INDIVISIBLE ULTRAMETRIC SPACES

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Abstract. A metric space is indivisible if for any partition of it into finitely many pieces one piece contains an isometric copy of the whole space. Continuing our investigation of indivisible metric spaces [\[1\]](#page-20-0), we show that a countable ultrametric space embeds isometrically into an indivisible ultrametric metric space if and only if it does not contain a strictly increasing sequence of balls.

INTRODUCTION

A metric space $\mathbb{M} := (M; d)$ is *indivisible* if for every partition of M into two parts, one of the two parts contains an isometric copy of M. If M is not indivisible then it is *divisible*. The notion of indivisibility was introduced for relational structures by R. Fraïssé in the fifties, see $[5]$ and also $[12]$, $[13]$. Results obtained since then are a part of what is now called Ramsey Theory. Recently, the study of extremely amenable groups pointed out to indivisible metric spaces. The first step was Pestov theorem asserting that the group $Iso(\mathbb{U})$ of isometries of the Urysohn space $\mathbb U$ is extremely amenable [\[11\]](#page-20-4). Next, the discovery by Keckris, Pestov and Todorcevic [\[7\]](#page-20-5) of the exact relationship between Fraisse limits, Ramsey classes and extremely amenable groups, followed by the introduction of the notion of *oscillation stable groups* and a characterization in terms of ε*-indivisibility*. In [\[9\]](#page-20-6), Nesetril proving the Ramsey property of the class of ordered finite metric spaces, suggested to look at the indivisibility properties of metric spaces. And, in [\[6\]](#page-20-7), Hjorth proved that $\mathbb{U}_{\mathbb{Q}}$, the Urysohn space with rational distances, is divisible and asked if the bounded Urysohn $\mathbb{U}_{\mathbb{Q}\leq 1}$ is also divisible. Prompted by the Hjorth question, we started in [\[1\]](#page-20-0) to investigate indivisible metric spaces. We proved that these spaces must be bounded and totally Cantor-disconnected (for countable spaces a condition stronger than totally Cantor-disconnedness must hold, indeed these spaces do not contain any *spider* [\[1\]](#page-20-0)). This implies that every Urysohn space \mathbb{U}_V with a subset of V dense in some initial segment of \mathbb{R}_+ is divisible, from which the divisibility of $\mathbb{U}_{\mathbb{O}\leq 1}$ follows. The fact that on every countable indivisible metric spaces there is a natural ultrametric distance, invited to look at ultrametric spaces. We proved that an indivisible ultrametric space does not contain an infinite strictly increasing

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sequence of balls. Furthermore, this condition, added to the fact that each nonterminal node in the tree associated to the space has an infinite degree, is necessary and sufficient for a countable homogeneous ultrametric to be indivisible [\[1\]](#page-20-0). From this follows that such a space is the ultrametric Urysohn with reversely well founded result (this latter result was also obtained by Nguyen Van Thé $[10]$). Here, we continue our investigation of countable indivisible ultrametric spaces, with the idea in mind that a complete description is not out of reach. We look first at spectra of indivisible ultrametric spaces (the spectrum of a metric space $\mathbb{M} := (M, d)$ is the set $Spec(\mathbb{M}) := \{d(x,y) : x,y \in M\}$. We show that beside the fact there are subsets of \mathbb{R}_+ containing 0, the only requirement imposed upon by the indivisibility is that they have a largest element (Proposition [2\)](#page-4-0). Spectra of indivisible homogeneous ultrametric spaces are reversely well ordered, hence theses spaces are quite rare. We introduce a notion of *endogeneous metric space*, generalizing the notion of homogeous metric space. We characterize countable endogeneous indivisible ultrametric spaces in a fashion similar to the homogeneous ones (Theorem 6). We prove that a countable ultrametric space M embeds isometrically into an indivisible ultrametric space if and only if it does not contain an infinite strictly increasing sequence of balls. Furthermore, when this condition holds, M embeds into a countable endogeneous indivisible ultrametric space with the same spectrum (Theorem [7\)](#page-10-0).

In Section 1 we record some facts we will use in the rest of the paper, the description of countable homogeneous ultrametric spaces and the special case of the indivisible ones. Except Proposition [2,](#page-4-0) they come from [\[1\]](#page-20-0). In Section 2 we present the notion of endogeneous ultrametric space, and criteria for the indivisibility of such spaces. In section 3 we present our result on the embeddability of an ultrametric space into an indivisible one.

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1. Ultrametric spaces, homogeneity and indivisibility

We recall the following notions. Let $\mathbb{M} := (M, d)$ be a metric space. If A is a subset of M, we denote by $d_{\uparrow A}$ the restriction of d to $A \times A$ and by $\mathbb{M}_{\uparrow A}$ the metric space $(A, d_{\restriction A})$, that we call the metric subspace of M *induced on* A. Let $a \in M$; for $r \in \mathbb{R}^+$, the *open*, resp. *closed*, *ball of center* a, *radius* r is the set $B(a,r) := \{x \in M : d(a,x) < r\},$ resp. $B'(a,r) := \{x \in M : d(a,x) \leq r\}.$ For a subset A of M, we set $B'(A,r) := \bigcup \{B'(a,r) : a \in A\}$. In the sequel, the term *ball* means an open or a closed ball. When needed, we denote by $\mathcal{B}all(\mathbb{M})$ the collection of balls of M. A ball is *non-trivial* if it has more than one element. The *diameter* of a subset B of M is $\delta(B) := \sup \{d(x, y) : x, y \in B\}$. Four others notions will be of importance:

Definitions 1. Let $a \in M$, the spectrum of a is the set $Spec(M, a) := \{d(a, x) :$ $x \in M$ *, The* multispectrum of M is the set $MSpec(M) := \{Spec(M, a) : a \in M\}$. *The* spectrum of M is the set $Spec(M) := \bigcup MSpec(M)$ $(= \{d(x, y) : x, y \in M\}).$ *The* nerve of M *is the set* $Nerv(M) := \{B'(a, r) : a \in M, r \in Spec(M, a)\}.$

1.1. The structure of ultrametric spaces. A metric space is an *ultrametric space* if it satisfies the strong triangle inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}\.$ See [\[8\]](#page-20-9) for

example. Note that a space is an ultrametric space if and only if $d(x, y) \geq d(y, z) \geq$ $d(x, z)$ implies $d(x, y) = d(y, z)$.

