A NOTE FOR EXTENSION OF ALMOST SURE CENTRAL LIMIT THEORY

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Abstract. Hörmann (2006) gave an extension of almost sure central limit theorem for bounded Lipschitz 1 function. In this paper, we show that his result of almost sure central limit theorem is also hold for any Lipschitz function under stronger conditions.

1. Introduction

The classical results on the almost sure central limit theorem (ASCLT) dealt with partial sums of random variables. A general pattern is that, if $X_1, X_2, \dots$ be a sequence of independent random variables with partial sums $S_n = X_1 + \cdots + X_n$ satisfying $(S_n - b_n)/a_n \xrightarrow{\mathcal{L}} H$ for some sequences $a_n > 0$, $b_n \in \mathbb{R}$ and some distribution function H, then under some mild conditions we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\Big\{(S_k - b_k)/a_k \le x\Big\} = H(x) \quad a.s.$$

for any continuity point x of H.

Several papers have dealt with logarithmic limit theorems of this kind and the above relation has been extended in various directions. Fahrner and Stadtmüller [5] gave an almost sure version of a maximum limit theorem. Berkes and Horváth [2] obtained a strong approximation for the logarithmic average of sample extremes. Berkes and Csáki [1] showed that not only the central limit theorem, but every weak limit theorem for independent random variables has an analogous almost sure version. For stationary Gaussian sequences with covariance r_n, Csáki and Gonchigdanzan [3] proved an almost sure limit theorem for the maxima of the sequences under the condition $r_n \log n (\log\log n)^{1+\varepsilon} = O(1)$. For some dependent random variables, Peligrad and Shao [7] and Dudziński [4] obtained corresponding results about the almost sure central limit theorem.

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Recently, Hörmann [6] gave an extension of almost sure central limit theory under some regularity condition as the following form:

$$\lim_{N\to\infty} D_N^{-1}\sum_{k=1}^{N} d_k f\Big(\frac{S_k}{a_k}-b_k\Big)=\int_{-\infty}^{\infty} f(x)dH(x)\quad a.s. \tag{1.1}$$

where f is a bounded Lipschitz 1 function and $D_N=\sum_{k=1}^N d_k$, $\{d_k\}_{k\ge1}$ is a sequence of positive constants. Using his method, we will show that for any Lipschitz function f, (1.1) holds under some additional conditions.

At first, we give our main result.

Theorem 1.1. *Let $X_1, X_2, \dots$ be independent random variables with partial sums S_n and assume that*

- **:** (C_1) *For some numerical sequences $a_n>0$ and b_n, we have $\dfrac{S_n}{a_n}-b_n \xrightarrow{\mathcal{L}} H$, where H is some (possibly degenerate) distribution function.*
- **:** (C_2) *$kd_k=O(1)$ and for some $0<\alpha<1$, $d_k k^\alpha$ is eventually non-increasing.*
- **:** (C_3) *For some $\rho>0$, $d_k=O\Big(\dfrac{D_k}{k(\log D_k)^\rho}\Big)$.*
- **:** (C_4) *There exist positive constants C, β, such that $a_k/a_l\le C(k/l)^\beta$ $(1\le k\le l)$. Furthermore, for some $0<r<\rho$,*

$$\mathbb{E}\Big|\frac{S_n}{a_n}-b_n\Big|^\mu=O(1),\quad for\ some\ integer\ \mu\ge(2\vee 4/(\rho-r)). \tag{1.2}$$

Then for any Lipschitz function f on the real line, we have (1.1).

Remarks 1.2. Obviously, for any bounded Lipschitz 1 function f, under the above assumptions, the equation (1.1) holds, i.e. we can obtain Theorem 1 in [6].

2. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1, according to the process of Hörmann in [6].

