of whose entries are $O_{n,d}(1)$ such that the automorphism of $\mathbb{P}_{\mathbb{Z}}^{n-1}$ induced by M maps $H_{\mathbf{b}}$ onto the hyperplane $X_n = 0$, which we identify with $\mathbb{P}^{n-2} = \text{Proj } \mathbb{Z}[X_1, \dots, X_{n-1}]$. Let \tilde{Z}_q be the image of $Z_q \cap H_{\mathbf{b}}$. Then

$$\tilde{Z}_q = \text{Proj } \mathbb{Z}[X_1, \dots, X_{n-1}]/(q, G_1, \dots, G_r)$$

where $G_i(X_1, \dots, X_{n-1}) = F_i(M^{-1}(X_1, \dots, X_{n-1}, 0))$ for $i = 1, \dots, r$, and each G_i is of the same degree as F_i . Obviously we have $\dim \text{Sing } \tilde{Z}_q = s - 1$. Moreover,

$$N_W(X, B, q) = \sum_{\mathbf{x}_q \in \tilde{X}_q} W\left(\frac{1}{B}\mathbf{x}\right) = \sum_{\mathbf{x}_q \in \tilde{X}_q} \tilde{W}\left(\frac{1}{B}\mathbf{x}\right),$$

where \tilde{X} is the image of X under the automorphism of \mathbb{A}^n induced by M and where $\tilde{W}(\mathbf{t}) = W(M^{-1}\mathbf{t})$. Then \tilde{W} is supported in a cube of side $L' \ll_{n,d} L$, so we can write

$$(11) \quad N_W(X, B, q) = \sum_{-BL' \leq c \leq BL'} \sum_{\substack{\mathbf{x}_q \in \tilde{X}_q \\ x_n = c}} \tilde{W}\left(\frac{1}{B}\mathbf{x}\right).$$

For each $c \in \mathbb{Z}$, the intersection of \tilde{X} with the hyperplane $x_n = c$ is isomorphic to

$$\tilde{X}_c = \text{Spec } \mathbb{Z}[X_1, \dots, X_{n-1}]/(g_1^c, \dots, g_r^c)$$

where $g_i^c(X_1, \dots, X_{n-1}) = f_i(X_1, \dots, X_{n-1}, c)$ for $i = 1, \dots, r$. For each c and i , the leading form of g_i^c is G_i , so our induction assumption applies to \tilde{X}_c, \tilde{Z}_q and the new weight function \tilde{W}_c on \mathbb{R}^{n-1} defined by $\tilde{W}_c(\mathbf{t}) = \tilde{W}(\mathbf{t}, c)$. We get

$$\begin{aligned} \sum_{\substack{\mathbf{x}_q \in \tilde{X}_q \\ x_n = c}} \tilde{W}\left(\frac{1}{B}\mathbf{x}\right) &= N_{\tilde{W}_c}(\tilde{X}_c, B, q) \\ &= q^{-r} N_{\tilde{W}_c}(\mathbb{A}^{n-1}, B, q) + O_{n,d,L}\left(D_{2n} B^s q^{(n-r-s-2)/2} (B + q^{1/2})\right). \end{aligned}$$

We shall now add the contributions from all c in the interval $[-BL', BL']$. Observe that

$$\begin{aligned} \sum_{-BL' \leq c \leq BL'} N_{\tilde{W}_c}(\mathbb{A}^{n-1}, B, q) &= \sum_{-BL' \leq c \leq BL'} \sum_{\mathbf{y} \in \mathbb{Z}^{n-1}} \tilde{W}\left(\frac{1}{B}(\mathbf{y}, c)\right) \\ &= \sum_{\mathbf{x} \in \mathbb{Z}^n} W\left(\frac{1}{B}M^{-1}\mathbf{x}\right) = \sum_{\mathbf{x}' \in \mathbb{Z}^n} W\left(\frac{1}{B}\mathbf{x}'\right) \\ &= N_W(\mathbb{A}^n, B, q), \end{aligned}$$

since M is unimodular. Thus, summing according to (11) we deduce that

$$N_W(X, B, q) = q^{-r} N_W(\mathbb{A}^n, B, q) + O_{n,d,L} \left(D_{2n} B^{s+1} q^{(n-r-s-2)/2} (B + q^{1/2}) \right)$$

and the induction step is finished. \square

4. PROOF OF THE MAIN RESULT

The aim of this section is to prove Theorem 2. Throughout the proof, any implicit constant is allowed to depend only on n and d , and we will omit the subscripts n, d from the O - and \ll -notation.

Note. It will suffice to prove the theorem under the somewhat weaker hypothesis that $p < 2B + 1 < q$, but with the additional assumption that $2B + 1$ is a multiple of p . We will now prove that the general case follows from this case. If p and q are given primes and B is an arbitrary real number such that $2p < 2B + 1 < q - p$, then there are integers B_1 and B_2 , with $B_1 \leq B \leq B_2$, such that $2B_1 + 1$ and $2B_2 + 1$ are multiples of p and $p < 2B_i + 1 < q$ for $i = 1, 2$. We have

$$\begin{aligned} N(X, B, pq) - \frac{(2B + 1)^n}{p^r q^r} &\leq N(X, B_2, pq) - \frac{(2B + 1)^n}{p^r q^r} \\ &= N(X, B_2, pq) - \frac{(2B_2 + 1)^n}{p^r q^r} + O(B^{n-1} p^{-r+1} q^{-r}), \end{aligned}$$

and similarly

$$N(X, B, pq) - \frac{(2B + 1)^n}{p^r q^r} \geq N(X, B_1, pq) - \frac{(2B_1 + 1)^n}{p^r q^r} + O(B^{n-1} p^{-r+1} q^{-r}).$$

Thus, if we assume Theorem 2 to be true for B_1 and B_2 , then we see that it must also hold for B , since $B_1, B_2 \asymp B$.