The essential property of ultrametric spaces is that balls are either disjoint or comparable w.r.t. inclusion. From this, one can look at ultrametric spaces as binary relational structures made of equivalence relations or as trees.

1.1.1. *Equivalences relations on ultrametric spaces.* Let M be an ultrametric space. Let $x, y \in M$ and $r \in \mathbb{R}^*_+$, resp. $r \in \mathbb{R}_+$, we set $x \equiv_{\leq r} y$, resp. $x \equiv_{\leq r} y$, if $d(x, y) < r$, resp. $d(x, y) \leq r$. Then:

- (a) The relation $\equiv_{\leq r}$, resp $\equiv_{\leq r}$, is an equivalence relation; the open, resp. closed, balls of radius r form a partition of M ; the blocks of this partition being the equivalence classes of the equivalence relation.
- (b) Let \equiv be one of the equivalences $\equiv_{\leq r}$, $\equiv_{\leq r}$. Then $x \equiv x'$, $y \equiv y'$ and $x \not\equiv y$ imply $d(x, y) = d(x', y')$.
- (c) The quotient M/\equiv can be equipped with a distance d_{\equiv} in such a way that the canonical map $M \to M/\equiv$ satisfies $d_{\equiv}(p(x), p(y)) = d(x, y)$ for all $x, y \in M$ such that $x \not\equiv y$.

1.1.2. *Valued trees.* Ultrametric spaces can be easily described in terms of realvalued trees. For that we recall some notions about ordered sets. Let P be an ordered set (poset). We denote by $max(P)$ the set of maximal elements of P. Let $x \in P$, an element y of P is an *immediate successor*, (or a cover) of x, if $x < y$ and there is no $z \in P$ such that $x < z < y$. One usually sets $\downarrow x := \{y \in P : y \leq x\}$ and similarly defines $\uparrow x$. We denote by $up(P)$ the collection of sets $\uparrow x$ where $x \in P$. The poset P is a *forest* if $\downarrow x$ is a chain for every $x \in P$; this is a *tree* if in addition every pair x, y of elements of P has a lower bound, and this is a *meet-tree* if x, y ∈ P has an infimum, denoted x ∧ y. We say that a poset P is *ramified* if for every $x, y \in P$ such that $x < y$ there is some $y' \in P$ such that $x < y'$ and y' incomparable to y . In the sequel, working with trees or forest, we will also use notations inherited from chains: sometimes, we will use the notation $(-x)$ instead of $\downarrow x$; we will set $[a, b] := \{x \in P : a < x \leq b\}, \{ \leftarrow a \mid := \{x \in P : x < a\}.$ The poset P is *well founded* if every non empty subset A of P contains some minimal element. As it is well known, if a poset P is well-founded, for every $x, y \in P$ such that $x < y$ there is some immediate successor z of x, such that $x < z \leq y$.

Definition 1. *An* ultrametric tree *is a pair* (P, v) *where* P *is a ramified meet-tree such that every element is below some maximal element and* v *is a strictly decreasing map from* P *to* \mathbb{R}_+ *with* $v(x) = 0$ *for each maximal element* x *of* P.

The following description given in [\[1\]](#page-20-0) is close from the one given by Lemin [\[8\]](#page-20-9) (who instead of $Nerv(\mathbb{M})$ considered $Ball(\mathbb{M})$).

- **Theorem 1.** (1) If $M := (M, d)$ is an ultrametric space, then the pair (P, v) , *where* $P := (Nerv(M), \supseteq), \delta$ *where* δ *is the diameter function is an ultrametric tree.*
	- (2) Conversely, if (P, v) is an ultrametric tree then $\mathbb{M} := (M, d)$ where $M :=$ $\max(P)$ *and* $d(x, y) := v(x \wedge y)$ *is an ultrametric space and* $Nerv(M)$ $up(P)_{\upharpoonright M}$ where $up(P)_{\upharpoonright M} := \{M \cap \uparrow x : x \in P\}.$
	- *(3) The two correspondences are inverse of each other.*

In [\[1\]](#page-20-0) we introduced the notion of *degree* of a node of a ramified meet-tree. If B is a member of the ramified meet-tree ($Nerv(M), \supseteq$), the degree of B is the number of sons of B that we define below.

Definition 2. Let $\mathbb{M} := (M; d)$ be an ultrametric space, $B \in Nerv(\mathbb{M})$ and $r :=$ $\delta(B)$ *. If* $r > 0$ *, a* son *of* B *is any open ball of radius* r *included into* B; we denote *by* Son(B) *the set of sons of* B*.*

Notice that according to Subsection [1.1.1,](#page-2-0) $Son(B)$ forms a partition of B. Also, notice that members of $Son(B)$ do not need to belong to $Nerv(M)$. But, if $Nerv(M)$, ordered by reverse of the inclusion, is well-founded then the members of $Son(B)$ are the immediate successors of B in the poset $(Nerv(B), 2)$ (hence the terminology we use).