Lemma 2.1. *(See Lemma 1 in [6]) Let (D_N) be a summation procedure, then the condition (C_3) of Theorem 1.1 implies that $D_N=o(N^\varepsilon)$ for any $\varepsilon>0$.*

Lemma 2.2. *Assume that condition (C_4) of Theorem 1.1 is satisfied and $b_n=0$. Then for every Lipschitz function $f:\mathbb{R}\to\mathbb{R}$ there exists constant $c>0$ such that*

$$\Big|Cov\Big(f\Big(\frac{S_k}{a_k}\Big), f\Big(\frac{S_l}{a_l}\Big)\Big)\Big|\le c(k/l)^\beta\quad(1\le k\le l), \tag{2.1}$$

where β is the same as in (C_4).

Proof. Firstly, we assume $f(0) = 0$. Denoting $\|f\|$ the Lipschitz constant of f, we get, by using the independence of S_k and $S_l - S_k$,

$$\begin{aligned}
&\Big|Cov\Big(f\Big(\frac{S_k}{a_k}\Big), f\Big(\frac{S_l}{a_l}\Big)\Big)\Big| = \Big|Cov\Big(f\Big(\frac{S_k}{a_k}\Big), f\Big(\frac{S_l}{a_l}\Big) - f\Big(\frac{S_l - S_k}{a_l}\Big)\Big)\Big| \\
\leq &\mathbb{E}\Big|f\Big(\frac{S_k}{a_k}\Big)\Big[f\Big(\frac{S_l}{a_l}\Big) - f\Big(\frac{S_l - S_k}{a_l}\Big)\Big]\Big| + \mathbb{E}\Big|f\Big(\frac{S_k}{a_k}\Big)\Big|\mathbb{E}\Big|f\Big(\frac{S_l}{a_l}\Big) - f\Big(\frac{S_l - S_k}{a_l}\Big)\Big| \\
\leq &\|f\|^2\frac{a_k}{a_l}\mathbb{E}\Big[\frac{S_k^2}{a_k^2}\Big] + \|f\|^2\frac{a_k}{a_l}(\mathbb{E}\Big[\frac{S_k}{a_k}\Big])^2 \\
\leq &2\|f\|^2\frac{a_k}{a_l}\mathbb{E}\Big[\frac{S_k^2}{a_k^2}\Big] \leq 2C\|f\|^2\mathbb{E}\Big[\frac{S_k^2}{a_k^2}\Big](k/l)^{\beta},
\end{aligned}$$

where the last inequality is due to condition (C_4). Since the equation (1.2), we can take $c > 0$ such that for any $k \geq 1$, $2C\|f\|^2\mathbb{E}\Big[\frac{S_k^2}{a_k^2}\Big] \leq c$. So (2.1) is obtained.

If $f(0) \neq 0$, we can define a function g, such that $g(x) = f(x) - f(0)$, then g is a Lipschitz function and $g(0) = 0$. And noting that,

$$Cov\Big(f\Big(\frac{S_k}{a_k}\Big), f\Big(\frac{S_l}{a_l}\Big)\Big) = Cov\Big(g\Big(\frac{S_k}{a_k}\Big), g\Big(\frac{S_l}{a_l}\Big)\Big)$$

we complete the proof of the lemma. □

Remarks 2.3. It is obvious to see that if we replace β by any $0 < \beta' < \beta$, the Lemma 2.2 also holds. Hence, without loss of generality, we can assume that β is the same as α in condition (C_2) of Theorem 1.1.

Next we will use the following notations,

$$\xi_l := f\Big(\frac{S_l}{a_l}\Big) - \mathbb{E}f\Big(\frac{S_l}{a_l}\Big), \quad \xi_{k,l} := f\Big(\frac{S_l - S_k}{a_l}\Big) - \mathbb{E}f\Big(\frac{S_l - S_k}{a_l}\Big). \tag{2.2}$$