From now on we assume that $2B + 1$ is a multiple of p between p and q . To facilitate the notation we introduce the characteristic function of the box $\mathbf{B} = [-B, B]^n \cap \mathbb{Z}^n$,

$$\chi_{\mathbf{B}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \max |x_i| \leq B, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$N := N(X, B, pq) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ pq | f_i(\mathbf{x})}} \chi_{\mathbf{B}}(\mathbf{x}) = \sum_{\substack{\mathbf{w} \in \mathbb{F}_p^n \\ p | f_i(\mathbf{w})}} \sum_{\substack{\mathbf{x} \equiv \mathbf{w}(p) \\ q | f_i(\mathbf{x})}} \chi_{\mathbf{B}}(\mathbf{x}).$$

The ‘‘expected value’’ of the inner sum is

$$K := p^{-n} q^{-r} (2B + 1)^n,$$

so let us write

$$N = \sum_{\substack{\mathbf{w} \in \mathbb{F}_p^n \\ p | f_i(\mathbf{w})}} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{w}(p) \\ q | f_i(\mathbf{x})}} \chi_{\mathbf{B}}(\mathbf{x}) - K \right) + K \sum_{\substack{\mathbf{w} \in \mathbb{F}_p^n \\ p | f_i(\mathbf{w})}} 1.$$

If we denote the first of these two sums by S , then, using Lemma 6, we get

$$(12) \quad \begin{aligned} N &= S + K \# X(\mathbb{F}_p) = S + K \left(p^{n-r} + O(p^{(n-r+1)/2}) \right) \\ &= \frac{(2B+1)^n}{p^r q^r} + S + O(B^n p^{-(n+r-1)/2} q^{-r}). \end{aligned}$$

Now we turn our attention to S . By Cauchy's inequality

$$S^2 \leq \left(\sum_{\substack{\mathbf{w} \in \mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} 1 \right) \left(\sum_{\substack{\mathbf{w} \in \mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{w}(p) \\ q|f_i(\mathbf{x})}} \chi_{\mathbf{B}}(\mathbf{x}) - K \right)^2 \right),$$

so that, if we denote the expression in the rightmost parentheses by Σ , and apply Lemma 5, we get

$$(13) \quad S \ll p^{(n-r)/2} \Sigma^{1/2}.$$

We estimate Σ by adding some extra (positive) terms:

$$\begin{aligned} \Sigma &\leq \sum_{\mathbf{w} \in \mathbb{F}_p^n} \sum_{\mathbf{a} \in \mathbb{F}_q^r} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{w}(p) \\ f_i(\mathbf{x}) \equiv a_i(q)}} \chi_{\mathbf{B}}(\mathbf{x}) - K \right)^2 \\ &= \sum_{\mathbf{w} \in \mathbb{F}_p^n} \sum_{\mathbf{a} \in \mathbb{F}_q^r} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{w}(p) \\ f_i(\mathbf{x}) \equiv a_i(q)}} \chi_{\mathbf{B}}(\mathbf{x}) \right)^2 - 2K \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathbf{B}}(\mathbf{x}) + p^n q^r K^2. \end{aligned}$$

The middle term here is just $-2p^n q^r K^2$, so, denoting the first sum by \mathcal{Z} we get

$$(14) \quad \Sigma \leq \mathcal{Z} - p^n q^r K^2.$$

To analyze \mathcal{Z} , we write

$$\mathcal{Z} = \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathbf{B}}(\mathbf{x}) \sum_{\substack{\mathbf{x}' \in \mathbb{Z}^n \\ \mathbf{x}' \equiv \mathbf{x}(p) \\ f_i(\mathbf{x}') \equiv f_i(\mathbf{x})(q)}} \chi_{\mathbf{B}}(\mathbf{x}').$$

We make the variable change $\mathbf{x}' = \mathbf{x} + p\mathbf{y}$ in the second sum, introducing the ‘‘differentiated’’ polynomials

$$f_i^{\mathbf{y}}(\mathbf{x}) = f_i(\mathbf{x} + p\mathbf{y}) - f_i(\mathbf{x}).$$

If $\mathbf{B}_{\mathbf{y}}$ denotes the new box $\mathbf{B} \cap (\mathbf{B} - p\mathbf{y}) = \{\mathbf{x} \in \mathbb{Z}^n; \mathbf{x} \in \mathbf{B}, \mathbf{x} + p\mathbf{y} \in \mathbf{B}\}$, we get

$$\begin{aligned} \mathcal{Z} &= \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathbf{B}}(\mathbf{x}) \sum_{\substack{\mathbf{y} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0(q)}} \chi_{\mathbf{B}}(\mathbf{x} + p\mathbf{y}) \\ &= \sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0(q)}} \chi_{\mathbf{B}_{\mathbf{y}}}(\mathbf{x}). \end{aligned}$$