1.2. Some examples of ultrametric spaces. Let λ be a chain and let \bar{a} := $(a_{\mu})_{\mu\in\lambda}$ such that $2 \le a_{\mu} \le \omega$. Set $\omega^{[\overline{a}]} := {\overline{b}} := (b_{\mu})_{\mu\in\lambda} : \mu \in \lambda \Rightarrow b_{\mu} < a_{\mu}$ and $supp(\overline{b}) := \{ \mu \in \lambda : b_{\mu} \neq 0 \}$ is finite }. If $a_{\mu} = \omega$ for every $\mu \in \lambda$, the set $\omega^{[\overline{a}]}$ is usually denoted $\omega^{[\lambda]}$. Add a largest element, denoted ∞ to λ . Given $\overline{b}, \overline{c} \in \omega^{[\overline{a}]}$, set $\Delta(\overline{b}, \overline{c}) := \infty$ if $\overline{b} = \overline{c}$, otherwise $\Delta(\overline{b}, \overline{c}) := \mu$ where μ is the least member of λ such that $b_{\mu} \neq c_{\mu}$. Suppose that λ embeds into R. Let $w : \lambda \cup {\infty} \rightarrow \mathbb{R}_{+}$ be a strictly decreasing map such that $w(\infty) = 0$, let $d_w := w \circ \Delta$ and let V be the image of w. Let $\omega^{\leq [\overline{a}]} := \{f_{\mid(\leftarrow \mu\mid} : f \in \omega^{[\overline{a}]}, \mu \in \lambda \cup \{\infty\}\}\)$ ordered by extension. Clearly, $\omega^{\leq [\overline{a}]}$ is a ramified meet-tree such that every element is below some maximal element. For $\mu \in \lambda \cup \{\infty\}$ and $f \in \omega^{\leq [\overline{a}]}$, set $v(f_{\upharpoonright (\leftarrow \mu[}) := w(\mu)).$

Lemma 3. [\[1\]](#page-20-0) *The pair* $\mathbb{M} := (\omega^{[\overline{a}]}, d_w)$ *is an ultrametric space,* $Spec(\mathbb{M}) = V$ *and the ultrametric tree associated to* M *is isomorphic to* $(\omega^{\leq [\overline{a}]}, v)$ *.*

Let M and M' be two metric spaces. A map $f : M \to M'$ is an *isometry from* M *into* M′ , or an *embedding*, if

(1)
$$
d'(f(x), f(y)) = d(x, y) \text{ for all } x, y \in M
$$

This map is an isometry from M *onto* M′ if it is surjective. We say that M *embeds* into M' if there is an embedding from M into M', that M and M' equimorphic if each embeds into the other and that M and M′ are *isometric* if there is an isometry from M onto M′ . A *local embedding from* M *into* M′ is any isometry from a subspace of M onto a subspace of M'. If $M = M'$, we will call it a local embedding of M.

We say that M is *point-homogeneous* if the group $Iso(\mathbb{M})$ of surjective isometries of M acts transitively on M. According to the terminology of Fraüssé $[5]$, a metric space M is *homogeneous* if every local embedding of M having a finite domain extends to an isometry of M onto M (in fact, for ultrametric spaces, the two notions coincide[\[2\]](#page-20-10)).

Theorem 2. [\[1\]](#page-20-0) *A countable ultrametric space* M *is homogeneous if and only if it is isometric to some* $(\omega^{[\overline{a}]}, d_w)$.

Let M be an ultrametric space, the *age* of M is the collection of finite metric spaces isometric to some subspace of M. Let V be a set such that $0 \in V \subseteq \mathbb{R}_+$. Let $Mult_V$ (resp. $Mult_{V\lt\omega}$) be the collection of ultrametric metric spaces (resp. finite ultrametric spaces) M whose spectrum is included into V. Then $Mult_{V\leq\omega}$ is closed under embeddability and has the amalgamation property. According to the famous theorem of Fraïssé (1954) [\[4\]](#page-20-11) p.383, if follows that if V is countable there is a countable homogeneous ultrametric space whose age is $Mult_{V,\leq w}$. It has spectrum V. We denote it $\mathbb{U}ll_{V}$ and we call it the *Urysohn ultrametric space with spectrum* V .

Proposition 1. [\[1\]](#page-20-0) *The space* $(\omega^{[\lambda]}, d_w)$ *is the countable homogeneous ultrametric space* \mathbb{U} *t*_V *with spectrum* V*.*

1.3. Indivisibility.

Definitions 2. Let $\mathbb{M} := (M; d)$ be a metric space. The sequence $a_0, a_1, \ldots, a_{n-1}, a_n$ *of elements of* M *is an* ϵ -chain joining a_0 and a_n *if* $d(a_i, a_{i+1}) \leq \epsilon$ *for all* $i \in n$ *. Let* $x, y \in M$ *. Set*

 $d^*(x, y) := \inf\{\epsilon > 0: x \text{ and } y \text{ are joined by an } \epsilon \text{-sequence }\}.$

Theorem 3. [\[1\]](#page-20-0) Let $\mathbb{M} := (M; d)$ be a countable homogeneous indivisible metric *space, then* M[∗] *is an homogeneous indivisible ultrametric space.*

Theorem 4. [\[1\]](#page-20-0) *If an ultrametric space is indivisible then the collection of balls, ordered by inclusion, is dually well-founded and the diameter is attained.*

Theorem 5. [\[1\]](#page-20-0) *Let* M *be a denumerable ultrametric space. The following properties are equivalent:*

- *(i)* \mathbb{M} *is isometric to some* \mathbb{U} *tv, where V is dually well-ordered.*
- *(ii)* M *is point-homogeneous,* Nerv(M) *ordered by reverse of the inclusion is well founded and every non-trivial* $B \in Nerv(\mathbb{M})$ *has infinitely many sons.*
- *(iii)* M *is homogeneous and indivisible.*

This result (in part) was obtained independently by L. Nguyen Van Thé $[10]$. The crucial part is the implication $(ii) \Rightarrow (iii)$. It is now a consequence of Theorem [6.](#page-8-0)

1.4. Spectrum of indivisible ultrametric spaces. The proposition below could be derived from Theorem [7.](#page-10-0) The proof we give uses only Theorem [5.](#page-4-1)