Lemma 2.4. *Assume that condition (C_4) of Theorem 1.1 is satisfied and $b_n = 0$, and define ξ_l and $\xi_{k,l}$ as in (2.2). If $\{d_k, k \geq 1\}$ are arbitrary positive weights, then we have for any $k \leq m \leq n$ and $p \in \mathbb{N}$, $p \leq \mu$,*

$$\mathbb{E}\Big|\sum_{l=m}^{n} d_l(\xi_l - \xi_{k,l})\Big|^p \leq E_p\Big(\sum_{l=m}^{n} d_l^2 l\Big)^{p/2},$$

where $E_p = c_p C^p\|f\|^p\Big[\Big(\frac{2^\kappa}{\kappa}\Big) \vee \Big(1 + \frac{1}{\kappa}\Big)\Big]^{p/2}$ *and* $\kappa = 2\beta$.

Proof. Let $Q(l) = Q(k,l) = \xi_l - \xi_{k,l}$, then

$$\begin{aligned}\mathbb{E}|Q(l)|^p =&\mathbb{E}\Big|f\Big(\frac{S_l}{a_l}\Big) - f\Big(\frac{S_l - S_k}{a_l}\Big) - \mathbb{E}\Big[f\Big(\frac{S_l}{a_l}\Big) - f\Big(\frac{S_l - S_k}{a_l}\Big)\Big]\Big|^p\\ \le&\|f\|^p(a_k/a_l)^p\mathbb{E}\Big(\frac{|S_k|}{a_k} + \mathbb{E}\Big(\frac{|S_k|}{a_k}\Big)\Big)^p\\ \le&C^p\|f\|^p\mathbb{E}\Big(\frac{|S_k|}{a_k} + \mathbb{E}\Big(\frac{|S_k|}{a_k}\Big)\Big)^p(k/l)^{p\beta}\\ \le&c_pC^p\|f\|^p(k/l)^{p\beta},\end{aligned}$$

where C is the same as in condition (C_4) and c_p is a positive constant such that for all k, $\mathbb{E}\Big(\frac{|S_k|}{a_k} + \mathbb{E}\Big(\frac{|S_k|}{a_k}\Big)\Big)^p \le c_p$. Thus by the Hölder inequality, we get

$$\begin{aligned}\mathbb{E}\Big|\sum_{l=m}^n d_l(\xi_l - \xi_{k,l})\Big|^p \le& \sum_{l_1=m}^n \cdots \sum_{l_p=m}^n d_{l_1}\cdots d_{l_p}(\mathbb{E}|Q(l_1)|^p \cdots \mathbb{E}|Q(l_p)|^p)^{1/p}\\ \le&c_pC^p\|f\|^pk^{p\beta}\sum_{l_1=m}^n \cdots \sum_{l_p=m}^n d_{l_1}\cdots d_{l_p}l_1^{-\beta}\cdots l_p^{-\beta}\\ =&c_pC^p\|f\|^pk^{p\beta}\Big(\sum_{l=m}^n d_l l^{-\beta}\Big)^p\\ \le&c_pC^p\|f\|^pm^{p\beta}\Big(\sum_{l=m}^n d_l^2 l\Big)^{p/2}\Big(\sum_{l=m}^n l^{-2\beta-1}\Big)^{p/2}.\end{aligned}$$

For $m \ge 2$, it is easy to see that

$$\begin{aligned}m^{p\beta}\Big(\sum_{l=m}^n l^{-2\beta-1}\Big)^{p/2} \le& m^{p\beta}\Big(\int_{m-1}^\infty l^{-2\beta-1}dl\Big)^{p/2}\\ \le&\Big(\frac{m}{m-1}\Big)^{p\beta}\Big(\frac{1}{2\beta}\Big)^{p/2} \le \Big(\frac{2^\kappa}{\kappa}\Big)^{p/2},\end{aligned}$$

where $\kappa := 2\beta$. Similarly, we get for $m = 1$

$$\Big(\sum_{l=1}^n l^{-2\beta-1}\Big)^{p/2} \le \Big(1 + \frac{1}{\kappa}\Big)^{p/2}.$$