Let us define

$$\Delta(\mathbf{y}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0(q)}} \chi_{B_{\mathbf{y}}}(\mathbf{x}) - q^{-r} \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{B_{\mathbf{y}}}(\mathbf{x}),$$

and write

$$\mathcal{Z} = \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}) + q^{-r} \sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{B_{\mathbf{y}}}(\mathbf{x}).$$

Now one sees that, since we are assuming $p \mid (2B + 1)$,

$$\begin{aligned} \sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{B_{\mathbf{y}}}(\mathbf{x}) &= \prod_{i=1}^n \left(\sum_{y_i \in \mathbb{Z}} \sum_{x_i \in \mathbb{Z}} \chi_{[-B, B]}(x_i) \chi_{[-B - py_i, B - py_i]}(x_i) \right) \\ &= \left(\frac{(2B + 1)^2}{p} \right)^n = p^n q^{2r} K^2. \end{aligned}$$

In other words, $\mathcal{Z} = \sum \Delta(\mathbf{y}) + p^n q^r K^2$, so we get by (14)

$$(15) \quad \Sigma \leq \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}).$$

Our task is now to estimate $\sum \Delta(\mathbf{y})$. To this end, denote the leading forms of $f_1^{\mathbf{y}}, \dots, f_r^{\mathbf{y}}$ by $F_1^{\mathbf{y}}, \dots, F_r^{\mathbf{y}}$ and let

$$\begin{aligned} X_{\mathbf{y}} &= \text{Spec } \mathbb{F}_q[x_1, \dots, x_n] / (f_1^{\mathbf{y}}, \dots, f_r^{\mathbf{y}}), \\ Z_{\mathbf{y}} &= \text{Proj } \mathbb{F}_q[x_1, \dots, x_n] / (F_1^{\mathbf{y}}, \dots, F_r^{\mathbf{y}}). \end{aligned}$$

Observe that for each $i = 1, \dots, r$ we have

$$F_i^{\mathbf{y}} = p\mathbf{y} \cdot \nabla F_i,$$

unless the right hand side vanishes identically (mod q) in \mathbf{x} . Due to the non-singularity of Z , this happens only if $\mathbf{y} \equiv 0 \pmod{q}$. Indeed, if $\mathbf{y} \cdot \nabla F_i$ is identically zero for some i , then, in the notation of Lemma 3, $S_{\mathbf{y}} = \mathbb{P}_{\mathbb{F}_q}^{n-1}$. Thus \mathbf{y} is a point on the affine cone over $T_{n-1} = \emptyset$.

Lemma 7.

$$\begin{aligned} \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}) &\ll B^{n+1} p^{-n} q^{(n-r-1)/2} (\log q)^n + B^{n+1} p^{-r} q^{-1/2} (\log q)^n \\ &\quad + B^n p^{-n} q^{(n-r)/2} (\log q)^n + B^n (\log q)^n. \end{aligned}$$

Proof. First, we note that $\Delta(\mathbf{y}) = 0$ for all \mathbf{y} with $|\mathbf{y}| \geq (2B + 1)/p$. Thus, we only need to sum over the set

$$\mathcal{B} = \{\mathbf{y} \in \mathbb{Z}^n; |\mathbf{y}| < (2B + 1)/p\}.$$

Let us decompose this set into subsets: $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \dots \cup \mathcal{B}_r$, where

$$\mathcal{B}_{\sigma} = \{\mathbf{y} \in \mathcal{B}; \text{codim } Z_{\mathbf{y}} = \sigma\}, \quad \sigma = 0, \dots, r.$$

For $\mathbf{y} \in \mathcal{B}_r$, we can use Theorem 1 of the Appendix [13] to get

$$\Delta(\mathbf{y}) \ll_{n,d} B^{s(\mathbf{y})+1} q^{(n-r-s(\mathbf{y})-2)/2} (B + q^{1/2}) (\log q)^n,$$

where $s(\mathbf{y}) = \dim \text{Sing}(Z_{\mathbf{y}})$. Next we need to find out how often each value of $s(\mathbf{y})$ arises. We consult Lemma 3. Since $Z_{\mathbf{y}}$ is a complete intersection of codimension r , the Jacobian Criterion implies that $\text{Sing}(Z_{\mathbf{y}}) = S_{\mathbf{y}}$. Thus,

the set of all \mathbf{y} such that $s(\mathbf{y}) = s$ is contained in the affine cone over the set T_s . By part (ii) of Lemma 3, T_s has projective dimension $n - s - 2$, so by part (iii) and Lemma 5, we get

$$\#\{\mathbf{y} \in \mathcal{B}_r; s(\mathbf{y}) = s\} \ll_{n,d} \left(\frac{B}{p}\right)^{n-s-1}.$$

Summing, we get

$$\begin{aligned} \sum_{\mathbf{y} \in \mathcal{B}_r} \Delta(\mathbf{y}) &\ll \sum_{s=-1}^{n-r-1} \left(\frac{B}{p}\right)^{n-s-1} B^{s+1} q^{(n-r-s-2)/2} (B + q^{1/2}) (\log q)^n \\ &\ll B^n (\log q)^n \left(B p^{-n} q^{(n-r-1)/2} + p^{-n} q^{(n-r)/2} + B p^{-r} q^{-1/2} + p^{-r} \right). \end{aligned}$$

It remains to consider the contribution from $\mathbf{y} \in \mathcal{B}_\sigma$, $\sigma < r$. We make a simple observation about the varieties $Z_{\mathbf{y}}$ originating from these values of \mathbf{y} : now the set $S_{\mathbf{y}}$ is very large.