Proposition 2. *A set* V *is the spectrum of an ultrametric space if and only if* $0 \in V \subseteq \mathbb{R}_+$. If this latter condition is fulfilled, V is the spectrum of an indivisible *ultrametric space if and only if* V *has a largest element. In this case* V *is the spectrum of an indivisible ultrametric space of size* $\aleph_0 + |V|$ *.*

Proof. If $V = Spec(M)$ for some metric space M then clearly $0 \in V \subseteq \mathbb{R}_+$. Conversely, let V such that $0 \in V \subseteq \mathbb{R}_+$. Define $d: V \times V \to \mathbb{R}_+$, setting $d(x, y) :=$ $\max\{x, y\}$ if $x \neq y$ and $d(x, y) := 0$ otherwise. Then $\mathbb{M} := (V, d)$ is an ultrametric space for which $Spec(\mathbb{M}) = V$. If M is an indivisible ultrametric space, its diameter is attained (Theorem [4\)](#page-4-2), that is $V := Spec(M)$ has a largest element. Conversely, let V, with a largest element r, such that $0 \in V \subseteq \mathbb{R}_+$. We set $\mathbb{M} := \mathbb{U} \mathcal{U}_V$, the Urysohn ultrametric space with age $\mathcal{M}_{V,\leq \omega}$, if V is finite. Otherwise, let $M :=$ $\bigcup \{Ult_F \times \{F\} : F \in D\}$ where D is the set of finite subsets F of $V \setminus \{r\}$ containing 0. For two elements $(x, F), (x', F') \in M$ set $d((x, F), (x', F')) := r$ if $F \neq F'$, otherwise set $d((x, F), (x', F')) := d(x, x')$ where d is the distance on $\mathbb{U}tr$. Clearly, M is an ultrametric space with spectrum V. If V is finite, M is indivisible by Theorem [5.](#page-4-1) Suppose that V is infinite. Let $f : M \to 2$. Set $g(F) = 0$ if there is some isometry $\psi_{F,0}: \mathbb{U}t_F \to \mathbb{M}_{|\mathcal{U}t_F \times \{F\} \cap f^{-1}(0)}$. Otherwise, set $g(F) = 1$, and since by Theorem [5,](#page-4-1)

Ult is indivisible, select an isometry $\psi_{F,1} : \mathbb{U}$ t $F \to \mathbb{M}_{|\mathcal{U}||F} \times \{F\} \cap f^{-1}(1)$. Ordered by inclusion, D is up directed. It follows that for some $i < 2$, $g^{-1}(i)$ is *cofinal* in D, that is every member of D is included into some member of $g^{-1}(i)$. In fact, as it is easy to see, more is true: there is a one to one map $\varphi: D \to g^{-1}(i)$ such that $F \subseteq \varphi(F)$ for every F in D. Since $\mathbb{U}lt_F$ embeds into $\mathbb{U}lt_{F'}$ by some map $e_{F,F'}$ whenever $F \subseteq F'$, we may define a map $\psi: M \to M$ by $\psi((x, F)) := (\psi_{\varphi(F), i}(e_{F, \varphi(F)}(x)), \varphi(F))$. This map is an isometry from M into $\mathbb{M}_{\uparrow f^{-1}(i)}$ proving that M is indivisible. \Box

2. Endogeneous and indivisible ultrametric spaces

2.1. Endogeneity.

Definition 4. *Let* M *and* M′ *be two metric spaces, a* local spectral-embedding*, in brief a* local spec-embedding*, is a local embedding from* M *into* M′ *such that:*

(2)
$$
Spec(M, x) \subseteq Spec(M', f(x)) \text{ for every } x \in Dom(f).
$$

If $M = M'$ *we will simply speak of local spec-embedding* of M.

Definitions 3. *Let* M *be a metric space.*

- *(a)* M *is* spec-endogeneous *if every local spec-embedding of* M *extends to an embedding of* M*.*
- *(b)* M *satisfies the* spec-extension property *if for every* $y \in M$ *, every local specembedding* g *of* M *defined on* y *extends to every other element* x *to a local spec-embedding of* M*.*
- *(c) If furthermore, there are infinitely many such extensions to* x *whose images are pairwise at distance* $d(x, y)$ *, then* M *satisfies the* infinite extension property.

Notations 5. Let $x \in M$, $r \in \mathbb{R}_+$ and B be a ball. We set $S(x,r) := \{y \in M :$ $d(x, y) := r$ *}*, $M(x) := \{y \in M : Spec(M, x) \subseteq Spec(M, y)\}\$, $M(\neg x) := M \setminus M(x)$ *and* $B(x) := B \cap M(x)$

With these notations, the definition (c) above requires that $M(x) \cap S(q(y), d(x, y))$ contains an infinite set whose elements are pairwise at distance $d(x, y)$.

Lemma 6. *If* $x \in B$ *then* $B(x) = \{y \in B : Spec(\mathbb{M}_{\upharpoonright B}, x) \subseteq Spec(\mathbb{M}_{\upharpoonright B}, y)\}$

We have easily:

Lemma 7. *Let* M *be an ultrametric space. The following properties are equivalent:*

- *(i)* M *satisfies the infinite spec-extension property.*
- *(ii) (a)* M *satisfies the spec-extension property.*
	- *(b) For every* $x, y \in M$ *, with* $x \neq y$ *, the set* $C_{y,x} := M(x) \cap S(y, d(y,x))$ *contains infinitely many elements at distance* $d(x, y)$ *from each other.*

Proposition 3. *A countable metric space* M *satisfying the infinite extension property is spec-endogeneous.*

Proof. We prove by induction on n that every local spec-embedding f of M, with domain A having size at most n, extends to every $x \in M \setminus A$ to a local specembedding \overline{f} of M. Since M is countable and every increasing union of local specembedding is a spec-embedding, this will insure that M is spec-endogeneous. Let $n < \omega, A \subseteq M$, with $|A| = n$ and $x \in M \setminus A$. If $n = 0$, the identity map provides the required extension. Suppose $n > 0$. Set $r := d(x, A) := \min\{d(x, y) := y \in A\}$ and $A_0 := \{y \in A : d(x, y) = r\}.$ Let $y \in A_0$. Since $d(x, y) = r, r \in Spec(\mathbb{M}, y)$ and, since f is a local spec-embedding, $Spec(\mathbb{M}, y) \subseteq Spec(\mathbb{M}, f(y))$, hence $r \in spec(\mathbb{M}, f(y))$, that is $B' := B'(f(y), r) \in Nerv(\mathbb{M})$. Since f is an isometry on A, B' is independent of u .

