Enumerating the Classes of Local Equivalency in Graphs

Mohsen Bahramgiri [*]   Salman Beigi[†]

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Abstract

There are local operators on (labeled) graphs G with labels (g_{ij}) coming from a finite field. If the filed is binary, in other words, if the graph is ordinary, the operation is just the local complementation. That is, to choose a vertex and complement the subgraph induced by its neighbors. But, in the general case, there are two different types of operators. The first type is the following. Let v be a vertex of the graph and $a \in \mathbf{F}_q$, the finite field of q elements. The operator is to obtain a graph with labels $g'_{ij} = g_{ij} + ag_{vi}g_{vj}$. For the second type of operators, let $0 \neq b \in \mathbf{F}_q$ and the resulted graph is a graph with labels $g''_{vi} = bg_{vi}$ and $g''_{ij} = g_{ij}$, for i, j unequal to v.

The local complementation operator (binary case) has appeared in combinatorial theory, and its properties have studied in the literature, [4, 5, 6]. Recently, a profound relation between local operators on graphs and quantum stabilizer codes has been found [7, 2], and it has become a natural question to recognize equivalency classes under these operators. In the present article, we show that the number of graphs locally equivalent to a given graph is at most q^{2n+1}, and consequently, the number of classes of local equivalency is $q^{\frac{n^2}{2} - o(n)}$.

1 Introduction

A *labeled graph* is a graph with labeled edges, with labels coming from a (finite) field. This covers the ordinary (simple) graphs, when one restricts the field to be the binary field, $\mathbf{F}_2$. For simplicity, we discuss the binary case separately to make the notion more clear. In the binary case, consider the following operator, called *local complementation*. Choose a vertex, and replace the graph induced by the neighbors of this vertex by its complement. Two graphs are called *locally equivalent* if one can be obtained from the other by applying some local operations described earlier.

In general, when the field is not binary, there are two independent types of operators involved. The first one is just the generalized version of the operation in binary case. Let the graph G be labeled with labels forming a symmetric matrix $G = (g_{ij})$ with zero diagonal, over $\mathbf{F}_q$ where q is a power of a prime number, and $\mathbf{F}_q$ is the field with q elements. Let v be a vertex of this graph, and $a \in \mathbf{F}_q$. We

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[*]Massachusetts Institute of Technology, Mathematics department & Computer Science and Artificial Intelligence Laboratory, m_bahram@mit.edu.

[†]Massachusetts Institute of Technology, Mathematics department, salman@math.mit.edu.

define the first type of operators in the following way. $G *_a v$ is defined to be a graph with labels $G' = (g'_{ij})$ such that $g'_{ij} = g_{ij} + a \cdot g_{vi} g_{vj}$. The second type of operators is multiplying the edges of some vertex by a non-zero number, $b \in \mathbf{F}_q$. In other words, $G \circ_b v$ is the graph with labels $G'' = (g''_{ij})$, where $g''_{vi} = b g_{vi}$ and $g''_{ij} = g_{ij}$ for i, j unequal to v. Similar to the previous situation, two graphs are called locally equivalent if one can obtain one of them by applying the operators $*$ and $\circ$ on the other one.

Studying and investigating the local equivalency of graphs has become a natural problem in quantum computing, and playing a significant role especially in *error-correcting codes*, due to the recent work of [1], [3], [7], [2] and [8] . Namely, in the quantum computing setting, some states, called *stabilizer states*, have a description as the common eigenvector of a subgroup of the *Pauli group*. Using stabilizer states, we may be able to create more preferable *quantum codes*, due to the property that the obtained codes, have relatively shorter description to handle the process. On the other hands, *graph states*, an important subset of stabilizer states, are defined based on graphs with labels in a finite field. Combining the theory of error-correcting codes and the tools in generalized graph theory, leads us to describe and investigate the properties of graph states more and more deeply.

Some stabilizer states may have similar properties. In fact, we can obtain one of the stabilizer states from the other by applying elements of *local Clifford group*. If two states are equivalent under local Clifford group, they present similar properties in quantum computing. The key point is that, any stabilizer state is equivalent to a graph state under the local Clifford group, and consequently, we may just consider the graph states. On the other hand, some of the graph states are equivalent under the local Clifford group. More precisely, shown in [7] and [2], two graph states are equivalent under the local Clifford group if their associated graphs are locally equivalent by the local operations described earlier. The properties of locally equivalent graphs have been deeply studied in the recent works, and an efficient algorithm to determine whether two graphs are locally equivalent or not is given. This algorithm for the binary case can be found in [4], and for the general case in [3].

The purpose of this article is solving a significantly crucial problem in studying the local operations: enumerating the graphs locally equivalent to a given one, as well as enumerating the equivalency classes.

1.1 Main results

The main results proven in the present paper are the followings.

First, The number of graphs locally equivalent to a given one is at most $(q - 1)(q^2 - 1)^n$, which is bounded above by q^{2n+1}, n being the number of vertices of the graph.

Second, $\mathcal{C}(n)$, the number of classes of local equivalency of graphs with n vertices satisfies:
$$q^{\frac{n^2}{2} - \frac{5n}{2} - 1} \leq \mathcal{C}(n) \leq q^{\frac{n^2}{2} - \frac{n}{2}}.$$

In other words,
$$\mathcal{C}(n) = q^{\frac{n^2}{2} - O(n)}.$$

In particular, for the usual (binary) graphs, i.e. when $q = 2$, the number of graphs locally equivalent to a graph is at most 3^n and the number of classes of local equivalency is
$$\mathcal{C}(n) = 2^{\frac{n^2}{2} - O(n)}.$$

1.2 Structure of the paper

This paper is organized as follows; In section 2, we introduce the geometrically known concept of *isotropic systems* and also a relation between isotropic systems and graphs. In fact, we correspond to every graph an isotropic system, and say that graph is a *graphic presentation* for the isotropic system. Then, we show that every isotropic system has a graphic presentation. This, somehow, says that the properties of graphs and isotropic systems are involved.

In section 3, after introducing the definitions of local operators, in theorem 3.1, we translate local equivalency into an algebraic equation, which is significantly helpful throughout this article. We will then prove that two fundamental graphs for an isotropic system are (up to a constant) locally equivalent.

Eulerian vectors, which, roughly speaking, are orthogonal vectors to an isotropic system, are introduced rigorously in section 4. The number of Eulerian vectors for a given isotropic system, say $\mathcal{L}$, is denoted by $\epsilon(\mathcal{L})$, and it is shown that, $\epsilon(\mathcal{L}) \geq 1$ for every $\mathcal{L}$. The notion of *switching property* is introduced in this section, and using its power, the exact number of graphic presentations is given in terms of $\epsilon(\mathcal{L})$.

In section 5, we introduce the notion of *index of an isotropic system*, denoted by $\lambda(\mathcal{L})$, and then we estimate it, from above as well as below, by the terms containing the dimension of the *bineighborhood* space, introduced and studied in this section.

Section 6 is dedicated to enumerating the number of graphs locally equivalent to a fixed one. It will be shown that this number is either $\frac{(q-1)\epsilon(\mathcal{L})}{\lambda(\mathcal{L})}$ or its half, depending on some issues discussed in the section.

Using this result, since we had estimated $\lambda(\mathcal{L})$, the only remained step is to approximate $\epsilon(\mathcal{L})$, the number of Eulerian vectors. This is done in section 7 using *Tutte-Martin polynomial*. Indeed, $\epsilon(\mathcal{L})$, number of Eulerian vectors can be written in term of this polynomial. So, using the known recursive formula of Tutte-Martin polynomial, we estimate $\epsilon(\mathcal{L})$.

In section 8, we put all these parts together to enumerate the classes of local equivalency.