Lemma 8. *Let G_1, \dots, G_r be forms in the variables X_1, \dots, X_n . Let*

$$V = \{G_1 = \dots = G_r = 0\} \subseteq \mathbb{P}^{n-1}$$

and let

$$W = \left\{ G_1 = \dots = G_r = 0, \text{rank} \left(\frac{\partial G_i}{\partial X_j} \right) < r \right\}.$$

Suppose that $\text{codim}(V) = \sigma < r$. Then W contains all irreducible components of V of dimension $n - 1 - \sigma$. In particular, $\dim W = n - 1 - \sigma$.

Proof. Let V' be an irreducible component of V with $\dim V' = n - 1 - \sigma$. Assume that there were a point $P \in V'$ such that $\text{rank} \left(\frac{\partial G_i}{\partial X_j} \right) (P) = r$. Then we would have

$$\dim T_P V' = n - 1 - r < n - 1 - \sigma = \dim V',$$

a contradiction. Thus $V' \subseteq W$. \square

We see that if $\mathbf{y} \in \mathcal{B}_\sigma$, then, by Lemma 8, $\dim S_{\mathbf{y}} = n - 1 - \sigma$. Recalling that, in the notation of Lemma 3, $T_{n-1-\sigma}$ has dimension less than or equal to $\sigma - 1$, we must have

$$|\mathcal{B}_\sigma| \ll \left(\frac{B}{p}\right)^\sigma.$$

Using Lemma 5 to get the trivial estimate $\Delta(\mathbf{y}) \ll B^{n-\sigma}$ for $\mathbf{y} \in \mathcal{B}_\sigma$, we compute the contribution from the \mathcal{B}_σ , $\sigma < r$:

$$\sum_{\sigma=0}^{r-1} \sum_{\mathbf{y} \in \mathcal{B}_\sigma} \Delta(\mathbf{y}) = \sum_{\sigma=0}^{r-1} \left(\frac{B}{p}\right)^\sigma B^{n-\sigma} = B^n \sum_{\sigma=0}^{r-1} p^{-\sigma} \ll B^n.$$

In sum, then,

$$\begin{aligned} \sum_{\mathbf{y} \in \mathcal{B}} \Delta(\mathbf{y}) &= \sum_{\sigma=0}^r \sum_{\mathbf{y} \in \mathcal{B}_\sigma} \Delta(\mathbf{y}) \\ &\ll B^{n+1} p^{-n} q^{(n-r-1)/2} (\log q)^n + B^{n+1} p^{-r} q^{-1/2} (\log q)^n \\ &\quad + B^n p^{-n} q^{(n-r)/2} (\log q)^n + B^n (\log q)^n, \end{aligned}$$

and Lemma 7 follows. \square

Working our way back through the estimates (15), (13) and (12), we now arrive at

$$(16) \quad N = \frac{(2B+1)^n}{p^r q^r} + O\left(B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} (\log q)^{n/2} \right. \\ \left. + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} (\log q)^{n/2} + B^{n/2} p^{-r/2} q^{(n-r)/4} (\log q)^{n/2} \right. \\ \left. + B^{n/2} p^{(n-r)/2} (\log q)^{n/2} + B^n p^{-(n+r-1)/2} q^{-r} \right).$$

This completes the proof of Theorem 2.

We shall now prove Corollary 1, where the modest dependence upon $\|F_i\|$ is due to the following lemma.

Lemma 9. *Let X and Z be defined as in Theorem 2, and assume that $Z_{\mathbb{Q}}$ is non-singular of dimension $n-1-r$. If $P \geq (\sum_{i=1}^r \log \|F_i\|)^{1+\delta}$, then there is a prime $p \asymp_{\delta} P$ such that Z_p is non-singular of dimension $n-1-r$.*

Proof. As in the proof of Lemma 2, let $\mathbf{P} = \mathbb{P}_1 \times \dots \times \mathbb{P}_r$, where \mathbb{P}_i is the projective space parametrizing all hypersurfaces of degree d_i in $\mathbb{P}_{\mathbb{Z}}^{n-1}$. By a semicontinuity argument analogous to that in the proof of Lemma 2, the subset $U \subseteq \mathbf{P}$ defined by

$$(G_1, \dots, G_r) \in U \Leftrightarrow V(G_1, \dots, G_r) \text{ is non-singular of codimension } r,$$

is Zariski open, its complement thus being defined by multihomogeneous polynomials H_1, \dots, H_t in the coefficients of G_1, \dots, G_r . Now by the hypotheses, for some j we must have $H_j(F_1, \dots, F_r) \neq 0$. We observe firstly that

$$\log |H_j(F_1, \dots, F_r)| \ll_{n,d} \sum_{i=1}^r \log \|F_i\|.$$

Secondly, for an arbitrary positive number A we have

$$\#\{p > AP; p \mid H_j(F_1, \dots, F_r)\} \ll \frac{\log |H_j(F_1, \dots, F_r)|}{\log AP}.$$