Pick $y_0 \in A_0$. Since M satisfies the infinite spec-extension property, the set $C := M(x) \cap S(f(y_0), r)$ contains infinitely many elements pairwise a distance r. The set $\bigcup \{B(f(y), r) : y \in A_0\}$ contains no more than $|A_0|$ elements at distance r, hence it does not cover C. Pick $x' \in C \setminus \bigcup \{B(f(y), r) : y \in A_0\}$. Extend f by setting $\overline{f}(x) := x'$.

Claim 1. \overline{f} *is a spec-embedding.*

Proof of Claim [1.](#page-6-0) This claim amounts to:

- (1) $d(\overline{f}(x), f(y)) = d(x, y)$ for all $y \in A$.
- (2) $Spec(\mathbb{M}, x) \subseteq Spec(\mathbb{M}, \overline{f}(x)).$

Item (1). Let $y \in A$. Set $r' := d(x, y)$. If $y \in A_0$, $r' = r$. Since $\overline{f}(x) \notin B(f(y), r)$ and $C \subseteq B'$, $d(\overline{f}(x), f(y)) = r'$. If $y \in A \setminus A_0$, then by the definition of $r, r' > r$. Since $d(x,y_0)=r$, we have $d(y_0,y)=r'$ hence $d(f(y_0),f(y))=r'$. Since $d(\overline{f}(x),f(y_0))=r'$ r, it follows that $d(\overline{f}(x), f(y)) = r'$, as required.

Item (2). This follows from the fact that $x' \in C$. With Claim [1](#page-6-0) the proof of Proposition [3](#page-5-0) is complete.

 \Box \Box

Corollary 1. *For a countable ultrametric space* M *the following properties are equivalent:*

- *(i)* M *satisfies the infinite spec-extension property.*
- *(ii) (a)* M *satisfies the spec-extension property.*
	- *(b)* For every $B \in Nerv(M)$ and every son B' of B there are infinitely many *sons* B'' *such that* $\mathbb{M}_{\restriction B'}$ *embeds into* $\mathbb{M}_{\restriction B''}$ *.*

Proof. If Property (ii)b holds then Property (ii)b of Lemma [7](#page-5-1) holds. Indeed, let $x, y \in M$. Set $r := d(x, y)$ and $B := B'(y, r)$ Then $B \in Nerv(M)$ and $B' := B(x, r)$ is a son of B. Moreover, if x' is the image of B' by some embedding g into B then $Spec(M, x) \subseteq Spec(M, x')$ (Lemma [6\)](#page-5-2). Now, if x', x'' are the images of x into two distinct sons, then $d(x', x'') = r$. Hence $C_{y,x}$ contains infinitely many elements, as claimed. Thus with the spec-extension property the infinite spec-extension holds. For the converse, let $B \in Nerv(M)$ and B' be a son of B. Pick $x \in B'$, $y \in B \setminus B'$. Property (2) b of Lemma [7](#page-5-1) asserts that x can be spec-embedded into infinitely many sons of B. Since M is countable, Proposition [3](#page-5-0) applies and $M_{\uparrow B'}$ embeds into these sons. \Box

2.2. Multispectrum, endogeneity and indivisibility.

Proposition 4. *Let* M *be a countable metric space such that every non trivial member of* Nerv(M) *has infinitely many sons. Then the following properties are equivalent:*

- *(i) (a) Every local spec-embedding of* M *defined on a singleton extends to an embedding of* M*.*
	- *(b)* For every $B \in Nerv(\mathbb{M})$, $MSpec(\mathbb{M}_{\restriction B})$ *is up-directed.*
- (*ii*) (*a*) For every $y, y', x \in M$, if $Spec(M, y) \subseteq Spec(M, y')$, there is some $x' \in$ $B'(y', d(x, y))$ *such that* $Spec(\mathbb{M}, x) \subseteq Spec(\mathbb{M}, x')$ *.*

(b) For every $B \in Nerv(\mathbb{M})$ and every $a \in B$, $\mathbb{M}_{\restriction B}$ embeds into $\mathbb{M}_{\restriction B(a)}$.

Proof. Suppose that (i) holds. First (ii)(a) holds trivially. Next (ii)(b) holds. For that we prove first that M has the infinite spec-extension property. We use Lemma [7.](#page-5-1) Let $x, y \in M$, with $x \neq y$. Let $\mathcal{B}_x := \{B' \in Son(B) : B' \cap M(x) \neq \emptyset\}$ and $C_{y,x} := M(x) \cap S(y, d(y,x))$. Clearly $C_{y,x}$ contains infinitely many elements at distance $d(x, y)$ from each other if and only if \mathcal{B}_x is infinite. Suppose that \mathcal{B}_x is finite. Let $B' \in Son(B) \setminus \mathcal{B}_x$. Pick $x' \in B'$. Since $MSpec(\mathbb{M}_{\upharpoonright B})$ is up-directed, there is some $z \in B$ such that $Spec(M, x) \cup Spec(M, x') \subseteq Spec(M, z)$ (use Lemma [6\)](#page-5-2). Since (i)(a) holds, there is an embedding f of M such that $f(x') = z$. This embedding maps each member of \mathcal{B}_x into a member of \mathcal{B}_x , and B' into a member of \mathcal{B}_x . This contradict the supposed finiteness of \mathcal{B}_x .

