

BLOW-UP OF SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATIONS ON STANDARD N-SPHERE AND HYPERBOLIC N-SPACE

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ABSTRACT. In this paper, we partially settle down the long standing open problem of the finite time blow-up property about the nonlinear Schrödinger equations on some Riemannian manifolds like the standard 2-sphere S^2 and the hyperbolic 2-space $H^2(-1)$. Using the similar idea, we establish such blow-up results on higher dimensional standard sphere and hyperbolic n -space. Extensions to n -dimensional Riemannian warped product manifolds with $n \geq 2$ are also given.

Keywords: Schrödinger equation, blow-up, Riemannian manifold

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1. INTRODUCTION

In this paper, we partially settle down the long standing open problem of the finite time blow-up property about the nonlinear Schrödinger equations on some Riemannian manifolds like the standard 2-sphere S^2 and the hyperbolic 2-space $H^2(-1)$. The nonlinear Schrödinger equations of the following form

$$(1) \quad iu_t = \Delta u + F(|u|^2)u$$

play an important role in many areas of applied physics, such as non-relativistic quantum mechanics, laser beam propagation, Bose-Einstein condensates and so on (see [18]). The initial value problems (IVP) or the initial-boundary value problems (IBVP) of (1) on \mathbb{R}^n have been extensively studied in the last two decades (see [8, 13, 21, 10, 19, 20]). In particular, the blow-up properties in finite time for IVP or IBVP have caught sufficient attention (see [11, 12, 16, 14, 15]). However, much less results have been known on bounded domains in \mathbb{R}^n or on compact manifolds (M, g) , with the notable exception of the works of H.Brézis and T.Gallouet [5], J.Bourgain [1, 2, 3] (In [1], the case $M = \mathbb{R}^2/\mathbb{Z}^2$ was discussed in detail), and N.Burq, P.Gérard and N.Tzvetkov [6, 7]. In particular, the blow-up in finite time of Schrödinger equations (1)

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posed on an arbitrary Riemannian manifold (M, g) is a long widely known open problem. To our knowledge, the only examples of such blow-up phenomena on Riemannian manifolds are given by the following result, attributed to Ogawa-Tsutsuni [17] if the dimension n of M equals to 1 and generalized to the case $n = 2$ by N.Burq, P.Gérard and N.Tzvetkov [6].

Theorem 1. *Let (M, g) be a compact Riemannian manifold of dimension $n = 1$ or $n = 2$. Assume there exist $x^0 \in M$ and a system of coordinates near x^0 in which*

$$g = \sum_{j=1}^d dx_j^2.$$

Then there exist smooth solutions $u \in C^\infty([0, T] \times M)$ of

$$iu_t = \Delta u + |u|^{4/n}u$$

such that as $t \rightarrow T$,

$$|u(t, x)|^2 \rightharpoonup \|Q\|_{L^2(\mathbb{R}^n)}^2 \delta(x - x^0),$$

where Q is the ground state solution on \mathbb{R}^n of

$$\Delta Q + Q^{1+4/n} = Q.$$

Even though the condition of Theorem 1 that the manifold near x^0 is flat is a very strong restriction, the result is also impressive.

In this paper, we concentrate on the analysis of the blow-up phenomena for IVP or BIVP of the Schrödinger equations posed on Riemannian manifolds. To be precise, the IVP and BIVP are of the following forms respectively

$$(2) \quad \text{IVP} \begin{cases} iu_t = \Delta u + F(|u|^2)u, & \text{on } M, \\ u(0, x) = u_0(x), \\ \partial M = \emptyset; \end{cases}$$

$$(3) \quad \text{BIVP} \begin{cases} iu_t = \Delta u + F(|u|^2)u, & \text{on } M, \\ u|_{\mathbf{R} \times \partial M} = 0, \\ u(0, x) = u_0(x), \end{cases}$$

where F is a real-valued smooth function on the n -dimensional Riemannian manifold (M, g) and F satisfies $F(s) \leq C(1 + s^{(p-1)/2})$ for some $p > 1$ on $[0, \infty)$. Here Δ is the Laplacian operator of the metric g with the sign $\Delta u = u''$ on the real line \mathbb{R} . Noticing that (3) reduces to (2) when $\partial M = \emptyset$, it's convenient to establish our blow-up results in the context of (3). The local wellposedness in $H^s(M)$ for $s > n/2$ to (3) with the interesting case $F(|u|^2)u = |u|^{p-1}u$ ($p > 1$) is a classical

consequence of energy estimates, and therefore, it's relaxed to assume that the solution $u(t)$ to (3) satisfies

$$(4) \quad u \in C^1([0, T], L^2) \cap C([0, T], H^2 \cap H_0^1 \cap L^{p+1}),$$

where T is the maximal existence time for the solution $u(t)$.

Before stating our main results, let's introduce some exact notations concerning Riemannian manifolds used below. Let (M, g) be a complete Riemannian manifold of dimension n with boundary ∂M or not. We denote by D the Levi-Civita connection, and by $TM = \bigcup_{x \in M} T_x M$, where $T_x M$ is the tangent space at $x \in M$. It's well known that the smooth sections of TM are just vector fields. For $f \in C^1(M)$, its gradient is defined as the unique vector field ∇f such that

$$\forall x \in M, \forall \xi \in TM, \quad g(\nabla f(x), \xi(x)) = (\xi f)(x).$$

The divergence $\operatorname{div} \mathbf{X}$ of a smooth vector field \mathbf{X} is defined as the unique smooth function on M such that

$$\forall f \in C_0^\infty(M), \quad \int f \operatorname{div} \mathbf{X} = - \int \mathbf{X} f.$$

The Laplace-Beltrami operator Δ on M is the second order differential operator defined by

$$\forall f \in C^2(M), \quad \Delta f = \operatorname{div}(\nabla f).$$

Corresponding to our analysis, we need to extend g to be defined on complex valued vector fields. For complex valued vectors $X_1 + iX_2, Y_1 + iY_2$, where X_1, X_2, Y_1, Y_2 are real, we define

$$\begin{aligned} &g(X_1 + iX_2, Y_1 + iY_2) \\ &= g(X_1, X_2) + ig(X_1, Y_1) + ig(X_2, Y_1) + g(Y_1, Y_2). \end{aligned}$$

It's easy to see that g defined in such a way is bilinear in the field \mathbb{C} and accordingly

$$\nabla f = \nabla \Re f + i \nabla \Im f, \quad \Delta f = \Delta \Re f + i \Delta \Im f.$$

When M has a nonempty boundary ∂M , we denote by \mathbf{v} the outer unit normal vector along ∂M .