2 Isotropic systems and graphic presentations

Assume that q is a power of p, which is an odd prime number, and $\mathbf{F}_q$ is the finite field with q elements. Also, let $\mathbf{K} = \mathbf{F}_q^2$ denote a two-dimensional vector space over $\mathbf{F}_q$, associated with the bilinear form $\langle.,.\rangle$ satisfying
$$\langle (x,y), (x',y') \rangle = xy' - x'y$$

for every $(x, y), (x', y') \in \mathbf{K}$. For a set V of n elements, define $\mathbf{K}^V$ to be the $2n$-dimensional vector space over $\mathbf{F}_q$, equipped with the bilinear form

$$\langle A, A' \rangle = \sum_{v \in V} \langle A(v), A'(v) \rangle.$$

For $X, Y \in \mathbf{F}_q^V$, let $X \times Y$ be a vector in $\mathbf{F}_q^V$ such that $(X \times Y)(v) = X(v)Y(v)$. Also, for $v \in V$ and $(x, y) \in \mathbf{K}$, let $E_{v,(x,y)}$ be a vector in $\mathbf{K}^V$ where its coordinates are all zero except the v-th one which is equal to (x, y), i.e., $E_{v,(x,y)}(w) = \delta_{vw}(x, y)$ for every $w \in V$. For any vector $A \in \mathbf{K}^V$, let $A_v = E_{v,A(v)}$.

For simplicity, we present a vector A in $\mathbf{K}^V$ as $A = (X, Y)$, where $X, Y \in \mathbf{F}_q^V$. Therefore, $A(v) = (X(v), Y(v))$. Also, for $X \in \mathbf{F}_q^V$ let $diag\ X$ be an $n \times n$ diagonal matrix where $(diag\ X)_{vv} = X(v)$, and for the $2n \times 2n$ matrix

$$D = \begin{pmatrix} diag\ X & diag\ Y \\ diag\ X' & diag\ Y' \end{pmatrix},$$

define

$$det\ D = diagX\ diagY' - diagX'\ diagY,$$

being an $n \times n$ diagonal matrix. Therefore, $\langle A, A' \rangle = tr(det\ D)$ where tr is the usual trace function. [1]

Definition 2.1 *An isotropic system $\mathcal{L}$ is an n-dimensional subspace of $\mathbf{K}^V$ where $|V| = n$, such that $\langle A, B \rangle = 0$ for every $A, B \in \mathcal{L}$. In other words, $\mathcal{L}$ is a subspace which is orthogonal to itself.*

Note that, $\langle ., . \rangle$ is non-degenerate and since $\mathcal{L}$ has dimension n, therefore

$$\mathcal{L} = \{A \in \mathbf{K}^V : \langle A, B \rangle = 0,\ \forall B \in \mathcal{L}\}.$$

In fact, if we fix a set of generators $\{A_1 = (X_1, Y_1), \dots A_n = (X_n, Y_n)\}$ for $\mathcal{L}$ and construct the $n \times 2n$ matrix

$$\mathcal{B} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} X_1 & | & Y_1 \\ X_2 & | & Y_2 \\ & \vdots & \\ X_n & | & Y_n \end{pmatrix}, \tag{1}$$

then $\mathcal{L}$ is an isotropic system if and only if $\mathcal{B}$ is a full-rank matrix (and hence $\dim \mathcal{L} = n$) and

$$\mathcal{B} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{B}^T = 0,$$

Because, the ij-th entry of this matrix is

$$X_i Y_j^T - Y_i X_j^T = \langle A_i, A_j \rangle.$$

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[1]Note that, the usual determinant of D is equal to $det(det\ D)$, which is the multiplication of diagonal entries of $det\ D$.

We will shortly prove that every isotropic system can be defined based on graphs. To start with, we introduce these graph-based isotropic systems. Suppose that $G = (V, E)$ is a simple labeled graph (without loops and multiple edges), where the label of each edge comes from $\mathbf{F}_q$. Thus, we can represent the graph by an $n \times n$ matrix $G = (g_{vw})$, where n is equal to $|V|$, the number of vertices in the graph, and for every $v, w \in V$, g_{vw} is equal to the label of the edge vw. So, G is a symmetric matrix with zero diagonal. Assume that $A = (X, Y)$ and $B = (Z, T)$ are in $\mathbf{K}^V$ such that $diagZ \ diagY - diagX \ diagT = cI$, where I is the identity matrix and $c \in \mathbf{F}_q$ is a non-zero constant. Denote by $\mathcal{L}$ the vector space generated by all vectors $g(v)(diagX \mid diagY) + B_v$ for $v \in V$, where $g(v)$ denotes the v-th row of G. In fact, rows of matrix

$$(I \mid G) \cdot D \tag{2}$$

form a basis for $\mathcal{L}$, where

$$D = \begin{pmatrix} diag \ Z & diag \ T \\ diag \ X & diag \ Y \end{pmatrix}.$$

In order to prove that $\mathcal{L}$ is an isotropic system, first note that $(I \mid G)$ is a full-rank matrix. Also, the determinant of D is c^n which is non-zero, and hence D is full-rank as well, and since D is a square matrix, $(I \mid G) \cdot D$ is full rank too. On the other hand, we need to show that the rows of $(I \mid G) \cdot D$ are orthogonal to each other, or equivalently,

$$(I \mid G) \cdot D \cdot \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \cdot \left((I \mid G) \cdot D \right)^T = 0.$$

But, the left hand side is equal to

$$
\begin{aligned}
&= (I \mid G) \begin{pmatrix} diag \ Z & diag \ T \\ diag \ X & diag \ Y \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} diag \ Z & diag \ X \\ diag \ T & diag \ Y \end{pmatrix} \begin{pmatrix} I \\ G \end{pmatrix} \\
&= (I \mid G) \begin{pmatrix} diag \ Z & diag \ T \\ diag \ X & diag \ Y \end{pmatrix} \begin{pmatrix} diag \ T & diag \ Y \\ -diag \ Z & -diag \ X \end{pmatrix} \begin{pmatrix} I \\ G \end{pmatrix} \\
&= (I \mid G) \begin{pmatrix} 0 & cI \\ -cI & 0 \end{pmatrix} \begin{pmatrix} I \\ G \end{pmatrix} \\
&= c(I \mid G) \begin{pmatrix} G \\ -I \end{pmatrix} = c(G - G) = 0.
\end{aligned}
$$

Therefore, $\mathcal{L}$ is an isotropic system.

Definition 2.2 *Suppose that $\mathcal{L}$ is an n-dimensional isotropic system for which there exist a graph G and vectors $A = (X, Y)$ and $B = (Z, T)$ in K^V such that, $det \ D(A, B) = cI$ for some $0 \neq c \in \mathbf{F}_q$, where*

$$D(A, B) = \begin{pmatrix} diagZ & diagT \\ diagX & diagY \end{pmatrix},$$

and $(I \mid G) \cdot D(A, B)$ is a basis for $\mathcal{L}$. If such G and A, B exist, we say (G, A, B) is a graphic presentation of $\mathcal{L}$, and G is a fundamental graph of $\mathcal{L}$.

Theorem 2.1 *Every isotropic system $\mathcal{L}$ has a graphic presentation.*

Proof: We have already shown that every subspace which addmits a graphic presentation is an isotropic system. Hence, it is sufficient to prove that every isotropic system has a basis of the form (2).

Consider an arbitrary basis for $\mathcal{L}$ and put them in the rows of an $n \times 2n$ matrix $\mathcal{B}$ to get a matrix of the form (1). Notice that, if we change the v-th column of the first block of $\mathcal{B}$ with v-th column of second block, we come up with a basis for another isotropic system and it is equivalent to multiplying this matrix with a $2n \times 2n$ matrix D_1 which consists of four $n \times n$ diagonal matrices (in fact, just two of these matrices are non-zero). Among all such matrices $\mathcal{B}D_1$, choose the one in which the rank of its first block is the maximum possible, namely r. Now note that, in K^V changing the order of coordinates is equivalent to changing the order of columns of $\mathcal{B}$ in the first and the second blocks. In fact, it is equivalent to multiplying $\mathcal{B}$ by a $2n \times 2n$ permutation matrix from the right hand side, i.e., by a matrix of the form

$$\Pi = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}, \tag{3}$$

where π is a permutation matrix over n elements. We find the permutation π such that in $\mathcal{B}D_1\Pi$ the first r columns of the first block are linearly independent. Then, there exists an invertible matrix U such that

$$U\mathcal{B}D_1\Pi = \begin{pmatrix} I_r & \alpha & | & \beta & \gamma \\ 0 & \zeta & | & \eta & \theta \end{pmatrix}.$$