Next, let $a \in B$. We prove by induction on n that every local spec-embedding f of M_{β} , with domain A having size at most n and range included into $B(a)$, extends to every $x \in B \setminus A$ to a local spec-embedding \overline{f} of $\mathbb{M}_{\upharpoonright B}$ with $\overline{f}(x) \in B(a)$. Since B is countable, this will insures that $\mathbb{M}_{\upharpoonright B}$ embeds into $\mathbb{M}_{\upharpoonright B(a)}$. We do exactly as is the proof of Proposition [3.](#page-5-0) At the final stage, we only have to check that the set $D := B(x) \cap B(a) \setminus \cup \{B(f(y), r) : y \in A_0\}$ is non empty. Since $MSpec(\mathbb{M}_{\upharpoonright B})$ is up-directed, there is some $c \in B$ such that $B(c) \subseteq B(x) \cap B(a)$. Since, from the proof of $ii(a)$ above, \mathcal{B}_c is infinite, D is nonempty.

Conversely, that (ii) holds. (i)(b) follows easily (i)(b). To get that (i)(a) holds it suffices from suppose that M satisfies properties (a) , (b) and (c) . From (b) $MSpec(\mathbb{M}_{\upharpoonright B})$ is up-directed, that is property (2) holds. To conclude, it suffices to prove that M has the infinite spec-extension property and to apply Proposition [3.](#page-5-0)

For that, let $y, y', x \in M$ such that $Spec(\mathbb{M}, y) \subseteq Spec(\mathbb{M}, y')$. Set $r := d(x, y)$, $B' := B'(y', r), C' := \{x' \in B' : Spec(M, x) \subseteq Spec(M, x')\}.$ Our aim is to show that $C' \cap \{x' \in B' : d(y', x') = r\}$ contains infinitely many elements at distance r of each other. This amounts to show that C' has this property. From (c) , C' is non empty. Let $a \in C'$. From (b) , $\mathbb{M}_{\upharpoonright B'}$ embeds into $\mathbb{M}_{\upharpoonright B'(a)}$. According to (a) , B' contains infinitely many elements at distance r of each other. Since $B'(a) \subseteq C'$, C' enjoy this property too. \Box

Lemma 8. *If* M *is indivisible then*

- (1) $M \in Nerv(\mathbb{M})$ *and for each son B of M there are infinitely many sons* B' *such that* $\mathbb{M}_{\restriction B}$ *embeds into* $\mathbb{M}_{\restriction B'}$ *.*
- (2) For every $x \in M$, M embeds into $\mathbb{M}_{\upharpoonright M(x)}$.

Proof. Item (1). The fact that $M \in Nerv(\mathbb{M})$ follows from Theorem [4.](#page-4-2) Let $r := \delta(M)$. If $r = 0$, M has no son and the property holds. So we may suppose $r \neq 0$. Since $M \in Nerv(\mathbb{M})$, r is attained, hence M has at least two sons. Let $B \in Son(M)$. Suppose that M has only finitely many sons B_1, \ldots, B_k such that $M_{\uparrow B}$ embeds into $\mathbb{M}_{\restriction B_i}$ for $i = 1, \ldots, k$. Let $\mathcal{B} := \{B' \in Son(M) : \mathbb{M}_{\restriction B}$ does not embed into $M_{\restriction B'}\}$ and $B_0 := \cup \mathcal{B}$. The sets B_0, \ldots, B_k form a partition of M. Since M is indivisible, M embeds into some $\mathbb{M}_{\restriction B_i}$. Since $\delta(\mathbb{M}) > \delta(M_{\restriction B_i})$ for $i > 0$, we have $i = 0$. But this is impossible, indeed, if g was an embedding, it would send two elements x and y of B into two different sons and we would have $d(x, y) < r = d(g(x), g(y))$. Item (2). We have $M = M(x) \cup M(\neg x)$. Trivially, M does not embeds into $\mathbb{M}_{\upharpoonright M(\neg x)}$. The conclusion follows with the indivisibility of M. \Box Definition 9. *A metric space* M *is* hereditarily indivisible *if* M *is indivisible and for every ball* B *,* $\mathbb{M}_{\restriction B}$ *is indivisible.*

We get for spec-endogeneous metric spaces the analog of the equivalence $(ii) \Leftrightarrow$ (*iii*) of Theorem [5.](#page-4-1)

Theorem 6. *A countable ultrametric space* M *is spec-endogeneous and hereditarily indivisible if and only if it satisfies the following properties:*

- *(1) Every local spec-embedding of* M *defined on a singleton extends to an embedding of* M*.*
- (2) (Nerv(M), ⊃) *is well founded.*
- *(3) Every non-trivial ball of* Nerv(M) *has infinitely many sons.*
- *(4) For every ball* $B \in Nerv(\mathbb{M})$ *,* $MSpec(\mathbb{M}_{\restriction B})$ *is up-directed.*

Proof. Suppose that M is spec-endogenous and hereditarily indivisible. We prove successively that properties (1) , (2) , (3) and (4) are satisfied.

Item (1). Follows from the fact that M is spec-endogeneous.

Item (2). Follows from the fact that M is indivisible, with the help of Theorem [4.](#page-4-2) Item (3). Since M is spec-endogeneous, it has the spec-extension property. Since it is hereditarily indivisible, each non-trivial ball in $Nerv(\mathbb{M})$ embeds into infinitely many sons (Lemma [8\)](#page-7-0).

Item (4). Follows from the fact that M_{β} is indivisible with the help of Lemma [8.](#page-7-0)

Conversely, suppose that M satisfies properties $(1), (2), (3), (4)$. First, from (1) . (3) and (4), M has the infinite spec-extension property (Proposition [4\)](#page-6-1). Since M is countable, M is spec-endogeneous (Proposition [3\)](#page-5-0). To conclude, we have to show that M is hereditarily indivisible. It suffices to prove that M is indivisible. Indeed, if $B \in Nerv(\mathbb{M})$, $\mathbb{M}_{\upharpoonright B}$ satisfies $(1), (2), (3)$ and (4) . Hence, by the same token, $\mathbb{M}_{\upharpoonright B}$ will be indivisible.