For the sake of simplicity, we omit the spatial integral variable $x \in M$ and omit the integral region when it's the whole space M , and we abbreviate $L^q(M), H^k(M)$ to L^q, H^k respectively. We write the integral $\int_M dV_M$ and $\int_{\partial M} dV_{\partial M}$ as \int and \oint respectively, and the norm of L^q as $\|\cdot\|_q$. We denote by S^n the standard sphere of dimension n and by $H^n(-1)$ the hyperbolic n -space respectively. We denote by N the north pole of S^n and by $\operatorname{dist}(N, x)$ the distance between N and $x \in S^n \setminus \{N\}$.

Our main results in 2-dimensions are the following two Theorems.

Theorem 2. *Consider the Schrödinger equation*

$$\begin{cases} iu_t = \Delta u + |u|^{p-1}u, & \text{on } S^2, \\ u(0) = u_0 \in H^1. \end{cases}$$

For $p \geq 5$, if $u_0(x) = u_0(r)$, where $r = \text{dist}(N, x)$ and $u_0(r)$ is an asymmetric function at $r = \frac{\pi}{2}$ with $u(\frac{\pi}{2}) = 0$ and $E_0 < 0$, then the asymmetric solution satisfying (4) blows up in finite time.

Theorem 3. *Consider the Schrödinger equations (2) on $M = H^2(-1)$. Assume there exists a constant $\kappa \geq 3$ such that*

$$sF(s) \geq \kappa G(s), \quad \forall s \geq 0.$$

Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

For higher dimensional results, please see Theorem 12 and 13. Here, we just want to point out that the range of the exponent p for blow-up of the solutions to the Schrödinger equation (1) with $F(s) = s^{(p-1)/2}$ on \mathbb{R}^n is $p \geq 1 + \frac{4}{n}$, on S^n is $p \geq 5$ and on $H^n(-1)$ is $p \geq 1 + \frac{4}{n-1}$.

We try to present very elementary proofs of our blow up results starting from 2-sphere. This paper is organized as follows. In section 2, we establish some new invariant quantities for the Schrödinger equations on general Riemannian manifolds, which generalize the corresponding classical results on \mathbb{R}^n . In section 3, we construct blow-up solutions on the unit sphere S^2 . In section 4, we establish the blow-up results on a class of noncompact manifolds. We discuss the blow-up properties on n -dim manifolds with $n \geq 3$ in section 5.

2. PRELIMINARY LEMMA

The following lemma is a generalization of the identities obtained by Glassey [11] (see also [12]). We define

$$G(u) = \int_0^u F(s)ds.$$

Lemma 4. *Suppose that (M, g) is a complete Riemannian manifold of dimension n with boundary ∂M or not, and \mathbf{v} is the outer unit normal vector along ∂M . Let u be a solution of (3) satisfying (4), ρ be an arbitrary smooth function on M , and \mathbf{X} be a real smooth vector field on M . Define $J(t) := \int \rho |u|^2$. Then we have*

- (A). $\|u(t)\|_2 = \|u_0\|_2$,
- (B). $\int (g(\nabla u, \nabla \bar{u}) - G(|u|^2)) \equiv \text{const} := E_0$,
- (C). $J'(t) = -2\Im \int g(\nabla \rho, \nabla u) \bar{u}$,

(D).

$$\begin{aligned}
& \frac{d}{dt} \Im \int g(\mathbf{X}, \nabla u) \bar{u} \\
&= -2 \int D\mathbf{X}(\nabla u, \nabla \bar{u}) + \frac{1}{2} \int (\Delta \operatorname{div} \mathbf{X}) |u|^2 \\
&\quad + \int (\operatorname{div} \mathbf{X})(F(|u|^2)|u|^2 - G(|u|^2)) \\
&\quad + \oint g(\nabla u, \nabla \bar{u}) g(\mathbf{X}, \mathbf{v}).
\end{aligned}$$

Proof. The facts that $\int \rho |u|^2$ and $\Im \int g(\mathbf{X}, \nabla u) \bar{u}$ are of $C^1[0, T]$ are straightforward and the reader can refer to [12] for details.

For (A), multiply both sides of (3) by $2\bar{u}$ and take the imaginary part to obtain

$$(5) \quad \frac{\partial}{\partial t} |u|^2 = 2\nabla \cdot \Im(\bar{u} \nabla u).$$

Integrating it over M we get (A).

For (B), multiply (3) by $2\bar{u}_t$, integrate, and take the real part of the resulting expression.

For (C), multiply (5) by ρ and integrate by parts over M .