Hence, we have

$$\mathbb{E}\Big|\sum_{l=m}^n d_l(\xi_l - \xi_{k,l})\Big|^p \le c_pC^p\|f\|^p\tau(\kappa)^{p/2}\Big(\sum_{l=m}^n d_l^2 l\Big)^{p/2},$$

where $\tau(\kappa) := \Big[\Big(\frac{2^\kappa}{\kappa}\Big) \vee \Big(1 + \frac{1}{\kappa}\Big)\Big]$. This completes the proof of our result. □

Lemma 2.5. *Assume that conditions $(C_2)-(C_4)$ of Theorem 1.1 are satisfied. Further let $b_n=0$ in condition (C_4) and f be a Lipschitz function. Then for every $p\le\mu$ and $p\in\mathbb{N}$ we have*

$$\mathbb{E}\Big|\sum_{k=1}^{N} d_k\Big(f\Big(\frac{S_k}{a_k}\Big)-\mathbb{E}f\Big(\frac{S_k}{a_k}\Big)\Big)\Big|^p \le C_p\Big(\sum_{1\le k\le l\le N} d_k d_l\Big(\frac{k}{l}\Big)^{\beta}\Big)^{p/2}, \tag{2.3}$$

where $C_p>0$ is a constant and β is the same as in (C_4).

Proof. At first, we set $C_p=(4\gamma)^{p^2}$ and

$$V_{m,n}:=\sum_{l=m}^{n} d_l l^{-\beta}\Big(\sum_{k=1}^{l} d_k k^{\beta}\Big),\quad (1\le m\le n).$$

For obtaining our result, it is enough to show that the following claim,
"*if the number γ is chosen large enough, then*

$$\mathbb{E}\Big|\sum_{k=m}^{n} d_k\xi_k\Big|^p \le C_p(V_{m,n})^{p/2},\quad \text{for all } 1\le m\le n, \tag{2.4}$$

where ξ_k is defined as in (2.2)."

We will use induction on p to show (2.4). By Lemma 2.2, we have

$$\mathbb{E}\Big|\sum_{k=m}^{n} d_k\xi_k\Big|^2 \le 2\sum_{m\le k\le l\le n} d_k d_l|\mathbb{E}\xi_k\xi_l| \le 2c\sum_{m\le k\le l\le n} d_k d_l (k/l)^{\beta} \le 2cV_{m,n}.$$

Hence if we choose γ so large that $(4\gamma)^4\ge 2c$, then (2.4) holds for $p=2$.

Assume now that (2.4) is true for $p-1\ge 2$. From $kd_k=O(1)$ it follows that there is a positive constant A such that $\sum_{k=1}^{l} d_k k^{\beta}\ge Al^{\beta}$. Then we get for $V_{m,n}\le\gamma$ as the proof of Lemma 2.4, there exists a constant A_p such that

$$\begin{aligned}\mathbb{E}\Big|\sum_{k=m}^{n} d_k\xi_k\Big|^p &\le \sum_{k_1=m}^{n}\cdots\sum_{k_p=m}^{n} d_{k_1}\cdots d_{k_p}(\mathbb{E}|\xi_{k_1}|^p\cdots\mathbb{E}|\xi_{k_p}|^p)^{1/p}\\ &\le A_p\|f\|^p\sum_{k_1=m}^{n}\cdots\sum_{k_p=m}^{n} d_{k_1}\cdots d_{k_p}\\ &= A_p\|f\|^p\Big(\sum_{k=m}^{n} d_k\Big)^p\\ &\le A_p\|f\|^p A^{-p}\Big(\sum_{k=m}^{n} d_k k^{-\beta}\Big(\sum_{l=1}^{k} d_l l^{\beta}\Big)\Big)^p.\end{aligned}$$

Now choose γ so large that the $C_p\le({A_p}^{1/p}\|f\|/A)^p\gamma^{p/2}$. In the case of $V_{m,n}\le\gamma$, we have shown (2.4) is valid.