Due to the properties of the matrix D_1 and the maximality assumption, we have $\zeta = 0$ and $\theta = 0$. Since, rows of matrix $U\mathcal{B}D_1\Pi$ form a basis for an isotropic system, and because of the orthogonality assumption, we conclude that $\eta = 0$. Therefore, the rank of whole matrix is r. But this is a basis for an isotropic system of dimension n. Thus, $r = n$, and we have

$$U\mathcal{B} = (I_n \mid \beta)\Pi^{-1}D_1^{-1} = (\pi^{-1} \mid \beta\pi^{-1})D_1^{-1}.$$

Therefore, $\pi U\mathcal{B} = (I \mid \pi\beta\pi^{-1})D_1^{-1}$. The matrix $\pi\beta\pi^{-1}$ may have non-zero diagonal entries, but by multiplying $(I \mid \pi\beta\pi^{-1})$ by a $2n \times 2n$ matrix with four diagonal blocks, one can obtain a matrix with the identity matrix in the first block, and the second block the same as before, except that the diagonal entries are all zero. Considering this multiplication, we end up with $\pi U\mathcal{B} = (I \mid G')D'$, where D' is the described $2n \times 2n$ matrix with four diagonal blocks. Note that both matrices π and U are invertible, so that the rows of $\pi U\mathcal{B}$ are still a basis for $\mathcal{L}$. Let

$$D' = \begin{pmatrix} diagZ' & diagT' \\ diagX' & diagY' \end{pmatrix}.$$

By considering the orthogonality assumption, we conclude that

$$
\begin{aligned}
0 &= (I \mid G')D' \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} D'^T (I \mid G')^T \\
&= (I \mid G') \begin{pmatrix} diagZ' & diagT' \\ diagX' & diagY' \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} diagZ' & diagX' \\ diagT' & diagY' \end{pmatrix} \begin{pmatrix} I \\ G'^T \end{pmatrix} \\
&= (I \mid G') \begin{pmatrix} 0 & detD' \\ -detD' & 0 \end{pmatrix} \begin{pmatrix} I \\ G'^T \end{pmatrix} \\
&= detD' \; G'^T - G' \; detD'.
\end{aligned}
$$

Therefore, $detD'G'^T = G' \; detD'$. It means that the matrix $G = G' \; detD'$ is symmetric. Moreover, it has a zero diagonal since G' does. We have

$$
\begin{aligned}
\pi U \mathcal{B} &= (I \mid G) \begin{pmatrix} I & 0 \\ 0 & detD'^{-1} \end{pmatrix} D' \\
&= (I \mid G) \begin{pmatrix} diagZ' & diagT' \\ detD'^{-1}diagX' & detD'^{-1}diagY' \end{pmatrix}.
\end{aligned}
$$

Hence, if we define

$$
D = \begin{pmatrix} diagZ' & diagT' \\ detD'^{-1}diagX' & detD'^{-1}diagY' \end{pmatrix},
$$

then $\pi U \mathcal{B} = (I \mid G)D$ and $det \; D = I$. This is a basis of the form (2), and the proof is completed.

$\square$

3 Fundamental graphs of isotropic systems

In the previous section we proved that every isotropic system admits a fundamental graph. But, this fundamental graph is not unique. In order to study these different fundamental graphs for an isotropic system, we present a couple of definitions.

Definition 3.1 *Let G be a graph over the vertex set V. For $v \in V$ and a number $r \in \mathbf{F}_q$, define $G *_r v$ to be a graph (more precisely, a symmetric matrix with zero diagonal) $G' = (g'_{uw})$, such that for every w, $g'_{vw} = g_{vw}$, and also for every u, w unequal to v,*

$$
g'_{uw} = g_{uw} + r g_{vu} g_{vw}.
$$

Moreover, for a non-zero number $s \in \mathbf{F}_q$, define $G \circ_s v$ to be a graph $G' = (g'_{uw})$, such that for each u, $g'_{uv} = s g_{uv}$, and also for each u, w unequal to v, $g'_{uw} = g_{uw}$.

Two graphs G and G' are called locally equivalent if there exists a sequence of the above operations, acting on G obtains G'.

Notice that, the operations $*$ and $\circ$ are invertible, so that local equivalency is really an equivalence relation.

The following theorem is proved in [3], and we do not repeat the proof here.

Theorem 3.1 *Two graphs G and G' are locally equivalent if and only if there exists an invertible $n \times n$ matrix U and a $2n \times 2n$ matrix D with four diagonal blocks satisfying $det D = I$ and $U(I \mid G)D = (I \mid G')$.*

<div align="right">□</div>

For a non-zero number $c \in \mathbf{F}_q$, let $G' = cG$ be the usual product of a matrix G and a constant c, i.e., $g'_{vw} = cg_{vw}$ for any v, w.

Theorem 3.2 *For any two fundamental graphs G and G' of an isotropic system, there exists a non-zero $c \in \mathbf{F}_q$ such that cG and G' are locally equivalent. Conversely, if for some non-zero number c, the graphs cG and G' are locally equivalent, then there is an isotropic system such that G and G' are its fundamental graphs.*

Proof: Suppose that (G, A, B) and (G', A', B') are two graphic presentations for the isotropic system $\mathcal{L}$. It means that, the rows of each of the matrices $(I \mid G)D(A, B)$ and $(I \mid G')D(A', B')$ form a basis for $\mathcal{L}$. Therefore, there exists an invertible matrix U such that

$$U(I \mid G)D(A, B) = (I \mid G')D(A', B').$$

Hence

$$U(I \mid G)D(A, B)D(A', B')^{-1} = (I \mid G').$$

Note that, both $det D(A, B)$ and $det D(A', B')$ are (non-zero) constant numbers, and $det D(A', B')^{-1}$ is also a constant. Thus, there exists a non-zero number $c \in \mathbf{F}_q$ such that $det\big(D(A, B)D(A', B')^{-1}\big) = cI$. Now let

$$D = \begin{pmatrix} I & 0 \\ 0 & c^{-1}I \end{pmatrix} D(A, B)D(A', B')^{-1}.$$

$det D = I$ and we have

$$U(I \mid G) \begin{pmatrix} I & 0 \\ 0 & cI \end{pmatrix} D = (I \mid G').$$

Then, $U(I \mid cG)D = (I \mid G')$ and the first part of the conclusion follows from theorem 3.1.

Conversely, suppose that cG and G' are locally equivalent. Therefore, there exist matrices U and D such that U is invertible, $det D = I$ and $U(I \mid cG)D = (I \mid G')$. Also, suppose that A and B are two vectors such that $D = D(A, B)$. Therefore, (G, cA, B) and (G', A', B') are two graphic presentations for the same isotropic system, where A', B' are defined so that $D(A', B') = I_{2n}$. More precisely, $A'(v) = (0, 1)$ and $B'(v) = (1, 0)$ for each v.

<div align="right">□</div>

Having this theorem in hand, we can now study the classes of local equivalency of graphs by investigating different graphic presentations of an isotropic system.

4 Eulerian vectors and local complementation

We call a vector $A \in \mathbf{K}^V$ *complete* if $A(v)$ is non-zero for all $v \in V$. Assume that $A \in \mathbf{K}^V$ is complete. We define $\hat{A}$ to be the vector subspace generated by vectors A_v for all $v \in V$, i.e., $\hat{A} = \langle A_v : v \in V \rangle$. In fact, if $A = (X, Y)$ then $\hat{A}$ is the vector space generated by rows of $(diagX, diagY)$.

Definition 4.1 *Let $\mathcal{L}$ be an isotropic system and $A \in \mathbf{K}^V$ be a complete vector. We call A an Eulerian vector for $\mathcal{L}$ if $\hat{A} \cap \mathcal{L} = 0$.*

Lemma 4.1 *Suppose that $\mathcal{L}$ is an isotropic system and (G, A, B) is a graphic presentation for $\mathcal{L}$. Then A is an Eulerian vector for $\mathcal{L}$.*

Proof: By definition of the graphic presentation we know that $detD(A, B)$ is a non-zero constant. Therefore, $A(v)$ is non-zero for any v. In fact, the v-th entry of the diagonal of $detD(A, B)$ is $\langle B(v), A(v) \rangle$, so that $A(v)$ should be non-zero and then A is complete. To get a contradiction, suppose that $\hat{A} \cap \mathcal{L}$ is non-zero. Therefore, if $A = (X, Y)$ then there exists a non-zero vector $S \in \mathbf{F}_q^n$ such that $S(diagX|diagY) \in \mathcal{L}$. S is non-zero, hence at least one of its coordinates , say v-th, is non-zero. By considering the orthogonality condition, we have

$$
\begin{aligned}
0 &= \langle S(diagX|diagY), g(v)(diagX|diagY) + B_v \rangle \\
&= \sum_{w \in V} S(w) g_{vw} \langle (X(w), Y(w)), (X(w), Y(w)) \rangle + \langle S(diagX|diagY), B_v \rangle \\
&= S(v) \langle A(v), B(v) \rangle.
\end{aligned}
$$

But, $S(v)$ is non-zero and also $\langle A(v), B(v) \rangle$ is non-zero since $detD(A, B)$ is so, which is a contradiction.