Claim 2. For each non-trivial $B \in Nerv(\mathbb{M})$ and every finite set C of sons of B, $\mathbb{M}_{\restriction B}$ *embeds into* $\mathbb{M}_{\restriction B\setminus\cup C}$ *.*

Proof of Claim [2.](#page-8-1) Since M has the infinite spec-extension property, for every ball B of M, $M_{\restriction B}$ has this property. Let $B \in Nerv(M)$. From Corollary [1,](#page-6-2) for every $B' \in Son(B)$ there are infinitely many $B'' \in Son(B)$ such that $\mathbb{M}_{\mid B'}$ embeds into $\mathbb{M}_{\mathbb{B}''}$. Since $Son(B)$ is countable, there is a one-to-one mapping $\psi : Son(B) \rightarrow$ $Son(B)\setminus\mathcal{C}$ such that $\mathbb{M}_{\restriction B'}$ embeds into $\mathbb{M}_{\restriction\psi(B')}$ for each $B\in Son(B)$. With the fact that $B = \cup Son(B)$, this implies that $\mathbb{M}_{\restriction B}$ embeds into $\mathbb{M}_{\restriction B\setminus C}$. \Box

Let $\chi : M \to \{0,1\}$ be a bicoloring of M. Let \mathcal{M}_0 denote the set of balls $B \in Nerv(\mathbb{M})$ such that there is some isometry φ_B from $\mathbb{M}_{\restriction B}$ into $\mathbb{M}_{\restriction B \cap {\setminus \{x^{-1}(0)\}}}$ and let $M_0 := \cup \mathcal{M}_0$. Observe that $M_0 \supseteq \chi^{-1}(0)$.

- **Claim 3.** (1) For every subset N of \mathcal{M}_0 , there is an isometry of M_⊾ \mathcal{M}_0 *into* $\mathbb{M}_{\upharpoonright(\cup\mathcal{N})\cap\chi^{-1}(0)}$.
	- *(2)* Let $B \in Nerv(\mathbb{M})$. If B is included in no member of \mathcal{M}_0 , then $\mathbb{M}_{\restriction B}$ does *not embed in* $\mathbb{M}_{\restriction B \cap M_0}$ *.*

Proof of Claim 3. Both parts rely on the fact that balls are either disjoint or comparable w.r.t. inclusion.

(1) Let \mathcal{N}' denote the set of maximal members of \mathcal{N} (maximal w.r.t. inclusion). Let $\varphi := \bigcup \{ \varphi_B : B \in \mathcal{N}' \}$. Since balls are either disjoint or comparable, φ is a function and, since $P := (Nerv(M), \supseteq)$ is well-founded, $\cup \mathcal{N}' = \cup \mathcal{N}$, hence the domain of φ is $\cup \mathcal{N}$.

(2) Since B is assumed to be included in no member of \mathcal{M}_0 , and balls are either disjoint or comparable, $B \cap M_0 = B \cap (\cup \mathcal{M}_0) = \cup \{B' \in \mathcal{M}_0 : B' \subseteq B\}.$ Hence, according to the first part of the present claim, $\mathbb{M}_{\vert B\cap M_0}$ embeds into $\mathbb{M}_{\restriction B\cap \chi^{-1}(0)}$. On the other hand $\mathbb{M}_{\restriction B}$ does not embed into $\mathbb{M}_{\restriction B\cap \chi^{-1}(0)}$, since we have supposed that $B \notin \mathcal{M}_0$. It follows that $\mathbb{M}_{\restriction B}$ does not embed into $\mathbb{M}_{\restriction B\cap M_0}.$

Now suppose that $M \notin \mathcal{M}_0$.

Claim 4. *Every local spec-embedding* f *of* M *with a finite domain* A *and its range included into* $M \setminus M_0$ *extends to every* $x \in M \setminus A$ *to a local spec-embedding* f *of* M *with range included into* $M \setminus M_0$ *.*

Proof of Claim [4.](#page-9-0) We argue by induction on $n := |A|$. We proceed as for the proof of Proposition [3.](#page-5-0) Suppose $n = 0$. Since $Nerv(\mathbb{M})$ is well-founded, $M \in Nerv(\mathbb{M})$ and we may apply Proposition [4.](#page-6-1) Thus M embeds into $\mathbb{M}_{\vert M(x)}$. Since M does not embed into $\mathbb{M}_{\upharpoonright M_0}$ (Claim 3), $M(x) \setminus M_0$ is non empty; choose any element x' in it and set $\overline{f}(x) := x'$.

Suppose $n > 0$. Set $r := d(x, A) := \min\{d(x, y) := y \in A\}$, $A_0 := \{y \in A$: $d(x,y) = r$ and $\mathcal{C} := \{B(f(y), r) : y \in A_0\}$. Our aim is to find some x' in the intersection of $M \setminus M_0$, $M(x)$ and \cap { $S(f(y), r) : y \in A_0$ }. Indeed, setting $\overline{f}(x) := x'$, the same argument as in Proposition [3](#page-5-0) yields that \overline{f} is a spec-embedding.

Let $y \in A_0$. Since $d(x, y) = r$, $r \in Spec(M, y)$ and, since f is a local specembedding, $Spec(M, y) \subseteq Spec(M, f(y))$, hence $r \in spec(M, f(y))$, that is $B' :=$ $B'(f(y), r) \in Nerv(\mathbb{M})$. Since $f_{\upharpoonright A_0}$ is an isometry, B' is independent of y.