The derivation of (D) is a bit involved. We first multiply (3) by $2D_{\mathbf{X}}\bar{u}$ to obtain

$$(6) \quad \begin{aligned} 2i(D_{\mathbf{X}}\bar{u})u_t &= 2(D_{\mathbf{X}}\bar{u})\Delta u + 2(D_{\mathbf{X}}\bar{u})F(|u|^2)u \\ &:= I_1 + I_2. \end{aligned}$$

Then, we take the real part of the left-hand side (LHS) of (6) to get

$$\begin{aligned}
\Re(\text{LHS}) &= i((D_{\mathbf{X}}\bar{u})u_t - (D_{\mathbf{X}}u)\bar{u}_t) \\
&= i((uD_{\mathbf{X}}\bar{u})_t - D_{\mathbf{X}}(u\bar{u}_t)) \\
&= \Re(i(uD_{\mathbf{X}}\bar{u})_t) - \Re(iD_{\mathbf{X}}(u\bar{u}_t)) \\
&= \frac{d}{dt} \Im(g(\mathbf{X}, \nabla u)\bar{u}) - \Re(iD_{\mathbf{X}}(u\bar{u}_t)).
\end{aligned}$$

Integrating this identity over M yields

$$(7) \quad \Re \int \text{LHS} = \frac{d}{dt} \Im \int g(\mathbf{X}, \nabla u) \bar{u} - \Re \int iD_{\mathbf{X}}(u\bar{u}_t).$$

Using integration by parts we have

$$\begin{aligned}
& \Re \int iD_{\mathbf{X}}(u\bar{u}_t) \\
&= -\Re \int (\operatorname{div} \mathbf{X})(iu\bar{u}_t) \\
&= -\Re \int (\operatorname{div} \mathbf{X})(-u\Delta\bar{u} - F(|u|^2)|u|^2) \\
&= -\Re \int (g(\nabla(\operatorname{div} \mathbf{X}), u\nabla\bar{u}) + (\operatorname{div} \mathbf{X})g(\nabla u, \nabla\bar{u})) + \int (\operatorname{div} \mathbf{X})F(|u|^2)|u|^2 \\
&= -\frac{1}{2} \int g(\nabla(\operatorname{div} \mathbf{X}), \nabla|u|^2) - \int (\operatorname{div} \mathbf{X})g(\nabla u, \nabla\bar{u}) \\
&\quad + \int (\operatorname{div} \mathbf{X})F(|u|^2)|u|^2 \\
&= \frac{1}{2} \int (\Delta \operatorname{div} \mathbf{X})|u|^2 - \int (\operatorname{div} \mathbf{X})g(\nabla u, \nabla\bar{u}) + \int (\operatorname{div} \mathbf{X})F(|u|^2)|u|^2.
\end{aligned}$$

Inserting this into (7) we obtain that

$$\begin{aligned}
(8) \quad \Re \int \text{LHS} &= \frac{d}{dt} \Im \int g(\mathbf{X}, \nabla u)\bar{u} - \frac{1}{2} \int (\Delta \operatorname{div} \mathbf{X})|u|^2 \\
&\quad + \int (\operatorname{div} \mathbf{X})g(\nabla u, \nabla\bar{u}) - \int (\operatorname{div} \mathbf{X})F(|u|^2)|u|^2.
\end{aligned}$$

To handle the right-hand side of (6), we use integration by parts again to obtain

$$\begin{aligned}
\Re \int I_1 &= 2\Re \int (D_{\mathbf{X}}\bar{u})\Delta u \\
&= -2\Re \int g(\nabla D_{\mathbf{X}}\bar{u}, \nabla u) + 2\Re \oint g((D_{\mathbf{X}}\bar{u})\nabla u, \mathbf{v}) \\
&= -2\Re \int (D\mathbf{X}(\nabla u, \nabla\bar{u}) + \frac{1}{2}D_{\mathbf{X}}g(\nabla u, \nabla\bar{u})) \\
&\quad + 2\Re \oint g((D_{\mathbf{X}}\bar{u})\nabla u, \mathbf{v}) \\
&= -2 \int D\mathbf{X}(\nabla u, \nabla\bar{u}) - \int D_{\mathbf{X}}g(\nabla u, \nabla\bar{u}) \\
&\quad + 2 \oint g(\nabla u, \nabla\bar{u})g(\mathbf{X}, \mathbf{v}).
\end{aligned}$$

The last "=" in the above expression follows from the fact that $u|_{\partial M} = 0$, which implies $\nabla u = g(\nabla u, \mathbf{v})\mathbf{v}$.

Noticing that

$$\int D_{\mathbf{X}}g(\nabla u, \nabla \bar{u}) = - \int (\operatorname{div} \mathbf{X})g(\nabla u, \nabla \bar{u}) + \oint g(\nabla u, \nabla \bar{u})g(\mathbf{X}, \mathbf{v}),$$

we then get

$$(9) \quad \Re \int I_1 = -2 \int D_{\mathbf{X}}(\nabla u, \nabla \bar{u}) + \int (\operatorname{div} \mathbf{X})g(\nabla u, \nabla \bar{u}) \\ + \oint g(\nabla u, \nabla \bar{u})g(\mathbf{X}, \mathbf{v}).$$

For I_2 we have

$$(10) \quad \Re \int I_2 = 2\Re \int (D_{\mathbf{X}}\bar{u})F(|u|^2)u = \int (D_{\mathbf{X}}|u|^2)F(|u|^2) \\ = \int D_{\mathbf{X}}G(|u|^2) = - \int (\operatorname{div} \mathbf{X})G(|u|^2).$$

Combining (8)-(10) with (6) we get (D), and the proof of the lemma is concluded. \square

Remark 5. *If we choose $\mathbf{X} = \nabla \rho$ in lemma 2 (D), we then arrive at*

$$(11) \quad J''(t) = 4 \int D^2 \rho(\nabla u, \nabla \bar{u}) - \int (\Delta^2 \rho)|u|^2 \\ - 2 \int (\Delta \rho)(F(|u|^2)|u|^2 - G(|u|^2)) \\ - 2 \oint g(\nabla u, \nabla \bar{u})g(\nabla \rho, \mathbf{v}).$$

This identity will play a vital role in our analysis. In particular, when $\nabla \rho = \mathbf{v}$, we have

$$g(\nabla u, \nabla \bar{u})g(\nabla \rho, \mathbf{v}) = g(\nabla u, \nabla \bar{u}) \geq 0,$$

which is an important fact in our proof.