$\square$

Corollary 4.1 *Every isotropic system has an Eulerian vector.*

Proof: By theorem 2.1, every isotropic system has a graphic presentation (G, A, B) and by lemma 4.1, A is an Eulerian vector for the isotropic system.

$\square$

Let $\mathcal{L}$ be an isotropic system and A an Eulerian vector for $\mathcal{L}$. Therefore $\mathcal{L} \cap \hat{A} = 0$. If A' is a vector which is equal to A at any coordinate except the v-th one, at which it is equal to a non-zero multiple of $A(v)$, then $\hat{A}' = \hat{A}$, and therefore A' is also an Eulerian vector for $\mathcal{L}$.

This observation gives us the motivation of defining $\mathbf{K}^*$ to be $\mathbf{K} \backslash \{0\}$ under the equivalency relation $(x, y) \sim (x', y')$ iff $(x, y) = r(x', y')$, for some non-zero r, (and hence $| \mathbf{K}^* | = q + 1$). Now by the above discussion if we replace each coordinate of A with something equivalent to it, we obtain another Eulerian vector. The set of Eulerian vectors of an isotropic system has even more useful properties.

4.1 Switching property

Definition 4.2 *We say that a subset $\sum \subseteq \mathbf{K}^V$ of complete vectors has the switching property if*

(i) Similar to Eulerian vectors, for each $A \in \sum$, one can replace each coordinate of A by its (scalar) multiple and it still remains in this subset.

(ii) In addition, for each $A \in \sum$ and $v \in V$, $A - A_v + rE_{v,(x,y)}$ is still in $\sum$ for each non-zero $r \in \mathbf{F}_q$ and every $(x,y) \in \mathbf{K}^*$ except one (x,y). In other words, in a set with switching property and a vector A in this set, we can replace a coordinate of A with exactly q elements of $\mathbf{K}^*$ so that it still remains in $\sum$.

We will observe shortly that switching at the vertex v is equivalent to a local complementation operation on this vertex.

Theorem 4.1 *The set of Eulerian vectors of an isotropic system has the switching property.*

Proof: Assume that A is an Eulerian vector for the isotropic system $\mathcal{L}$, and to lead to a contradiction, suppose that (x_i, y_i), $i = 1, 2$, are two different vectors in $\mathbf{K}^*$ and $A_i = A - A_v + E_{v,(x_i,y_i)}$, $i = 1, 2$, are not Eulerian (we know that all of these vectors $A - A_v + E_{v,(x,y)}$, $(x,y) \in \mathbf{K}^*$ can not be Eulerian, since if so, for every $C \in \mathcal{L}, C(v) = 0$ and the dimension of $\mathcal{L}$ could not be equal to n). Therefore there exist non-zero vectors $C_i \in \hat{A}_i \cap \mathcal{L}$ for $i = 1, 2$. The v-th coordinate of C_i can not be zero, because otherwise, $C_i \in \hat{A} \cap \mathcal{L}$, which is not possible since A is Eulerian. Therefore, the v-th coordinate of C_i is a non-zero multiple of (x_i, y_i), $i = 1, 2$. Now notice that (x_1, y_1) and (x_2, y_2) are different elements of $\mathbf{K}^*$, thus they are linearly independent and there exist $r_1, r_2 \in \mathbf{F}_q$ such that $A(v) = r_1(x_1, y_1) + r_2(x_2, y_2)$. Therefore $r_1 C_1 + r_2 C_2$ is a non-zero vector in $\hat{A} \cap \mathcal{L}$, which is a contradiction.
$\square$

Theorem 4.2 *For every isotropic system $\mathcal{L}$ and a graphic presentation (G, A, B) of it, A is an Eulerian vector. Conversely, for every Eulerian vector A, there exists a graphic presentation (G, A, B). Also, this graphic presentation is unique up to a (non-zero) constant, i.e., if (G', A, B') is another graphic presentation for $\mathcal{L}$ then there exists a non-zero number $c \in \mathbf{F}_q$ such that $G' = cG$ and $B' = cB$.*

Proof: The first part of the theorem was already proved in lemma 4.1. For the second part, suppose that $A = (X, Y)$ is an Eulerian vector of the isotropic system $\mathcal{L}$. By the switching property, for every $v \in V$, there exists some $(z_v, t_v) \in \mathbf{K}$ such that $(z_v, t_v) \not\sim A(v)$ (meaning that (z_v, t_v) and $A(v)$ are not scalar multiples of each other) and there is a vector of the form $C_v + E_{v,(z_v,t_v)}$ in $\mathcal{L}$, where C_v is in $\hat{A}$ and $C_v(v) = 0$. Since $(z_v, t_v) \not\sim A(v)$, we have $\langle (z_v, t_v), A(v) \rangle \neq 0$ and by considering an scalar multiple (if necessary) of (z_v, t_v), we may assume that

$$\langle (z_v, t_v), A(v) \rangle = 1. \tag{4}$$

We know that $C_v \in \hat{A}$ and $C_v(v) = 0$, thus there exists a matrix $G = (g_{uw})$ over $\mathbf{F}_q$ such that $g_{vv} = 0$ and $C_v = g(v)(diagX|diagY)$, for every v. Here, by $g(v)$ we mean the v-th row of G.

Now, Let $B = (Z, T)$ be the vector in $\mathbf{K}^V$ with $Z(v) = z_v$ and $T(v) = t_v$ for every $v \in V$. Due to the equation (4), we observe that $detD(A, B) = I$ and also $D(A, B)$ is an invertible matrix. Using this notation, the rows of $(I \mid G)D(A, B)$ are all in $\mathcal{L}$. Since, this matrix is a full-rank one, its rows form a basis for $\mathcal{L}$. Therefore, once we show that G is the matrix for a graph, we will end up with the presentation (G, A, B) for $\mathcal{L}$.

To show that G is a graph, first note that $g_{vv} = 0$ by its definition. For proving that G is symmetric, consider again the orthogonality assumption. We have

$$
\begin{aligned}
0 &= (I \mid G) \begin{pmatrix} diagZ & diagT \\ diagX & diagY \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} diagZ & diagX \\ diagT & diagY \end{pmatrix} \begin{pmatrix} I \\ G^T \end{pmatrix} \\
&= (I \mid G) \begin{pmatrix} diagZ & diagT \\ diagX & diagY \end{pmatrix} \begin{pmatrix} diagT & diagY \\ -diagZ & -diagX \end{pmatrix} \begin{pmatrix} I \\ G^T \end{pmatrix} \\
&= (I \mid G) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I \\ G^T \end{pmatrix} \\
&= G^T - G.
\end{aligned}
$$

Therefore, G is a graph, and (G, A, B) is a graphic presentation of $\mathcal{L}$.