We now want to show that if for any given $X \geq \gamma$ and the inequality (2.4) holds for $V_{m,n} \leq X$, then it will also hold for $V_{m,n} \leq 3X/2$ and this will show that (2.4) holds for any value of $V_{m,n}$, i.e. complete the induction step.

Assume $V_{m,n} \leq 3X/2$ and set

$$S_1 + S_2 := \sum_{k=m}^{w} d_k \xi_k + \sum_{k=w+1}^{n} d_k \xi_k, \quad T_2 := \sum_{k=w+1}^{n} d_k \xi_{w,k}, \quad (m \leq w \leq n).$$

From the discussion of Lemma 4 in Hörmann, S. [6] (2006), and Lemma 2.1, for a fixed m and n we choose w in such a way that

$$V_{m,w} \leq X, \quad V_{w+1,n} \leq X \text{ and } \frac{V_{w+1,n}}{V_{m,w}} = \lambda \in [1/2, 1].$$

From the mean value theorem we get

$$|S_2^j - T_2^j| \leq j|S_2 - T_2|(|S_2|^{j-1} + |T_2|^{j-1}) \quad (j \geq 1). \tag{2.5}$$

Since condition (C_2) and Remarks 2.3, there exists a constant $B > 0$ such that for all $l \geq 1$,

$$B \sum_{k=1}^{l} d_k k^{\beta} \geq l^{1+\beta} d_l.$$

This also shows that

$$\sum_{l=m}^{n} l d_l^2 \leq B V_{m,n}, \quad \text{for all } 1 \leq m \leq n.$$

By Lemma 2.4, we get for all $j \geq 1$,

$$\mathbb{E}|S_2 - T_2|^j \leq F_j (V_{w+1,n})^{j/2},$$

where $F_j = B^{j/2} E_j$ and E_j is the constant in Lemma 2.4.

From the induction hypothesis in the case of $1 \leq j \leq p-1$ and from the validity of (2.4) for $V_{m,n} \leq X$ in the case of $j = p$, we have

$$\mathbb{E}|S_1|^j \leq C_j (V_{m,w})^{j/2}, \quad (1 \leq j \leq p) \tag{2.6}$$

and

$$\mathbb{E}|S_2|^j \leq C_j (V_{w+1,n})^{j/2} \leq C_j \lambda^{j/2} (V_{m,w})^{j/2}, \quad (1 \leq j \leq p). \tag{2.7}$$

The remains of the proof are the same as in Lemma 4 in Hörmann, S. [6] (2006), but for completeness, we still give the proof. By C_r inequality, we have

$$\mathbb{E}|T_2|^j \leq 2^j C_j \lambda^{j/2} (V_{m,w})^{j/2}, \quad (1 \leq j \leq p). \tag{2.8}$$

Furthermore, from Hölder inequality the following result is easy,

$$\begin{aligned} \mathbb{E}|S_1|^j |S_2 - T_2| |S_2|^{p-j-1} &\leq (\mathbb{E}|S_1|^p)^{j/p} (\mathbb{E}|S_2 - T_2|^p)^{1/p} (\mathbb{E}|S_2|^p)^{(p-j-1)/p} \\ &\leq C_p^{(p-1)/p} F_p^{1/p} \lambda^{(p-j)/2} (V_{m,w})^{p/2}. \end{aligned} \tag{2.9}$$

The last inequality remains valid, with an extra factor 2^{p-j-1} on the right hand side, if $|S_2|^{p-j-1}$ on the left hand side is replaced by $|T_2|^{p-j-1}$. Since S_1 and T_2 are independent, we get

$$\mathbb{E}|S_1+S_2|^p \le \mathbb{E}|S_1|^p + \mathbb{E}|S_2|^p + \sum_{j=1}^{p-1} G_p^j(\mathbb{E}|S_1|^j|S_2^{p-j} - T_2^{p-j}| + \mathbb{E}|S_1|^j\mathbb{E}|T_2|^{p-j}),$$