3. BLOW-UP PHENOMENA ON S^2

To investigate the blow-up nature of the solution u , one method is to observe the long time behavior of $J(t) := \int \rho |u|^2$. If there exists $\rho \geq 0$ such that $J(t)$ becomes negative after some finite time T due to the conservation of the L^2 norm and the energy E_0 , then u must blow up before the time T . It's classical that $\rho(x) = |x|^2$ when $M = \mathbb{R}^d$. But for an arbitrary manifold, the sharp ρ adopted to the blow-up properties is unknown explicitly. It seems that $\Delta |x|^2 = 2n$ is a nice property for us to use (11) on $M = \mathbb{R}^n$. For noncompact manifolds and compact manifolds with boundary, it's possible to find ρ such that $\Delta \rho = \text{const}$.

We put this idea in practice on the half sphere $S_+^2 := S^2 \cap \{x^1 \geq 0\}$ and the hyperbolic space $M = H^2(-1)$. We now state the result for S_+^2 .

Theorem 6. *Consider the Schrödinger equations (3) on $M = S_+^2$. Assume there exists a constant $\kappa \geq 3$ such that*

$$sF(s) \geq \kappa G(s), \quad \forall s \geq 0.$$

Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

Proof. In \mathbb{R}^3 , $S_+^2 = \{(x^1)^2 + (x^2)^2 + (x^3)^2 = 1\} \cap \{x^1 \geq 0\}$. We want to construct a function ρ such that

- (a). $\rho \in C^4(S_+^2)$;
- (b). $\rho > 0$ on $S_+^2 \setminus \{N\}$;
- (c). $\Delta \rho = 1$ on S_+^2 ;
- (d). $D^2 \rho(\nabla u, \nabla \bar{u}) \leq g(\nabla u, \nabla \bar{u})$, $\forall u \in C^1(S_+^2)$.

We now give the form of ρ exactly. To make the calculations clear, we recall the expressions of ∇f , Δf and $D^2 f$ for $f \in C^2(M)$ in local coordinates. We use Einstein's convention. Let $g = g_{ij} dx^i dx^j$, $G = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Then

$$(12) \quad \nabla f = g^{ij} f_i \partial_j,$$

$$(13) \quad \Delta f = \frac{1}{\sqrt{G}} \partial_i (g^{ij} \sqrt{G} f_j),$$

$$(14) \quad D^2 f = (f_{ij} - \Gamma_{ij}^k f_k) dx^i \otimes dx^j,$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

In our situation, we use the geodesic polar coordinate (r, θ) at the north pole N for S_+^2 , i.e.,

$$\begin{cases} x^1 = \cos r, \\ x^2 = \sin r \cos \theta, \\ x^3 = \sin r \sin \theta, \end{cases}$$

where $r \in (0, \pi/2]$, $\theta \in [0, 2\pi)$. In this coordinate,

$$\begin{cases} dx^1 = -\sin r dr, \\ dx^2 = \cos r \cos \theta dr - \sin r \sin \theta d\theta, \\ dx^3 = \cos r \sin \theta dr + \sin r \cos \theta d\theta, \end{cases}$$

and hence

$$g = \sum_{i=1}^3 (dx^i)^2 = dr^2 + \sin^2 r d\theta^2,$$

$$\Gamma_{11}^1 = \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \quad \Gamma_{22}^1 = -\sin r \cos r.$$

By (13) we have for $\varrho = \varrho(r) \in C^2(0, \pi/2]$ that

$$\Delta \varrho(r) = \frac{1}{\sin r} (\sin r \varrho'(r))' = \varrho''(r) + \varrho'(r) \cot r.$$

Solving the ODE

$$\begin{cases} \varrho''(r) + \varrho'(r) \cot r = 1, & 0 < r \leq \pi/2, \\ \varrho(r) > 0, & 0 < r \leq \pi/2, \\ \varrho'(\pi/2) = 1, \end{cases}$$

we get a solution $\varrho(r) = -2 \log \cos(r/2)$. We then define $\rho(r)$ as

$$\rho(r) = \begin{cases} \varrho(r), & 0 < r \leq \pi/2, \\ 0, & r = 0. \end{cases}$$

It's then easy to see that $\rho(r) \in C^4[0, \pi/2]$, i.e. $\rho \in C^4(S_+^2)$.

From (12) and (14), we have for any $u \in C^1(S_+^2)$,

$$g(\nabla u, \nabla \bar{u}) = |u_r|^2 + \frac{1}{\sin^2 r} |u_\theta|^2$$

and

$$\begin{aligned} D^2 \rho(\nabla u, \nabla \bar{u}) &= \rho''(r) |u_r|^2 + \rho'(r) \frac{\cos r}{\sin^3 r} |u_\theta|^2 \\ &= \frac{1 - \cos r}{\sin^2 r} |u_r|^2 + \frac{(1 - \cos r) \cos r}{\sin^4 r} |u_\theta|^2. \end{aligned}$$

It's obvious that $D^2 \rho(\nabla u, \nabla \bar{u}) \leq g(\nabla u, \nabla \bar{u})$ provided $0 < r \leq \pi/2$. By a standard approximation process we get

$$\int_{S_+^2} D^2 \rho(\nabla u, \nabla \bar{u}) \leq \int_{S_+^2} g(\nabla u, \nabla \bar{u}), \quad \forall u \in H^1(S_+^2).$$