For the uniqueness, suppose that (G', A, B') is another graphic presentation for $\mathcal{L}$ such that $detD(A, B') = cI$. It means that $(I \mid G')D(A, B')$ is also a basis for $\mathcal{L}$. Let $B' = (Z', T')$ and by considering the orthogonality assumption, once again we have

$$
\begin{aligned}
0 &= (I \mid G) \begin{pmatrix} diagZ & diagT \\ diagX & diagY \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} diagZ' & diagX \\ diagT' & diagY \end{pmatrix} \begin{pmatrix} I \\ G' \end{pmatrix} \\
&= (I \mid G) \begin{pmatrix} detD(B', B) & detD(A, B) \\ -detD(A, B') & 0 \end{pmatrix} \begin{pmatrix} I \\ G' \end{pmatrix} \\
&= (I \mid G) \begin{pmatrix} detD(B', B) + G' \\ -cI \end{pmatrix} \\
&= detD(B', B) + G' - cG.
\end{aligned}
$$

Therefore, $detD(B', B) + G' - cG = 0$ and since $detD(B', B)$ is a diagonal matrix and the diagonals of G and G' are both equal to zero, we have $detD(B', B) = 0$ and $G' = cG$. Hence, it just remains to show $B' = cB$. Because of the equation

$$
0 = detD(B', B) = diagZ \ diagT' - diagZ' \ diagT,
$$

for any v there exists a $c_v \in \mathbf{F}_q$ such that $(Z'(v), T'(v)) = c_v(Z(v), T(v))$. Therefore $D(A, B')_{vv} = c_v D(A, B)_{vv}$. On the other hand $detD(A, B) = I$ and $detD(A, B') = cI$, so that $c_v = c$ for any v. Hence $B' = cB$.

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$\square$

Using this theorem, if we denote by $\epsilon(\mathcal{L})$ the number of Eulerian vectors of the isotropic system $\mathcal{L}$, we conclude the following corollary.

Corollary 4.2 *The number of graphic presentations of an isotropic system is equal to $(q-1)\epsilon(\mathcal{L})$.*

4.2 Switching property in terms of local complementation

The following theorem, explains the relationship between the switching property and local complementation.

Theorem 4.3 *Suppose that (G, A, B) is a graphic presentation for the isotropic system $\mathcal{L}$, and $v \in V$.*

(i) If $r \in \mathbf{F}_q$, then $(G *_r v, A + rB_v, B + rg(v)^2 \times A)$ is also a graphic presentation of $\mathcal{L}$. Therefore switching A at v is equivalent to a local complementation operator.

(ii) If $s \in \mathbf{F}_q$ is non-zero, then $(G \circ_s v, A + (s^{-1} - 1)A_v, B + (s-1)B_v)$ is also a graphic presentation of $\mathcal{L}$.

Proof: We prove first part, and the second part is similar. It is easy to check that $det D(A + rB_v, B + rg(v)^2 \times A)$ is constant. Hence, it is sufficient to show that all of the rows of $(I \mid G *_r v)D(A + rB_v, B + rg(v)^2 \times A)$ are in $\mathcal{L}$. Let $G' = G *_r v$ and $w \in V$, $w \neq v$. We have $g'(w) = g(w) + rg_{vw}g(v) - rg_{vw}^2\delta_w$, thus the w-th row of $(I \mid G')D(A + rB_v, B + rg(v)^2 \times A)$ is equal to

$$
\begin{aligned}
&= \; g'(w) \times (A + rB_v) + (B_w + rg_{vw}^2 A_w) \\
&= \; (g(w) + rg_{vw}g(v) - rg_{vw}^2\delta_w) \times (A + rB_v) + (B_w + rg_{vw}^2 A_w) \\
&= \; g(w) \times A + rg_{vw}g(v) \times A - rg_{vw}^2 A_w + rg_{vw}B_v + B_w + rg_{vw}^2 A_w \\
&= \; (g(w) \times A + B_w) + rg_{vw}(g(v) \times A + B_v),
\end{aligned}
$$

which is in $\mathcal{L}$. Also, for the v-th row, $g'(v) = g(v)$ and

$$g(v) \times (A + rB_v) + (B_v + rg^2(v) \times A_v) = g(v) \times A + B_v$$

which is an element of $\mathcal{L}$.

$\square$

5 Index of an isotropic system

We are now in the position of introducing the notion of *index* for an isotropic system.

Theorem 5.1 *For any isotropic system $\mathcal{L}$, there exists a number $\lambda(\mathcal{L})$ such that for any fundamental graph G for $\mathcal{L}$, there are exactly $\lambda(\mathcal{L})$ pairs (A, B) such that (G, A, B) is a graphic presentation of $\mathcal{L}$. This number is called the index of the isotropic system $\mathcal{L}$.*

Proof: Suppose that G is a fundamental graph of $\mathcal{L}$, and there exist exactly k graphic presentations of the form (G, A_i, B_i), $i = 1, \ldots, k$ for $\mathcal{L}$. It is sufficient to show that for any other fundamental graph H for $\mathcal{L}$, there are also k graphic presentation with H as a fundamental graph. Since, G and H are fundamental graphs of the same isotropic system, by theorems 3.1 and 3.2, there exist invertible matrices U and D, such that D consists of four diagonal blocks and $det D = cI$ for some non-zero $c \in \mathbf{F}_q$, and moreover,

$$U(I \mid H)D = (I \mid G). \tag{5}$$

Since, (G, A_i, B_i) is a graphic presentation of $\mathcal{L}$ for $i = 1, \ldots, k$, the rows of $(I \mid G)D(A_i, B_i)$ form a basis for $\mathcal{L}$. Now using (5), we conclude that the rows of $U(I \mid H)D\ D(A_i, B_i)$ and hence the rows of $(I \mid H)D\ D(A_i, B_i)$ form bases for $\mathcal{L}$.

Notice that $det D\ D(A_i, B_i)$ is a constant. Therefore, it gives us a graphic presentation of $\mathcal{L}$ with fundamental graph H, for $i = 1, \ldots, k$. Also, since the matrices U, D and $D(A_i, B_i)$ are invertible, the described k presentations are different. Moreover, for any presentation with fundamental graph H, we can convert it to a presentation with fundamental graph G. Thus, for any fundamental graph of $\mathcal{L}$, there exist exactly $\lambda(\mathcal{L}) = k$ graphic presentations with this graph as a fundamental graph.
□

Theorem 5.2 *Assume that $\mathcal{L}$ is an isotropic system admitting G as a fundamental graph. Then $\lambda(\mathcal{L})$ is equal to the number of matrices of the form $D(A, B)$, with non-zero constant determinant, and*

$$(I \mid G)D(A, B)\begin{pmatrix} G \\ -I \end{pmatrix} = 0. \tag{6}$$

In fact, $\lambda(\mathcal{L}) = \lambda(G)$, meaning that the index of an isotropic system just depends on any arbitrary fundamental graph.

Proof: As in the proof of theorem 5.1, suppose that (G, A_i, B_i), $i = 1, \ldots, k$, are all graphic presentations of $\mathcal{L}$ with fundamental graph G. Using the orthogonality assumption we have

$$
\begin{aligned}
0 &= (I \mid G)D(A_1, B_1)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}D(A_i, B_i)^T\begin{pmatrix} I \\ G \end{pmatrix} \\
&= (I \mid G)D(A_1, B_1)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}D(A_i, B_i)^T\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} G \\ -I \end{pmatrix}.
\end{aligned}
$$

On the other hand, the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}D(A_i, B_i)^T\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

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consists of four diagonal matrices and has non-zero constant determinant, and also,

$$D(A_1, B_1) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} D(A_i, B_i)^T \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

satisfies all of these properties. Therefore, for each i, where $i = 1, \ldots, k$, we find a solution of (6). Conversely, since all of the above equations can be inverted, for each solution $D(A, B)$ we can find one of the graphic presentations (G, A_i, B_i).

$\square$

Corollary 5.1 *For an isotropic system that admits G as a fundamental graph, the number of its graphic presentations is equal to $\lambda(G)$ times the number of graphs that are locally equivalent to cG for some non-zero $c \in \mathbf{F}_q$.*

5.1 Bineighborhood space and index of a graph

For a graph G, we call a pair vw of vertices an *edge*, if $g_{vw} \neq 0$. Suppose that, C is an even cycle (a cycle with an even number of edges) consisting of vertices $v_1, v_2, \ldots, v_{2l}$. Set

$$\nu(C) = \sum_{i=1}^{2l} (-1)^i g_{v_i v_{i+1}} g(v_i) \times g(v_{i+1}).$$

Definition 5.1 *Suppose that G is a graph. The bineighborhood space of G, denoted by $\nu(G)$, is a subspace of $\mathbf{F}_q^V$ defined by*

$$\nu(G) = span\{\nu(C) : C \text{ even cycle}\} \cup \{g(v) \times g(w) : g_{vw} = 0\}.$$

We assume that the graphs we consider are connected, and restate a couple of theorems, which will be used shortly. These theorems are all proved in [3].