where G_p^j denote the combination, i.e., $G_p^j = p![j!(p-j)!]^{-1}$. Substituting (2.5)−(2.9) (using also the analogue of (2.9) with $|T_2|^{p-j-1}$) in the above inequality and get

$$\begin{aligned}\mathbb{E}|S_1+S_2|^p \le & C_p(V_{m,w})^{p/2}\Big(1+\lambda^{p/2} + C_p^{-1/p}F_p^{1/p}\sum_{j=1}^{p-1} 2^{p-j}G_p^j(p-j)\lambda^{(p-j)/2} \\ & + C_p^{-1}\sum_{j=1}^{p-1} 2^{p-j}G_p^j\lambda^{(p-j)/2}C_jC_{p-j}\Big).\end{aligned}$$

Note that

$$C_p^{-1/p}F_p^{1/p} \le const\cdot\tau(\kappa)^{1/2}c_p^{1/p}(4\gamma)^{-p}$$

and

$$C_jC_{p-j}/C_p \le (4\gamma)^{-p},\ (1\le j\le p-1),$$

thus, by $\lambda \le 1$, we have

$$C_p^{-1/p}F_p^{1/p}\sum_{j=1}^{p-1} 2^{p-j}G_p^j(p-j)\lambda^{(p-j)/2} \le const\cdot\tau(\kappa)^{1/2}p\,c_p^{1/p}\gamma^{-p}$$

and

$$C_p^{-1}\sum_{j=1}^{p-1} 2^{p-j}G_p^j\lambda^{(p-j)/2}C_jC_{p-j} \le const\cdot\gamma^{-p}.$$

Since $\lambda \ge 1/2$ we have shown that for a large γ the relation $\mathbb{E}|S_1+S_2|^p \le C_p(1+\lambda)^{p/2}(V_{m,w})^{p/2} = C_p(V_{m,n})^{p/2}$ is true, i.e., for $V_{m,n}\le 3X/2$, (2.4) is valid. □

Lemma 2.6. *(See Lemma 5 in [6]) Assume the condition (C_3) of Theorem 1.1 is satisfied. Then for any $\alpha>0$ and any $\eta<\rho$, we have*

$$\sum_{1\le k\le l\le N} d_kd_l\Big(\frac{k}{l}\Big)^{\alpha} = O\Big(\frac{D_N^2}{(\log D_N)^{\eta}}\Big).$$

Proof of Theorem 1.1. Without loss of generality, from Lemma 2.5 and Lemma 2.6, we have, for any $\varepsilon>0$, $p\le\mu$ and $p\in\mathbb{N}$,

$$\mathbb{P}\Big(\Big|\sum_{k=1}^N d_k\xi_k\Big| > \varepsilon D_N\Big) \le c(p,\varepsilon)(\log D_N)^{-p\eta/2}, \quad for\ \ N\ge N_0.$$

Since $\mu \geq (2 \vee 4/(\rho - r))$ for some $0 < r < \rho$, we can choose suitable $\eta < \rho$ and p such that $p\eta > 4$. By (C_3), we have $D_{N+1}/D_N \to 1$, thus we can choose (N_j) such that $D_{N_j} \sim \exp\{\sqrt{j}\}$. Applying Borel-Cantelli lemma, we get

$$\lim_{j\to\infty} D_{N_j}^{-1} \sum_{k=1}^{N_j} d_k \xi_k = 0 \quad a.s..$$

For $N_j \leq N \leq N_{j+1}$, we have

$$D_N^{-1} \Big| \sum_{k=1}^{N} d_k \xi_k \Big| \leq D_{N_j}^{-1} \Big| \sum_{k=1}^{N_j} d_k \xi_k \Big| + 2(D_{N_{j+1}}/D_{N_j} - 1) \quad a.s..$$

The convergence of the subsequence implies that whole sequence converges a.s., since $D_{N_{j+1}}/D_{N_j} \to 1$. This complete the proof of the theorem. □

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