Notice that $\nabla \rho = \mathbf{v}$ on the boundary ∂S_+^2 . Then by (11) we have

$$\begin{aligned} (15) \quad J''(t) &\leq 4 \int_{S_+^2} D^2 \rho(\nabla u, \nabla \bar{u}) - 2 \int_{S_+^2} (F(|u|^2) |u|^2 - G(|u|^2)) \\ &\quad - 2 \oint_{\partial S_+^2} g(\nabla u, \nabla \bar{u}) g(\nabla \rho, \mathbf{v}) \\ &\leq 4 \int_{S_+^2} g(\nabla u, \nabla \bar{u}) - 2(\kappa - 1) \int_{S_+^2} G(|u|^2). \end{aligned}$$

Combining Lemma 2 (B) with (15) we obtain

$$J''(t) \leq 4E_0 + (6 - 2\kappa) \int_{S_+^2} G(|u|^2) \leq 4E_0 < 0,$$

which implies that $J(t)$ becomes negative after some finite time T , i.e., the solution u satisfying $E_0 < 0$ blows up in finite time. \square

Remark 7. *Here we only established the blow-up result for the Schrödinger equations on S_+^2 . The method above can't be applied to the whole sphere S^2 or some other compact manifolds because of the nonexistence of subharmonic functions on compact manifolds. But this result allows us to construct blow-up solutions for the Schrödinger equations posed on S^2 . See the proof of Theorem 2.*

Remark 8. *If $F(s) = s^{\frac{p-1}{2}}$ where $p > 1$, then $G(s) = \frac{2}{p+1}s^{\frac{p+1}{2}}$, and the condition $sF(s) \geq \kappa G(s)$ on $[0, +\infty)$ for some $\kappa \geq 3$ is equivalent to $p \geq 5$. It's already known that when $1 < p < 3$, the Schrödinger equation $iu_t = \Delta u + |u|^{p-1}u$ with $u_0 \in H^s$ ($s \geq 1$) on S^2 has a unique global solution $u \in C(\mathbb{R}, H^s)$ (see [7]). We point out when $p \geq 5$, blow-up phenomena may occur by our Theorem 2.*

Proof of Theorem 2: If $u_0(r)$ is asymmetric in r , then by the symmetry of the Schrödinger equation, the solution $u(t, x)$ is also asymmetric, that is, $u(t, r)$ is asymmetric with respected to r , which implies that $u(t, \pi/2) \equiv 0$. We then cut the sphere S^2 into two parts S_+^2 and S_-^2 , where

$$\begin{aligned} S_+^2 &= \{(x^1)^2 + (x^2)^2 + (x^3)^2 = 1\} \cap \{x^1 \geq 0\}, \\ S_-^2 &= \{(x^1)^2 + (x^2)^2 + (x^3)^2 = 1\} \cap \{x^1 \leq 0\}. \end{aligned}$$

We write by ρ_+ the function obtained in the proof of Theorem 6 on S_+^2 , and define

$$\rho_-(x) := \rho_+(-x) \quad \text{for } x \in S_-^2.$$

We denote by \mathbf{v}_+ and \mathbf{v}_- the outer normal vector along ∂S_+^2 and ∂S_-^2 . Then $\mathbf{v}_+ = -\mathbf{v}_-$, and $\nabla \rho_+ = \mathbf{v}_+$, $\nabla \rho_- = \mathbf{v}_-$.

We now define

$$J(t) := \int_{S_+^2} \rho_+ |u|^2 + \int_{S_-^2} \rho_- |u|^2 := J_1 + J_2.$$

Since $u = 0$ on $\partial S_+^2 = \partial S_-^2$, we can use (11) directly for both J_1 and J_2 to get that

$$\begin{aligned} (16) \quad J_1''(t) &\leq 4 \int_{S_+^2} D^2 \rho_+(\nabla u, \nabla \bar{u}) - 2 \int_{S_+^2} (F(|u|^2)|u|^2 - G(|u|^2)) \\ &\quad - 2 \oint_{\partial S_+^2} g(\nabla u, \nabla \bar{u}) g(\nabla \rho_+, \mathbf{v}_+) \\ &\leq 4 \int_{S_+^2} g(\nabla u, \nabla \bar{u}) - 2(\kappa - 1) \int_{S_+^2} G(|u|^2), \end{aligned}$$

and

$$\begin{aligned}
(17) \quad J_2''(t) &\leq 4 \int_{S_-^2} D^2 \rho_-(\nabla u, \nabla \bar{u}) - 2 \int_{S_-^2} (F(|u|^2)|u|^2 - G(|u|^2)) \\
&\quad - 2 \oint_{\partial S_-^2} g(\nabla u, \nabla \bar{u}) g(\nabla \rho_-, \mathbf{v}_-) \\
&\leq 4 \int_{S_-^2} g(\nabla u, \nabla \bar{u}) - 2(\kappa - 1) \int_{S_-^2} G(|u|^2).
\end{aligned}$$

We obtain from (16)+(17) that

$$\begin{aligned}
J''(t) &\leq 4 \int_{S^2} g(\nabla u, \nabla \bar{u}) - 2(\kappa - 1) \int_{S^2} G(|u|^2) \\
&= 4E_0 + (6 - 2\kappa) \int_{S^2} G(|u|^2) \\
&\leq 4E_0 < 0
\end{aligned}$$

provided $\kappa \geq 3$, i.e., $p \geq 5$, which implies that the solution u blows up in finite time. This is the end of proof.

Remark 9. *When $3 \leq p < 5$, the blow-up property of the solution on S^2 leaves open.*

4. BLOW-UP ON NONCOMPACT 2-DIM MANIFOLDS

By the method of introducing some proper weight function ρ , we can get similar blow-up results for noncompact manifolds. We first prove the result for the hyperbolic 2-space $H^2(-1)$, i.e., Theorem 3, and then generalize it to a class of noncompact manifolds.