Thus, if we choose A large enough, there are fewer than

$$a := \left\lceil \sum_{i=1}^r \log \|F_i\| \right\rceil$$

such primes. Hence among the a first prime numbers greater than AP , there must be one prime p such that $p \nmid H_j(F_1, \dots, F_r)$. By Chebyshev's Theorem it is possible to find an interval $[AP, c_{\delta} AP]$ that contains more than $P^{1/(1+\delta)}$ primes. Since $P \geq a^{1+\delta}$, this interval must contain p . \square

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Theorem 2 yields in particular that

$$\begin{aligned} N(X, B, pq) \ll_{n,d} & \left[\frac{B^n}{p^r q^r} + B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} \right. \\ & + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} + B^{n/2} p^{-r/2} q^{(n-r)/4} + B^{n/2} p^{(n-r)/2} \\ & \left. + B^n p^{-(n+r-1)/2} q^{-r} + B^{n-1} p^{-r+1} q^{-r} \right] (\log q)^{n/2} \end{aligned}$$

Thus we want to optimize the expression

$$\begin{aligned} & \frac{B^n}{p^r q^r} + B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} \\ & + B^{n/2} p^{-r/2} q^{(n-r)/4} + B^{n/2} p^{(n-r)/2} + B^n p^{-(n+r-1)/2} q^{-r} + B^{n-1} p^{-r+1} q^{-r} \end{aligned}$$

by choosing appropriate p and q . It turns out that

$$(17) \quad p \asymp B^{1 - \frac{5nr - r^2 - 5r}{n^2 + 4nr - n - r^2 - r}}, \quad q \asymp B^{2 - \frac{2(4nr - r^2)}{n^2 + 4nr - n - r^2 - r}}.$$

would be an optimal choice. (Note that the last two terms in the expression are dominated by the first term, so the optimization consists of trying to get the first five terms to be of approximately equal order of magnitude.) The restriction $n \geq 4r + 2$ ensures that (17) is compatible with the requirement that $2p < 2B + 1 < q - p$. The trouble is now to make sure that the intervals specified in (17) contain “good” primes, that is, primes such that both Z_p and Z_q are non-singular of dimension $n - 1 - r$.

For B large enough, (17) is a valid choice. Indeed, if

$$\begin{aligned} B & \geq \left(\sum_{i=1}^r \log \|F_i\| \right)^{e_1}, \text{ where} \\ e_1 & = \left(1 - \frac{5nr - r^2 - 5r}{n^2 + 4nr - n - r^2 - r} \right)^{-1} \left(1 + \frac{1}{2r} \right), \end{aligned}$$

then by Lemma 9 (with $\delta = (2r)^{-1}$) we can choose p and q , satisfying (17), such that Theorem 2 holds. For these B , and with p and q subject to (17), Theorem 2 implies that

$$N(X, B) \ll_{n,d} N(X, B, pq) \ll_{n,d} B^{n-3r+r^2 \frac{13n-3r-5}{n^2+4nr-n-r^2-r}} (\log B)^{n/2}.$$

For $B < (\sum_{i=1}^r \log \|F_i\|)^{e_1}$, we use the trivial estimate

$$N(X, B) \ll_{n,d} B^{n-r}$$

obtained by Lemma 5 to get

$$\begin{aligned} N(X, B) & \ll_{n,d} B^{n-3r+r^2 \frac{13n-3r-5}{n^2+4nr-n-r^2-r}} \left(\sum_{i=1}^r \log \|F_i\| \right)^{e_2}, \text{ where} \\ e_2 & = e_1 \left(2r - r^2 \frac{13n - 3r - 5}{n^2 + 4nr - n - r^2 - r} \right) \leq 2r + 1. \end{aligned}$$

This proves the theorem. \square

Remark. If we are content with just an upper bound for $N(X, B, pq)$ in Theorem 2, we can get rid of the factor $(\log q)^{n/2}$ and thus prove a slightly sharpened version of Theorem 1, without the factor $(\log B)^{n/2}$. This can be achieved by introducing an infinitely differentiable weight function into the proof of Theorem 2, as in [7], and using Theorem 3 in the place of [13, Thm. 1]. More precisely, if instead of $N(X, B, pq)$ we consider the weighted counting function

$$N_W(X, B, pq) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \mathbf{x}_p \in X_p \\ \mathbf{x}_q \in X_q}} W\left(\frac{1}{2B}\mathbf{x}\right),$$

where W is a non-negative, infinitely differentiable weight function on \mathbb{R}^n supported in $[-1, 1]^n$, we can prove an asymptotic formula for $N_W(X, B, pq)$ where the main term is

$$p^{-r}q^{-r} \sum_{\mathbf{x} \in \mathbb{Z}^n} W\left(\frac{1}{2B}\mathbf{x}\right).$$

The error term would then consist of the first four error terms of Theorem 2 with the factor $(\log q)^{n/2}$ removed, the fifth error term unchanged, and an additional term which is $o(p^{-r}q^{-r}B^n)$ and thus negligible for the application of Theorem 1. To prove this asymptotic formula one imitates the proof of Theorem 2, with $\chi_B(\mathbf{x})$ replaced by $W\left(\frac{1}{2B}\mathbf{x}\right)$ and K by

$$K_W = p^{-n}q^{-r} \sum_{\mathbf{x} \in \mathbb{Z}^n} W\left(\frac{1}{2B}\mathbf{x}\right).$$