Lemma 5.1 *If $D(A, B)$, where $A = (X, Y)$ and $B = (Z, T)$, satisfies (6) then $Y + Z$ is a scalar multiple of $(1, 1, \ldots, 1)$. On the other hand, for any such vectors A, B, the matrix $D(A, B) + cI_{2n}$ satisfies (6) for each $c \in \mathbf{F}_q$.*

$\square$

Theorem 5.3 *Suppose that $D(A, B)$ satisfies (6) and $A = (X, Y)$. Then $X \in \nu(G)^\perp$. On the other hand for any $X \in \nu(G)^\perp$,*

(i) *if G has an odd cycle, then there exist a unique Y and a unique T such that $D(A, B)$ satisfies (6), where $A = (X, Y)$ and $B = (-Y, T)$.*

(ii) *if G does not have an odd cycle, then there exist exactly q pairs of Y_i, T_i, $i = 1, \ldots, q$, such that $D(A_i, B_i)$ satisfies (6), where $A = (X, Y_i)$ and $B = (-Y_i, T_i)$.*

$\square$

Theorem 5.4 *If $D(A, B)$ satisfies (6) for some vectors A and B, then it has a constant determinant.*

$\square$

Having all of these theorems in hand, we conclude the following statement.

Theorem 5.5

(i) *If G has an odd cycle, then for any $X \in \nu(G)^{\perp}$ there are exactly q (A, B)'s, where the first component of A is X, and $D(A, B)$ satisfies (6). Among all of these q solutions, either all of them or $q - 2$ of them have non-zero constant determinant.*

(ii) *If G has no odd cycle, then for any $X \in \nu(G)^{\perp}$ there are exactly q^2 (A, B)'s where the first component of A is X and $D(A, B)$ satisfies (6). Among all of these q^2 solutions, at least $q(q - 2)$ of them have non-zero constant determinant.*

Proof: Suppose that G has an odd cycle, and fix some $X \in \nu(G)^{\perp}$. By lemma 5.1 and theorem 5.3, there are $A_0 = (X, Y_0)$ and $B_0 = (-Y_0, T)$ such that $D(A_0, B_0)$ satisfies (6), and any other solution $D(A, B)$ of (6), where the first component of A is X, is of the form $D(A, B) = D(A_0, B_0) + c I_{2n}$, for any $c \in \mathbf{F}_q$. Hence, there are q solutions with this property. Notice that, $det D(A, B) = det D(A_0, B_0) + c^2 I$, and by theorem 5.4, $det D(A_0, B_0) = d_0 I$ is constant. Thus, depending on whether $-d_0 \in \mathbf{F}_q$ is a perfect square or not, there are either $q - 2$ or q different values of $c \in \mathbf{F}_q$ such that $d_0 + c^2$ is non-zero.

The proof of (ii) is exactly the same. The only difference is that in this case there are q solutions of the form $A = (X, Y)$, $B = (-Y, T)$ for (6).

□

We can now give an estimation on $\lambda(G)$ for a graph G.

Corollary 5.2 *If a graph G has an odd cycle then*

$$(q - 2)q^{dim\,\nu(G)^{\perp}} \leq \lambda(G) \leq q^{dim\,\nu(G)^{\perp}+1},$$

and if not

$$(q - 2)q^{dim\,\nu(G)^{\perp}+1} \leq \lambda(G) \leq q^{dim\,\nu(G)^{\perp}+2}.$$

6   The number of graphs locally equivalent to a fixed one

In this section we plan to compute $l(G)$, the number of graphs which are locally equivalent to the graph G. Once again, we assume that all the graphs we consider are connected.

Lemma 6.1 $l(G) = l(cG)$ *for any non-zero $c \in \mathbf{F}_q$.*

Proof: It is not hard to see that, for any graph H locally equivalent to G, cH is locally equivalent to cG.

□

Lemma 6.2 *If $c \in \mathbf{F}_q$ is a non-zero perfect square, then G and cG are locally equivalent.*

Proof: Suppose that $c = d^2$, and apply the operation $\circ_d v$ on G for all $v \in V$, and the resulted graph is cG.

$\square$

Theorem 6.1 *The number of graphs locally equivalent to some non-zero scalar multiple of the graph G is either equal to $l(G)$ or $2l(G)$.*

Proof: Suppse that cG is locally equivalent to G for any non-zero c. Then for any graph locally equivalent to cG for some c, it is also locally equivalent to G. Therefore $l(G)$ is the number of graphs that are locally equivalent to cG for some c.

Next, suppose that there exists a non-zero c_0 such that $c_0 G$ is not locally equivalent to G. By lemma 6.2, c_0 is not a perfect square. Assume that, H is a graph that is locally equivalent to cG, for some c. If c is a perfect square, then by lemma 6.2, H is also locally equivalent to G. On the other hand, if c is not a perfect square, then $c_0^{-1}c$ is a perfect square. Hence $c_0^{-1}H$ is locally equivalent to G. Therefore, for any such H, it is locally equivalent to either G or $c_0 G$. Also, by lemma 6.1, $l(c_0 G) = l(G)$. Therefore, in this case, the number of graphs locally equivalent to cG for some c, is $2l(G)$.

$\square$

Using theorem 6.1, we can relate $l(G)$ to the number of graphs that are locally equivalent to cG for some non-zero c. Since, this number appears in counting the number of graphic presentations of an isotropic system, we can obtain some useful information about $l(G)$. For this purpose, fix an isotropic system $\mathcal{L}$ admitting G as a fundamental graph.

Theorem 6.2 *The number of graphs locally equivalent to cG for some non-zero $c \in \mathbf{F}_q$ is equal to*

$$\frac{(q-1)\epsilon(\mathcal{L})}{\lambda(\mathcal{L})}.$$

Proof: By corollary 4.2, the number of graphic presentations of $\mathcal{L}$ is equal to $(q-1)\epsilon(\mathcal{L})$. On the other hand, by corollary 5.1, the number of graphic presentations is equal to $\lambda(\mathcal{L})$ times the number of graphs that are locally equivalent to cG, for some non-zero c. By letting these two values be equal, one obtains the described conclusion.

$\square$

The following corollary is a direct consequence of theorems 6.1 and 6.2.

Corollary 6.1 *Either $l(G)$ or $2l(G)$ is equal to*

$$\frac{(q-1)\epsilon(\mathcal{L})}{\lambda(\mathcal{L})}.$$

Using corollaries 5.2 and 6.1, giving a bound for $\epsilon(\mathcal{L})$ can lead us to a bound for $l(G)$.

Remark. *In [4],[5] and [6], it has been shown that in the binary case, $l(G) = \frac{\epsilon(\mathcal{L})}{\lambda(\mathcal{L})}$. So, corollary 6.1 is valid for binary case, too.*

7 The number of Eulerian vectors

In this section, we include the binary case. To be precise, we assume that q is either 2 or a power of an odd prime number. In addition, we do *not* assume the connectivity of graphs. In order to compute $\epsilon(\mathcal{L})$, the number of Eulerian vectors of an isotropic system, we use a well-known polynomial, so called *Tutte-Martin polynomial* which is defined for an isotropic system $\mathcal{L}$ as follows:

$$\mathcal{M}(\mathcal{L}; t) = \sum_{C:\ complete} (t - q)^{\ dim(\mathcal{L} \cap \hat{C})},$$

where the summation is over all complete vectors $C \in \mathbf{K}^V$. By the definition of Eulerian vector,

$$\epsilon(\mathcal{L}) = \mathcal{M}(\mathcal{L}; q).$$

So, in order to compute $\epsilon(\mathcal{L})$, it suffices to be able to compute the Tutte-Martin polynomial. To do that, we give a recursion formula for this polynomial.

Definition 7.1 *Let $\mathcal{L}$ be an isotropic system, $v \in V$ and $x \in \mathbf{K}^*$ (or $x = 0$). Define*

$$\mathcal{L}_x^v = \{C \in \mathcal{L} : \langle C(v), x \rangle = 0\} = \{C \in \mathcal{L} : C(v) \sim x \ or \ C(v) = 0\},$$

and

$$\mathcal{L}|_x^v = projection \ of \ \mathcal{L}_x^v \ on \ \mathbf{K}^{V-\{v\}}.$$

Also, let $\mathcal{L}_0^v = \{C \in \mathcal{L} : C(v) = 0\}$, and for $C \in \mathbf{K}^V$, we set $C|^v$ to be the projection of C on $\mathbf{K}^{V-\{v\}}$.