Proof of Theorem 3: For $H^2(-1)$, it's standard that for $s \in [0, \infty)$ and $\theta \in [0, 2\pi)$,

$$g = \frac{1}{1+s^2} ds^2 + s^2 d\theta^2.$$

Choose $r(s) = \sinh^{-1}(s)$, then we have $dr = \frac{1}{\sqrt{1+s^2}} ds$, and thus

$$g = dr^2 + \sinh^2(r) d\theta^2.$$

Calculating by (13), we get for $\rho = \rho(r) \in C^2(0, \infty)$,

$$\Delta \rho(r) = \frac{1}{\sinh r} (\sinh r \rho'(r))' = \rho''(r) + \rho'(r) \coth r.$$

Solving the ODE

$$\begin{cases} \rho''(r) + \rho'(r) \coth r = 1, & 0 < r < \infty, \\ \rho(r) > 0, & 0 < r < \infty, \end{cases}$$

we get a solution $\rho(r) = 2 \log \cosh(\frac{r}{2})$. It's easy to see that by defining $\rho(0) = 0$, $\rho \in C^4[0, \infty)$, i.e., $\rho \in C^4(H^2(-1))$.

From (12) and (14) we have for any $u \in C^1(H^2(-1))$,

$$g(\nabla u, \nabla \bar{u}) = |u_r|^2 + \frac{1}{\sinh^2 r} |u_\theta|^2,$$

and

$$\begin{aligned} D^2 \rho(\nabla u, \nabla \bar{u}) &= \rho''(r) |u_r|^2 + \frac{\cosh r}{\sinh^3 r} |u_\theta|^2 \\ &= \frac{\cosh r - 1}{\sinh^2 r} |u_r|^2 + \frac{\cosh r (\cosh r - 1)}{\sinh^4 r} |u_\theta|^2. \end{aligned}$$

Noticing that when $r > 0$ we have

$$\frac{\cosh r - 1}{\sinh^2 r} = \frac{1}{2 \cosh^2 r/2} \leq \frac{1}{2}$$

and

$$\frac{\cosh r (\cosh r - 1)}{\sinh^2 r} = \frac{\cosh r}{2 \cosh^2 r/2} \leq 1,$$

we obtain that

$$D^2 \rho(\nabla u, \nabla \bar{u}) \leq g(\nabla u, \nabla \bar{u}).$$

The remainder of the proof is the same as in Theorem 6. This completes the proof of Theorem 3.

The next result generalize our analysis to a class of 2-dim noncompact manifolds.

Theorem 10. *Let (M, g) be a 2-dim Riemannian manifold such that M can be covered by only one coordinate system in which*

$$g = dr^2 + h^2(r) d\theta^2,$$

where $h \in C^4[0, \infty)$ satisfying for some constants $\tau_1, \tau_2 \in [0, 1]$,

$$\begin{cases} h(r) \in C^4[0, \infty), \\ h(r) > 0, \text{ on } (0, \infty); \quad h(0) = 0; \quad h'(0) > 0, \\ \tau_1 h^2(r) \leq h'(r) \int_0^r h(s) ds \leq \tau_2 h^2(r), \text{ on } (0, \infty). \end{cases}$$

Consider the Schrödinger equations (2) on M . Assume there exists a constant $\kappa \geq 2 \max\{1 - \tau_1, \tau_2\} + 1$ such that

$$sF(s) \geq \kappa G(s), \quad \forall s \geq 0.$$

Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

Proof. In above coordinate,

$$\Gamma_{11}^1 = \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \quad \Gamma_{22}^1 = -h(r)h'(r).$$

By (13) we have for $\rho = \rho(r) \in C^2(0, \infty)$ that

$$\Delta\rho(r) = \frac{1}{h(r)}(h(r)\rho'(r))' = \rho''(r) + \rho'(r)\frac{h'(r)}{h(r)}.$$

Solving the ODE

$$\begin{cases} \rho''(r) + \rho'(r)\frac{h'(r)}{h(r)} = 1, & 0 < r < \infty, \\ \rho(r) > 0, & 0 < r < \infty, \end{cases}$$

we get a solution

$$\rho(r) = \int_0^r \left(\int_0^s h(t)dt \right) (h(s))^{-1} ds.$$

We then define $\rho(0) = 0$, and thus it's easy to see that $\rho(r) \in C^4[0, \infty)$, i.e, $\rho \in C^4(M)$.

From (12) and (14), we have for any $u \in C^1(M)$,

$$g(\nabla u, \nabla \bar{u}) = |u_r|^2 + \frac{1}{h^2(r)} |u_\theta|^2$$

and

$$\begin{aligned} D^2\rho(\nabla u, \nabla \bar{u}) &= \rho''(r)|u_r|^2 + \rho'(r)\frac{h'(r)}{h^3(r)}|u_\theta|^2 \\ &= \left(1 - \frac{h'(r) \int_0^r h(s)ds}{h^2(r)}\right)|u_r|^2 + \frac{h'(r) \int_0^r h(s)ds}{h^4(r)}|u_\theta|^2 \\ &\leq (1 - \tau_1)|u_r|^2 + \frac{\tau_2}{h^2(r)}|u_\theta|^2 \\ &\leq \max\{1 - \tau_1, \tau_2\}g(\nabla u, \nabla \bar{u}). \end{aligned}$$

The remainder of the proof is the same as in Theorem 6. \square

5. SOME RESULTS FOR HIGHER DIMENSIONAL MANIFOLDS

In this section, we make some calculations for S^n and $H^n(-1)$. We first compute the case $S_+^n = S^2 \cap \{x^1 \geq 0\}$. As above, for $0 < r \leq \pi/2$, $0 \leq \theta_1, \dots, \theta_{n-1} < 2\pi$, we have

$$\begin{cases} x^1 = \cos r, \\ x^2 = \sin r \cos \theta_1, \\ x^3 = \sin r \sin \theta_1 \cos \theta_2, \\ \dots \dots \\ x^n = \sin r \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1}, \\ x^{n+1} = \sin r \sin \theta_1 \dots \sin \theta_{n-2} \sin \theta_{n-1}. \end{cases}$$