One is then led to estimate expressions

$$\Delta_W(\mathbf{y}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0(q)}} W_{\mathbf{y}}(\mathbf{x}) - q^{-r} \sum_{\mathbf{x} \in \mathbb{Z}^n} W_{\mathbf{y}}(\mathbf{x}),$$

where $W_{\mathbf{y}}(\mathbf{x}) = W\left(\frac{1}{2B}\mathbf{x}\right)W\left(\frac{1}{2B}(\mathbf{x} + p\mathbf{y})\right)$. At this point we invoke Theorem 3. Here the error term, in contrast to the unweighted formula of Theorem 1 in the Appendix, contains no factor $(\log q)^n$, whence the promised improvement of the upper bound. The only main divergence from the proof of Theorem 2 lies in the calculation of the sum $\sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\mathbf{x} \in \mathbb{Z}^n} W_{\mathbf{y}}(\mathbf{x})$. This can be done by means of Poisson summation (see [7, p. 20]) and gives rise to the additional error term mentioned above.

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APPENDIX

PER SALBERGER

The aim of this note is to count \mathbb{F}_q -points in boxes on affine varieties. If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ and q is a prime, then we set $\mathbf{x}_q = (x_1 + q\mathbb{Z}, \dots, x_n + q\mathbb{Z}) \in \mathbb{F}_q^n$. If \mathbf{B} is a box in \mathbb{R}^n and W a closed subscheme of $\mathbb{A}_{\mathbb{Z}}^n$, then we let

$$N(W, \mathbf{B}, q) = \# \{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbf{B} \cap \mathbb{Z}^n : \mathbf{x}_q \in W(\mathbb{F}_q) \}.$$

Lemma 1. *Let q be a prime and \mathbf{B} be a box in \mathbb{R}^n such that each side has length at most $2B < q$. Let $f_1, \dots, f_r, l_1, \dots, l_{s+1}$ be polynomials in $\mathbb{Z}[x_1, \dots, x_n]$, $r + s + 1 \leq n$ such that the leading forms F_1, \dots, F_r of f_1, \dots, f_r are of degree ≥ 2 and the leading forms L_1, \dots, L_{s+1} of l_1, \dots, l_{s+1} are of degree 1. Let*

$$\begin{aligned} X &= \text{Spec } \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r, l_1, \dots, l_{s+1}), \\ \Lambda &= \text{Spec } \mathbb{Z}[x_1, \dots, x_n]/(l_1, \dots, l_{s+1}) \text{ and} \\ Z &= \text{Proj } \mathbb{Z}[x_1, \dots, x_n]/(F_1, \dots, F_r, L_1, \dots, L_{s+1}). \end{aligned}$$

Suppose that $Z_q = Z_{\mathbb{F}_q}$ is non-singular of codimension $r + s + 1$ in $\mathbb{P}_{\mathbb{F}_q}^{n-1}$. Then

$$N(X, \mathbf{B}, q) = q^{-r} N(\Lambda, \mathbf{B}, q) + O_{n,d}(q^{(n-r-s-2)/2}(B + q^{1/2})(\log q)^n),$$

where $d = \max_i \deg F_i$.

Proof. If $r + s + 1 = n$, then $\#X(\mathbb{F}_q) \leq d^n$ by the theorem of Bezout and hence $N(X, \mathbf{B}, q) - q^{-r} N(\Lambda, \mathbf{B}, q) \ll_{n,d} 1 \leq q^{(n-r-s-2)/2}(B + q^{1/2})$. If $r + s + 1 = n - 1$, then $N(X, \mathbf{B}, q) = O_{n,d}(B)$ by Lemma 5 in [4] so that $N(X, \mathbf{B}, q) - q^{-r} N(\Lambda, \mathbf{B}, q) \ll_{n,d} B \leq q^{(n-r-s-2)/2}(B + q^{1/2})$. We may thus assume that $r + s + 1 \leq n - 2$. Then, Z_q is geometrically connected since it is a complete intersection of dimension ≥ 1 (see [1, Ex. II.8.4(c)]). It is thus geometrically integral since it is non-singular. Therefore, by the homogeneous Nullstellensatz we obtain that a linear form $\mathbf{a} \cdot \mathbf{x} = a_1 x_1 + \dots + a_n x_n$, $(a_1, \dots, a_n) \in \mathbb{F}_q^n$ vanishes on Z_q if and only if $\mathbf{a} \cdot \mathbf{x}$ belongs to the linear \mathbb{F}_q -space V of linear forms in (x_1, \dots, x_n) generated by the reductions of $L_1, \dots, L_{s+1} \pmod{q}$. We now follow the approach of [3]. Let $S_1(\mathbf{a}) = \sum_{\mathbf{b} \in \mathbf{B} \cap \mathbb{Z}^n} e_q(-\mathbf{a} \cdot \mathbf{b})$ and $S_2(\mathbf{a}) = \sum_{\mathbf{x} \in X(\mathbb{F}_q)} e_q(\mathbf{a} \cdot \mathbf{x})$ for $\mathbf{a} \in \mathbb{F}_q^n$. Then,

$$N(X, \mathbf{B}, q) = q^{-n} \sum_{\mathbf{a} \in \mathbb{F}_q^n} S_1(\mathbf{a}) S_2(\mathbf{a}).$$