Lemma 7.1 *$\mathcal{L}|_x^v$ is an isotropic system.*

Proof: First notice that, $\mathcal{L}_x^v$ is a subspace of $\mathcal{L}$ and is self-orthogonal. On the other hand, the v-th coordinate of each vector in $\mathcal{L}_x^v$ is either 0 or equivalent to x. Then the v-th coordinate of any two vectors in $\mathcal{L}_x^v$ are orthogonal, and hence if we delete the v-th coordinate, all of the vectors remain orthogonal to each other. In other words, $\mathcal{L}|_x^v$ is again self-orthogonal. Thus, it remains to show $dim\,\mathcal{L}|_x^v = n - 1$. We consider two cases:

(i) There exists $C_0 \in \mathcal{L}$ such that $\langle C_0(v), x \rangle \neq 0$. In this case, we have $\mathcal{L}_x^v \neq \mathcal{L}$, and therefore, $dim\mathcal{L}_x^v = n - 1$. Notice that, $E_{v,x}$ is not orthogonal to C_0, therefore $E_{v,x}$ is not in $\mathcal{L}_x^v$. Then, the projection of $\mathcal{L}_x^v$ on $\mathbf{K}^{V-\{v\}}$ is injective, and $dim\mathcal{L}|_x^v$ is also $n - 1$.

(ii) For any $C \in \mathcal{L}$, $\langle C(v), x \rangle = 0$. Then $E_{v,x}$ is in $\mathcal{L}$, and for any $C \in \mathcal{L}$, $C(v) = 0$ or $C(v) \sim x$. Thus, $\mathcal{L}_x^v = \mathcal{L}$, and $\mathcal{L} = \mathcal{L}_0^v \oplus \langle E_{v,x} \rangle$. It means that, by removing the v-th coordinate of $\mathcal{L}_0^v$, we end up with $\mathcal{L}|_x^v$. Therefore, $dim\mathcal{L}|_x^v = dim\mathcal{L}_0^v = n - 1$.

$\square$

Remark. *Let G be a fundamental graph of $\mathcal{L}$ and (G, A, B) be a graphic presentation. Then, $\mathcal{M}(\mathcal{L}; t)$ just depends on G. In fact, if $\mathcal{L}'$ is another isotropic*

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system with a graphic presentation (G, A', B'), *then multiplication by the matrix* $D(A, B)^{-1}D(A', B')$ *maps* $\mathcal{L}$ *to* $\mathcal{L}'$. *Also, it maps* $\mathcal{L} \cap \hat{C}$ *to* $\mathcal{L}' \cap \hat{C}'$, *where* $C' = CD(A, B)^{-1}D(A', B')$. *Therefore,* $\mathcal{M}(\mathcal{L}'; t) = \mathcal{M}(\mathcal{L}'; t)$.

Using this remark, we can talk about the Tutte-Martin polynomial of a graph, $\mathcal{M}(G; t)$. Also, if G and H are two locally equivalent graphs, they are fundamental graphs of the same isotropic system, and then $\mathcal{M}(G; t) = \mathcal{M}(H; t)$.

Next, if a vertex v is isolated in G, i.e., v has no neighbor, then $E_{v, B(v)}$ is in $\mathcal{L}$ and $\mathcal{L} = \mathcal{L}_0^v \oplus \langle E_{v, B(v)} \rangle$. This case is actually studied in the second part, in the proof of lemma 7.1. In this case, as we already mentioned, for all x, $\mathcal{L}|_x^v$ is the same and equal to $\mathcal{L}|_0^v$.

Theorem 7.1

$$\mathcal{M}(\mathcal{L}; t) = \begin{cases} (q-1)t\mathcal{M}(\mathcal{L}|_0^v; t) & \textit{if } v \textit{ is isolated in } G, \\ (q-1)\sum_{x \in \mathbf{K}^*} \mathcal{M}(\mathcal{L}|_x^v; t) & \textit{otherwise.} \end{cases}$$

Proof: Suppose that v is isolated is G. Then $\mathcal{L} = \mathcal{L}_0^v \oplus \langle E_{v, B(v)} \rangle$, and for any complete vector $C \in \mathbf{K}^V$, we have

$$\mathcal{L} \cap \hat{C} = \left(\mathcal{L}|_0^v \cap \hat{C}|^v \right) \oplus \left(\langle E_{v, B(v)} \rangle \cap \langle E_{v, C(v)} \rangle \right).$$

Therefore,

$$\begin{aligned}
\mathcal{M}(\mathcal{L}; t) &= \sum_{C \in \mathbf{K}^V} (t-q)^{\ dim(\mathcal{L} \cap \hat{C})} \\
&= \sum_{C \in \mathbf{K}^V} (t-q)^{\ dim\left(\mathcal{L}|_0^v \cap \hat{C}|^v \right) \oplus \left(\langle E_{v, B(v)} \rangle \cap \langle E_{v, C(v)} \rangle \right)} \\
&= \sum_{C \in \mathbf{K}^V} (t-q)^{\ dim\ (\mathcal{L}|_0^v \cap \hat{C}|^v)}(t-q)^{\delta(B(v) \sim C(v))} \\
&= \sum_{C \in \mathbf{K}^{V-\{v\}}} \left((q^2 - q) + (q-1)(t-q) \right)(t-q)^{\ dim\ (\mathcal{L}|_0^v \cap \hat{C})} \\
&= (q-1)t\mathcal{M}(\mathcal{L}|_0^v; t),
\end{aligned}$$

where, all summations are over the complete vectors.

Now, assume that v is not isolated and g_{vw} is non-zero for some $w \in V$. Hence, there exists a vector $C_1 \in \mathcal{L}$ such that $C_1(v) \sim A(v)$. Also, we already know that for some $C_2 \in \mathcal{L}$, we have $C_2(v) \sim B(v)$, and $A(v) \not\sim B(v)$. Therefore, the v-th coordinates of vectors in $\mathcal{L}$ cover the whole space $\mathbf{K}$. Thus,

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$$
\begin{aligned}
\mathcal{M}(\mathcal{L}; t) &= \sum_{C \in \mathbf{K}^V} (t-q)^{dim(\mathcal{L} \cap \hat{C})} \\
&= \sum_{x \in \mathbf{K}^*} \sum_{C, C(v) \sim x} (t-q)^{dim(\mathcal{L} \cap \hat{C})} \\
&= \sum_{x \in \mathbf{K}^*} \sum_{C, C(v) \sim x} (t-q)^{dim(\mathcal{L}_x^v \cap \hat{C})} \\
&= \sum_{x \in \mathbf{K}^*} \sum_{C, C(v) \sim x} (t-q)^{dim(\mathcal{L}|_x^v \cap \hat{C}|^v)} \\
&= \sum_{x \in \mathbf{K}^*} \sum_{C \in \mathbf{K}^{V-\{v\}}} (q-1)(t-q)^{dim(\mathcal{L}|_x^v \cap \hat{C})} \\
&= \sum_{x \in \mathbf{K}^*} (q-1)\mathcal{M}(\mathcal{L}|_x^v; t),
\end{aligned}
$$

(once again, all summations are over the complete vectors).

$\square$

Consider a graph G. As usual, by $G - \{v\}$, we mean the graph obtained from G by deleting vertex v.

Lemma 7.2 *Let $\mathcal{L}$ be an isotropic system with graphic presentation (G, A, B). Then $G - \{v\}$ is a fundamental graph of $\mathcal{L}|_{A(v)}^v$.*

Proof: We already know that the rows of $(I \mid G)D(A, B)$ form a basis for $\mathcal{L}$. The v-th coordinate of each row, except the v-th row, is either zero or equivalent to $A(v)$. Therefore, the rows of $(I \mid G)D(A, B)$, except the v-th row form a basis for $\mathcal{L}_{A(v)}^v$. Thus, by deleting the v-th row and the v-th column of G, and also the v-th coordinates of A and B, we come up with a graphic presentation of $\mathcal{L}|_{A(v)}^v$, meaning that $G - \{v\}$ is a fundamental graph of $\mathcal{L}|_{A(v)}^v$.