By (13) we get

$$g = dr^2 + \sin^2 r (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} d\theta_{n-1}^2),$$

and

$$\Delta \rho(r) = \frac{(\rho'(r) \sin^{n-1} r)'}{\sin^{n-1} r}.$$

Solving the ODE

$$\Delta \rho(r) = 1 \quad \text{on } (0, \pi/2]$$

we get the desired positive solution

$$\rho(r) = \int_0^r \left(\int_0^s \sin^{n-1} \tau d\tau \right) (\sin^{n-1} s)^{-1} ds.$$

By defining $\rho(0) = 0$ we see that $\rho \in C^4(S_+^n)$.

From (12) and (14) we get for all $u \in C^1(S_+^n)$

$$g(\nabla u, \nabla \bar{u}) = |u_r|^2 + \frac{|u_{\theta_1}|^2}{\sin^2 r} + \cdots + \frac{|u_{\theta_{n-1}}|^2}{(\sin r \sin \theta_1 \cdots \sin \theta_{n-2})^2},$$

and

$$\begin{aligned} D^2 \rho(r) &= \rho''(r) dr \otimes dr + \rho'(r) \sin r \cos r d\theta_1 \otimes d\theta_1 \\ &\quad + \rho'(r) \sin r \cos r \sin^2 \theta_1 d\theta_2 \otimes d\theta_2 \\ &\quad + \cdots + \rho'(r) \sin r \cos r \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} d\theta_{n-1} \otimes d\theta_{n-1}, \end{aligned}$$

i.e.,

$$\begin{aligned} &D^2 \rho(\nabla u, \nabla \bar{u}) \\ &= \rho''(r) |u_r|^2 + \rho'(r) \frac{\cos r}{\sin^3 r} \left(|u_{\theta_1}|^2 + \cdots + \frac{|u_{\theta_{n-1}}|^2}{(\sin \theta_1 \cdots \sin \theta_{n-2})^2} \right) \\ &= (1 - (n-1) \frac{\cos r}{\sin^n r} \int_0^r \sin^{n-1} s ds) |u_r|^2 \\ &\quad + \frac{\cos r}{\sin^n r} \int_0^r \sin^{n-1} s ds \left(\frac{|u_{\theta_1}|^2}{\sin^2 r} + \cdots + \frac{|u_{\theta_{n-1}}|^2}{(\sin r \sin \theta_1 \cdots \sin \theta_{n-2})^2} \right) \\ &\leq g(\nabla u, \nabla \bar{u}). \end{aligned}$$

Following the analysis in section 3 and 4, we can easily obtain Theorem 11 and 12.

Theorem 11. *Consider the Schrödinger equations (3) on $M = S_+^n$. Assume there exists a constant $\kappa \geq 3$ such that*

$$sF(s) \geq \kappa G(s), \quad \forall s \geq 0.$$

Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

Theorem 12. *Consider the Schrödinger equation*

$$\begin{cases} iu_t = \Delta u + |u|^{p-1}u, & \text{on } S^n, \\ u(0) = u_0 \in H^1(S^n). \end{cases}$$

For $p \geq 5$, if $u_0(x) = u_0(r)$, where $r = \text{dist}(N, x)$ and $u_0(r)$ is an asymmetric function at $r = \frac{\pi}{2}$ with $u(\frac{\pi}{2}) = 0$ and $E_0 < 0$, then the asymmetric solution satisfying (4) blows up in finite time.

We next compute the case $M = H^n(-1)$. As above, for $0 < r < \infty$, $0 \leq \theta_1, \dots, \theta_{n-1} < 2\pi$, we have

$$g = dr^2 + \sinh^2 r (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} d\theta_{n-1}^2),$$

and

$$\Delta \rho(r) = \frac{(\rho'(r) \sinh^{n-1} r)'}{\sinh^{n-1} r}.$$

Solving the ODE

$$\Delta \rho(r) = 1 \quad \text{on } (0, \pi/2]$$

we get the desired positive solution

$$\rho(r) = \int_0^r \left(\int_0^s \sinh^{n-1} \tau d\tau \right) (\sinh^{n-1} s)^{-1} ds.$$

By defining $\rho(0) = 0$ we see that $\rho \in C^4(S_+^n)$.

From (12) and (14) we get for all $u \in C^1(S_+^n)$

$$(18) \quad g(\nabla u, \nabla \bar{u}) = |u_r|^2 + \frac{|u_{\theta_1}|^2}{\sinh^2 r} + \dots + \frac{|u_{\theta_{n-1}}|^2}{(\sinh r \sin \theta_1 \dots \sin \theta_{n-2})^2}$$

and

$$\begin{aligned} D^2 \rho(r) &= \rho''(r) dr \otimes dr + \rho'(r) \sinh r \cosh r d\theta_1 \otimes d\theta_1 \\ &\quad + \rho'(r) \sinh r \cosh r \sin^2 \theta_1 d\theta_2 \otimes d\theta_2 \\ &\quad + \dots + \rho'(r) \sinh r \cosh r \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} d\theta_{n-1} \otimes d\theta_{n-1}, \end{aligned}$$

i.e.,

(19)

$$\begin{aligned} &D^2 \rho(\nabla u, \nabla \bar{u}) \\ &= \rho''(r) |u_r|^2 + \rho'(r) \frac{\cosh r}{\sinh^3 r} (|u_{\theta_1}|^2 + \frac{|u_{\theta_2}|^2}{\sin^2 \theta_1} + \dots + \frac{|u_{\theta_{n-1}}|^2}{(\sin \theta_1 \dots \sin \theta_{n-2})^2}) \\ &= (1 - (n-1) \frac{\cosh r}{\sinh^n r} \int_0^r \sinh^{n-1} s ds) |u_r|^2 \\ &\quad + \frac{\cosh r}{\sinh^n r} \int_0^r \sinh^{n-1} s ds \left(\frac{|u_{\theta_1}|^2}{\sinh^2 r} + \dots + \frac{|u_{\theta_{n-1}}|^2}{(\sinh r \sin \theta_1 \dots \sin \theta_{n-2})^2} \right). \end{aligned}$$