Let $\Pi_{\mathbf{a}} = \text{Proj } \mathbb{F}_q[x_1, \dots, x_n]/(a_1x_1 + \dots + a_nx_n)$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}_q^n$. Then,

$$\begin{aligned} q^{-(s+1)} \sum_{\mathbf{a} \in V} S_1(\mathbf{a})S_2(\mathbf{a}) &= q^{-(s+1)} \sum_{\mathbf{a} \in V} \sum_{\mathbf{x} \in X(\mathbb{F}_q)} \sum_{\mathbf{b} \in \mathbf{B} \cap \mathbb{Z}^n} e_q(\mathbf{a} \cdot (\mathbf{x} - \mathbf{b})) \\ &= \sum_{\mathbf{x} \in X(\mathbb{F}_q)} \sum_{\mathbf{b} \in \mathbf{B} \cap \mathbb{Z}^n} \prod_{i=1}^{s+1} \left(\frac{1}{q} \sum_{a \in \mathbb{F}_q} e_q(aL_i(\mathbf{x} - \mathbf{b})) \right) \\ &= \# \{(\mathbf{x}, \mathbf{b}) \in X(\mathbb{F}_q) \times (\mathbf{B} \cap \mathbb{Z}^n) : L_1(\mathbf{x} - \mathbf{b}) \equiv \dots \equiv L_{s+1}(\mathbf{x} - \mathbf{b}) \equiv 0 \pmod{q}\} \\ &= \# \{(\mathbf{x}, \mathbf{b}) \in X(\mathbb{F}_q) \times (\mathbf{B} \cap \mathbb{Z}^n) : l_1(\mathbf{b}) \equiv \dots \equiv l_{s+1}(\mathbf{b}) \equiv 0 \pmod{q}\} \\ &= \#X(\mathbb{F}_q)N(\Lambda, \mathbf{B}, q). \end{aligned}$$

Here $\#X(\mathbb{F}_q) = q^{n-r-s-1} + O_{n,d}(q^{(n-r-s)/2})$ by Lemma 6 in [4]. There is also a set of $n-s-1$ indices $i(1), \dots, i(n-s-1) \in \{1, \dots, n\}$ such that any $\mathbf{b} = (b_1, \dots, b_n) \in \mathbf{B} \cap \mathbb{Z}^n$ with $\mathbf{b}_q \in \Lambda(\mathbb{F}_q)$ is uniquely determined by $(b_{i(1)}, \dots, b_{i(n-s-1)})$. Hence, $\#N(\Lambda, \mathbf{B}, q) \ll_n B^{n-s-1}$. We have thus shown that

$$\begin{aligned} q^{-n} \sum_{\mathbf{a} \in V} S_1(\mathbf{a})S_2(\mathbf{a}) &= q^{-(n-s-1)} \#X(\mathbb{F}_q)N(\Lambda, \mathbf{B}, q) \\ &= q^{-r}N(\Lambda, \mathbf{B}, q) + O_{n,d}(q^{-(n-s-1)+(n-r-s)/2}B^{n-s-1}). \end{aligned}$$

As $q^{-(n-s-1)+(n-r-s)/2}B^{n-s-1} < q^{(n-r-s-2)/2}B$, we conclude that

$$q^{-n} \sum_{\mathbf{a} \in V} S_1(\mathbf{a})S_2(\mathbf{a}) = q^{-r}N(\Lambda, \mathbf{B}, q) + O_{n,d}(q^{(n-r-s-2)/2}B).$$

We now estimate $q^{-n} \sum_{\mathbf{a} \in \mathbb{F}_q^n \setminus V} S_1(\mathbf{a})S_2(\mathbf{a})$. Since $\dim Z_q \cap \Pi_{\mathbf{a}} < \dim Z_q$ for $\mathbf{a} \notin V$, we obtain from the theorem of Katz (cf. [3]) that

$$S_2(\mathbf{a}) \ll_{n,d} q^{(n-r-s+\delta)/2}$$

where $\delta = \dim \text{Sing}(Z_q \cap \Pi_{\mathbf{a}}) < \dim Z_q \in \{-1, 0\}$. As

$$\sum_{\mathbf{a} \in \mathbb{F}_q^n} |S_1(\mathbf{a})| \ll_{n,d} q^n (\log q)^n$$

(see [3]), we get that the total contribution to $q^{-n} \sum_{\mathbf{a} \in \mathbb{F}_q^n \setminus V} S_1(\mathbf{a})S_2(\mathbf{a})$ from all $\mathbf{a} \in \mathbb{F}_q^n \setminus V$ where $Z_q \cap \Pi_{\mathbf{a}}$ is non-singular is $O_{n,d}(q^{(n-r-s-1)/2}(\log q)^n)$.