$\square$

Theorem 7.2

(i) *If v is an isolated vertex in G, then $G - \{v\}$ is a fundamental graph of $\mathcal{L}|_0^v$.*

(ii) *If w is a neighbor of v in G then q graphs $G *_r v - \{v\}$, $r \in \mathbf{F}_q$, together with $G *_{-g_{vw}^{-2}} w *_1 v - \{v\}$ are fundamental graphs of $\mathcal{L}|_x^v$ for $x \in \mathbf{K}^*$.*

Proof: Part (i) is a direct consequece of lemma 7.2. To prove (ii), using lemma 7.2, we should show that for each $x \in \mathbf{K}^*$, there exists an Eulerian vector related to one of these graphs, such that its v-th coordinate is x.

Suppose that (G, A, B) is a graphic presentation of $\mathcal{L}$. By theorem 4.3, for any $r \in \mathbf{F}_q$, $(G *_r v, A + rB_v, B + rg^2(v) \times A)$ is also a graphic presentation of $\mathcal{L}$. The v-th coordinate of the Eulerian vector of this presentation is $A(v) + rB(v)$. Therefore, $G *_r v - \{v\}$ is a fundamental graph of $\mathcal{L}|_{A(v)+rB(v)}^v$.

Now notice that, $\langle A(v), B(v) \rangle \neq 0$. Therefore, for r varies in $\mathbf{F}_q$, $A(v) + rB(v)$ are all different elements of $\mathbf{K}^*$, and there are q of them. The only element of $\mathbf{K}^*$ not obtained in this way is $B(v)$. Consider the fundamental graph $G *_s w *_1 v$, where $s = -g_{vw}^{-2}$. By theorem 4.3, $A + sB_w + (B_v + sg_{vw}^2 A_v)$ is an Eulerian vector for this fundamental graph, and the v-th coordinate of this vector is $(sg_{vw}^2 + 1)A(v) + B(v) = B(v)$. Thus, $G *_s w *_1 v - \{v\}$ is a fundamental graph of $\mathcal{L}|_{B(v)}^v$.

$\square$

Corollary 7.1 *If v is isolated in G then*

$$\mathcal{M}(G; t) = (q-1)t\mathcal{M}(G - \{v\}; t),$$

otherwise, if w is a neighbor of v then

$$\mathcal{M}(G; t) = (q-1)\left[\mathcal{M}(G *_{-g_{vw}^{-2}} w *_1 v; t) + \sum_{r \in \mathbf{F}_q} \mathcal{M}(G *_r v - \{v\}; t)\right].$$

7.1   Estimation of $\epsilon(G)$

The final step to evaluate $\epsilon(G)$ is the following one, which is valid due to the fact that all of the graphs described in the right hand side of the formula in corollary 7.1 have $n - 1$ vertices. Indeed, one can observe that the number of graphs in the right hand side is equal to $q + 1$, and hence,

$$\max_{G: |G|=n} |\mathcal{M}(G; t)| \leq (q-1) \cdot \max\{t, q+1\} \cdot \max_{H: |H|=n-1} |\mathcal{M}(H; t)|.$$

Moreover, when the graph G has only one vertex, i.e., $n = 1$, we have

$$|\mathcal{M}(G; t)| \leq (q^2 - 1) \cdot \max\{|t - q|, 1\}.$$

Putting together these two statements, we conclude the following corollary.

Corollary 7.2 *For a graph G with n vertices, the Tutte-Martin polynomial $\mathcal{M}(G; t)$ can be estimated as follows:*

$$|\mathcal{M}(G; t)| \leq (q^2 - 1) \cdot [(q-1) \cdot \max\{t, q+1\}]^{(n-1)} \cdot \max\{|t - q|, 1\},$$

and by setting $t = q$, we obtain that:

$$\epsilon(G) \leq (q^2 - 1)^n.$$

We can even derive a lower bound for $\epsilon(G)$ as well.

$$\min_{G: |G|=n} \mathcal{M}(G; q) \geq (q-1) \cdot \min\{t, q+1\} \cdot \min_{H: |H|=n-1} |\mathcal{M}(H; q)|.$$

On the other hand, when the graph has just one vertex, $\epsilon(G) \geq 1$, and hence in general,

$$\epsilon(G) \geq (q^2 - q)^{n-1}.$$

Thus, the proof of the following theorem is now complete.

Theorem 7.3 *The number of Eulerian vectors for a graph (or equivalently for an isotropic system) with n vertices satisfies the following property:*

$$(q^2 - q)^{n-1} \leq \epsilon(G) \leq (q^2 - 1)^n.$$

In particular, when the graph is ordinary (binary), i.e., when $q = 2$, we have:

$$2^{n-1} \leq \epsilon(G) \leq 3^n.$$

$\square$

8　The number of classes of local equivalency

As mentioned earlier, we use the formula given in corollary 6.1, as well as the estimations for the number of Eulerian vectors given in the previous section, in order to give a bound for $l(G)$, the number of graphs locally equivalent to G.

By corollary 6.1, if G is connected, then

$$l(G) \leq \frac{(q-1)\epsilon(\mathcal{L})}{\lambda(\mathcal{L})} \leq (q-1)\epsilon(\mathcal{L})$$

Taking into account the estimation of $\epsilon(\mathcal{L}) = \epsilon(G)$ presented in the theorem 7.3, we come up with an upper bound for $l(G)$, given that the number of graphs with n vertices is exactly $q^{\frac{n^2}{2} - \frac{n}{2}}$.

Theorem 8.1

(i) *The number of graphs locally equivalent to a connected graph is at most $(q - 1)(q^2 - 1)^n$ which is bounded above by q^{2n+1}, n being the number of vertices of the graph.*

(ii) *$\mathcal{C}(n)$, the number of classes of local equivalency of connected graphs with n vertices satisfies:*

$$q^{\frac{n^2}{2} - \frac{5n}{2} - 1} \leq \mathcal{C}(n) \leq q^{\frac{n^2}{2} - \frac{n}{2}}.$$

In other words,

$$\mathcal{C}(n) = q^{\frac{n^2}{2} - O(n)}.$$

In particular, for the usual (binary) graphs, i.e., when $q = 2$, the number of graphs locally equivalent to a graph is at most 3^n and the number of classes of local equivalency is

$$\mathcal{C}(n) = 2^{\frac{n^2}{2} - O(n)}.$$

$\square$

9 Conclusion

We developed a method to compute number of graphs locally equivalent to a given one. Using this method, we bounded this number for an arbitrary graph. Also, we found an approximation of the number of equivalency classes. That is, $\mathcal{C}(n) = q^{\frac{n^2}{2} - O(n)}$. Notice that, number of all graphs is $q^{\frac{n^2}{2} - \frac{n}{2}}$. Therefore, this estimation says that number of equivalency classes is almost the same as number of all graphs.

We got these results by developing a reach theory of isotropic systems and also local operation over graphs, and it seems that this theory can be used for other problems in this area, too.

References

[1] A. Ashikhmin and E. Knill, *Nonbinary quantum stabilizer codes*, IEEE Trans. Info. Theory, 47 (2001), 3065-3072.

[2] M. Bahramgiri, S. Beigi, *Graph States Under the Action of Local Clifford Group in Non-Binary Case,* quant-ph/0610267.

[3] M. Bahramgiri, S. Beigi, *An Efficient Algorithm to Recognize Locally Equivalent Graphs in Non-Binary Case,* cs/0702057.

[4] A. Bouchet, *An efficient algorithm to recognize locally equivalent graphs*, Combinatorica, 11 (1991), 315-329.

[5] A. Bouchet, *Transforming trees by successive local complementations*, J. Graph Theory, 12 (1988), 195-207.

[6] A. Bouchet, *Recognizing locally equivalent graphs*, Discrete Math., 114 (1993), 75-86.

[7] J. Dehaene, M. Van den Nest and B. De Moor, *Graphical description of the action of local Clifford trasformations on graph states*, quant-ph/0308151.

[8] M. Hein, W. Dur, J. Eisert, R. Raussendorf, M. Van den Nest and H. J. Briegel, *Entanglement in graph states and its applications*, quant-ph/0602096.