Claim:

$$\frac{1}{n} < \frac{\cosh r}{\sinh^n r} \int_0^r \sinh^{n-1} s ds < \frac{1}{n-1}, \quad \forall r > 0.$$

Proof. For $r > 0$, we have

$$\frac{\cosh r}{\sinh^n r} \int_0^r \sinh^{n-1} s ds > \frac{1}{\sinh^n r} \int_0^r \sinh^{n-1} s \cosh s ds = \frac{1}{n},$$

and the left-hand side is proved. For the right-hand side, we write

$$\phi(r) = \frac{\cosh r}{\sinh^n r} \int_0^r \sinh^{n-1} s ds.$$

Assume that $\phi(r)$ achieves its maximum at $r = r_0$, then we have

$$\begin{aligned} \phi'(r_0) &= \sinh^{n-1} r_0 (\cosh r_0 \sinh^n r_0 + \sinh^2 r_0 \int_0^{r_0} \sinh^{n-1} s ds \\ &\quad - n \cosh^2 r_0 \int_0^{r_0} \sinh^{n-1} s ds) / \sinh^{2n} r_0 \\ &= 0, \end{aligned}$$

which gives that

$$\int_0^{r_0} \sinh^{n-1} s ds = \frac{\cosh r_0 \sinh^n r_0}{n \cosh^2 r_0 - \sinh^2 r_0}.$$

Therefore at $r = r_0$, we have

$$\begin{aligned} \phi(r_0) &= \frac{\cosh^2 r_0}{n \cosh^2 r_0 - \sinh^2 r_0} \\ &= \frac{\cosh^2 r_0}{(n-1) \cosh^2 r_0 + 1} < \frac{1}{n-1}. \end{aligned}$$

The upper bound $\frac{1}{n-1}$ is sharp due to the obvious fact

$$\lim_{r \rightarrow \infty} \phi(r) = \frac{1}{n-1}.$$

□

Comparing (18) and (19) with the help of the **Claim**, we obtain that

$$D^2 \rho(\nabla u, \nabla \bar{u}) \leq \frac{1}{n-1} g(\nabla u, \nabla \bar{u}), \quad \forall u \in C^2(H^n(-1)).$$

Following the analysis in section 3 and 4, we can easily obtain Theorem 13 below on the hyperbolic space $H^n(-1)$.

Theorem 13. *Consider the Schrödinger equations (2) on $M = H^n(-1)$. Assume there exists a constant $\kappa \geq 1 + \frac{2}{n-1}$ such that*

$$sF(s) \geq \kappa G(s), \quad \forall s \geq 0.$$

Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

If $F(s) = s^{\frac{p-1}{2}}$ where $p > 1$, the condition $sF(s) \geq \kappa G(s)$ on $[0, +\infty)$ for some $\kappa \geq 1 + \frac{2}{n-1}$ is equivalent to $p \geq 1 + \frac{4}{n-1}$. We state this result as Theorem 14.

Theorem 14. *Consider the Schrödinger equation*

$$\begin{cases} iu_t = \Delta u + |u|^{p-1}u, & \text{on } H^n(-1), \\ u(0) = u_0 \in H^1, \end{cases}$$

where $p \geq 1 + \frac{4}{n-1}$. Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

We remark that the similar result is also true for the complete warped product manifold $M := \mathbb{R}_+ \times \mathbb{B}^{n-1}$ with the metric $g = dr^2 + k(r)^2 ds^2$. Here (\mathbb{B}^{n-1}, ds^2) is a closed manifold of dimension $n - 1$ and $k(r)$ is a non-negative smooth function in $[0, \infty)$ with $k(0) = 0$, $k(r) > 0$ for all $r > 0$, and $k'(0) > 0$. Assume that (M, g) has bounded geometry. Then we choose the function

$$\rho = \rho(r) := \int_0^r k^{-(n-1)}(s) \left(\int_0^s k(\tau)^{n-1} d\tau \right) ds$$

as the weight function in

$$J(u) = \int_M \rho |u|^2.$$

As showed in page 31 of [9], we have

$$\Delta \rho = 1, \quad \rho(0) = 0, \quad \rho' \geq 0,$$

with uniform bounded Hessian

$$D^2 \rho \leq cg$$

for some positive constant c . Then, using the similar argument, we have the following result:

Theorem 15. *Consider the Schrödinger equations (2) on the warped product M as above. Assume there exists a constant $\kappa \geq 2c + 1$ such that*

$$sF(s) \geq \kappa G(s), \quad \forall s \geq 0.$$

Then any solution satisfying (4) with $E_0 < 0$ blows up in finite time.

Finally, we remark that our blow-up result can be extended to the following Schrödinger equations with harmonic potential on the warped product manifold M above:

$$\begin{cases} iu_t = \Delta u + \rho u + |u|^{p-1}u, & \text{on } M, \\ u(0) = u_0 \in H^1(M). \end{cases}$$

However, we prefer not to give the detailed statements here. Using the similar method, we can also set up the blow-up result for Klein-Gordon equations on Riemannian manifolds. We believe our method can also be used to some other evolution systems (see [4]).

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