To estimate the contribution from the remaining $\mathbf{a} \in \mathbb{F}_q^n$, we use that there exists a form $\Phi \in \mathbb{Z}[y_1, \dots, y_n]$ of degree $O_{n,d}(1)$ in the dual coordinates (y_1, \dots, y_n) of (x_1, \dots, x_n) such that $\Phi(\mathbf{a}) = 0$ in $\mathbb{Z}/q\mathbb{Z}$ for all n -tuples \mathbf{a} where $Z_q \cap \Pi_{\mathbf{a}}$ is singular (cf. Lemma 2 in [4]). Hence,

$$\sum_{\substack{\mathbf{a} \in \mathbb{F}_q^n \\ \text{Sing}(Z_q \cap \Pi_{\mathbf{a}}) \neq \emptyset}} |S_1(\mathbf{a})| \leq \sum_{\substack{\mathbf{a} \in \mathbb{F}_q^n \\ \Phi(\mathbf{a})=0}} |S_1(\mathbf{a})| \ll_{n,d} q^{n-1} B (\log q)^{n-1},$$

where the last inequality comes from an argument in [3]. The n -tuples \mathbf{a} where $Z_q \cap \Pi_{\mathbf{a}}$ is singular will therefore contribute with

$$O_{n,d}(q^{(n-r-s-2)/2}B(\log q)^{n-1})$$

to $q^{-n} \sum_{\mathbf{a} \in \mathbb{F}_q^n} S_1(\mathbf{a})S_2(\mathbf{a})$. This completes the proof of the lemma. \square

For a linear form $L = a_1x_1 + \dots + a_nx_n \in \mathbb{Z}[x_1, \dots, x_n]$, we will write $\|L\| = \sup(|a_1|, \dots, |a_n|)$.

Theorem 1. *Let q be a prime and \mathbf{B} be a box in \mathbb{R}^n such that each side has length at most $2B < q$. Let f_1, \dots, f_r be polynomials in $\mathbb{Z}[x_1, \dots, x_n]$, $r < n$ with leading forms F_1, \dots, F_r of degree ≥ 2 . Let*

$$X = \text{Spec } \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r) \text{ and}$$

$$Z = \text{Proj } \mathbb{Z}[x_1, \dots, x_n]/(F_1, \dots, F_r)$$

Suppose that $Z_q = Z_{\mathbb{F}_q}$ is a closed subscheme of $\mathbb{P}_{\mathbb{F}_q}^{n-1}$ of codimension r with singular locus of dimension s . Then,

$$N(X, \mathbf{B}, q) = q^{-r} N(\mathbb{A}_{\mathbb{Z}}^n, \mathbf{B}, q) + O_{n,d}(B^{s+1} q^{(n-r-s-2)/2} (B + q^{1/2}) (\log q)^n),$$

where $d = \max_i \deg F_i$.

Proof. It is enough to prove the statement for q greater than some constant q_0 depending only on n and d , since for $q \ll_{n,d} 1$ we have $B \ll_{n,d} 1$ and thus, trivially, $N(X, \mathbf{B}, q) - q^{-r} N(\mathbb{A}_{\mathbb{Z}}^n, \mathbf{B}, q) \ll_{n,d} 1$. Thus, assuming that q is large enough, we choose $s+1$ linear forms $L_1, \dots, L_{s+1} \in \mathbb{Z}[x_1, \dots, x_n]$ such that $\|L_i\| = O_{d,n}(1)$ and such that

$$Z_q^i = \text{Proj } \mathbb{Z}[x_1, \dots, x_n]/(q, F_1, \dots, F_r, L_1, \dots, L_i)$$

is a closed subscheme of codimension $r+i$ in $\mathbb{P}_{\mathbb{F}_q}^{n-1}$ with singular locus of dimension $s-i$ for $i = 1, \dots, s+1$. Such forms were used already in [2] and one gets a proof of their existence from Lemma 2 in [4].

Let $I = L(\mathbf{B} \cap \mathbb{Z}^n)$ for the map $L : \mathbb{Z}^n \rightarrow \mathbb{Z}^{s+1}$ which sends $\mathbf{b} = (b_1, \dots, b_n)$ to $(L_1(\mathbf{b}), \dots, L_{s+1}(\mathbf{b}))$. Then $\#I = O_{n,d}(B^{s+1})$. Moreover, if $\mathbf{c} = (c_1, \dots, c_{s+1}) \in \mathbb{Z}^{s+1}$, then we may apply Lemma 1 to the affine subscheme $X_{\mathbf{c}}$ of $\mathbb{A}_{\mathbb{Z}}^n$ defined by $(f_1, \dots, f_r, L_1 - c_1, \dots, L_{s+1} - c_{s+1})$ and conclude that

$$N(X_{\mathbf{c}}, \mathbf{B}, q) = q^{-r} N(\Lambda_{\mathbf{c}}, \mathbf{B}, q) + O_{n,d}(q^{(n-r-s-2)/2} (B + q^{1/2}) (\log q)^n)$$

for $\Lambda_{\mathbf{c}} = \text{Spec } \mathbb{Z}[x_1, \dots, x_n]/(L_1 - c_1, \dots, L_{s+1} - c_{s+1})$. If we sum over all $\mathbf{c} = (c_1, \dots, c_{s+1}) \in I$, then we get the desired asymptotic formula for $N(X, \mathbf{B}, q)$. This finishes the proof. \square

Remark. Note that $q^{-r} N(\mathbb{A}_{\mathbb{Z}}^n, \mathbf{B}, q) = q^{-r} \#(\mathbf{B} \cap \mathbb{Z}^n)$, since different elements in $\mathbf{B} \cap \mathbb{Z}^n$ are non-congruent (mod q) by the assumption on \mathbf{B} .

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