Our aim reduces to find some $C \in Son(B') \setminus C$ and such that $C(x) := C \cap M(x)$ is not included into M_0 . For that, it suffices to prove that

(3)
$$
\mathbb{M}_{\restriction B'} \text{ embeds into } \mathbb{M}_{\restriction B'(x)\setminus\cup\mathcal{C}}
$$

Indeed, according to Claim 3, $\mathbb{M}_{\vert B'}$ does not embed into $\mathbb{M}_{\vert B'\cap N_0}$. Hence $B'(x)\setminus\cup\mathcal{C}$ is not included into M_0 . Since $B'(x) \setminus \cup \mathcal{C} = \cup \{C(x) : C \in Son(B) \setminus \mathcal{C}\}\)$, there is some $C \in Son(B) \setminus C$ such that $C(x)$ is not included into M_0 .

To get (4), we prove first that $\mathbb{M}_{\restriction B'}$ embeds into $\mathbb{M}_{\restriction B'(x)}$. Indeed, pick $y_0 \in A_0$. Since M satisfies the infinite spec-extension property, there is some $x' \in M$ such that $d(x', f(y_0)) = d(x, y_0)$ and $Spec(x, M) \subseteq Spec(M, x')$. Let $B' := \{x' \in B' :$ $Spec(\mathbb{M}_{\upharpoonright B'}, x) \subseteq Spec(\mathbb{M}_{\upharpoonright B'}, x')$. We have $B'' = B'(x)$, hence, from Proposition [4,](#page-6-1) $\mathbb{M}_{\vert B'}$ embeds into $\mathbb{M}_{\vert B'(x)}$. Next, applying Claim [2](#page-8-1) we get that $\mathbb{M}_{\vert B'}$ embeds into $\mathbb{M}_{\{B\}\cup\mathcal{C}}$. If g and h are two such embeddings, in this order, then h∘g is an embedding of $\mathbb{M}_{\restriction B'}$ into $\mathbb{M}_{\restriction B'(x)\setminus\cup\mathcal{C}}$. \Box

Since M is countable, Claim 3 insures that M embeds into $\mathbb{M}_{\mid M\setminus M_0}$. Since M \setminus $M_0 \subseteq \chi^{-1}(1)$, M embeds into $\mathbb{M}_{\restriction \chi^{-1}(0)}$. This proves that M is indivisible.

3. Extensions of indivisible ultrametric spaces

The purpose of this section is to prove:

 \Box

 \Box

Theorem 7. *A countable ultrametric space* M *embeds into a countable indivisible ultrametric space if and only if it does not contain an infinite strictly increasing sequence of balls. Furthermore, when this condition holds* M *embeds into a countable spec-endogeneous indivisible ultrametric space with the same spectrum*

The fact that the condition on balls is necessary follows from Theorem [4.](#page-4-2) For the sufficiency, we construct an extension of M to which we can apply Theorem [6.](#page-8-0)

The key notions are these:

Definitions 4. *Let* M *be a metric space; a binary operation, denoted* +*, on* M *is* compatible *if*

(4)
$$
d(z+x, z+y) = d(x,y) = d(x+z, y+z)
$$

for all x, y, z ∈ M*. An* ultrametric monoid *is an ultrametric space* M *endowed with a compatible operation* + *such that* M *with this operation is a monoid.*

Indeed, we will prove that M extends to an ultrametric monoid M[∗] with the same spectrum, and the same condition on balls (Theorem [8\)](#page-11-0) and having infinitely many sons. Next, we will extend \mathbb{M}^* to an other ultrametric monoid, $Path(\mathbb{M}^*)$, such that each of its balls can be also endowed with a structure of ultrametric monoid (Theorem [9,](#page-14-0) and Theorem [10\)](#page-16-0). Finally we will prove that this space is spec-endogeneous and hereditary indivisible (Theorem [11\)](#page-19-0).

3.1. Monoids extensions of an ultrametric space. Let $\mathbb{M} := (\omega^{[\lambda]}, d_w)$, as defined in Subsectio[n1.2.](#page-3-0) For $f, g \in \omega^{[\lambda]}$ let $f + g$ be defined by $(f + g)(\mu) =$ $f(\mu)+g(\mu)$, and let 0 be the constant map equal to 0. With this, M is a comutative monoid. Furthermore, the operation is compatible. Thus M is a commutative ultrametric monoid.

The set $\omega^{\leq [\lambda]}$ ordered by extension is a ramified mee-tree in which every element is below some maximal one. For $x, x' \in \omega^{\leq [\lambda]}$, we denote by $x \wedge x'$ the meet of x, x' . Let $X \subseteq \omega^{[\lambda]}$. Set X^* for the set of finite sums of members of X with 0 included. Let $T(X) := \{e \wedge e' : e, e' \in X\}$. It is easy to show that $T(X)$ is a meet-tree; we call it *the meet-tree generated by* X.

Lemma 10. Let $X \subseteq \omega^{[\lambda]}$. If $T(X)$ is well-founded then $T(X^*)$ is well-founded.

Proof. Suppose that $T(X^*)$ is not well-founded. Let $Y \subseteq X^*$ and let $y_0 > y_1 \cdots$ $y_n > \cdots$ be an infinite strictly decreasing sequence of members of $T(Y)$. Let $\overline{y} \in Y$ such that $\overline{y} \geq y_0$.

Claim 1. There is an infinite sequence $\overline{y}_0, \overline{y}_1, \ldots, \overline{y}_n, \ldots$ of members of Y such that $y_n = \overline{y} \wedge \overline{y}_n$ for all $n \in \mathbb{N}$.

Proof of Claim 1. Since $y_n \in T(Y)$ there are $e_n, e'_n \in Y$ such that $y_n = e_n \wedge e'_n$. Since $\overline{y} \ge y_n$, we have either $y_n = \overline{y} \wedge e_n$ or $y_n = \overline{y} \wedge e'_n$. In the first case set $\overline{y}_n := e_n$ and in the second case $\overline{y}_n := e'_n$. \Box

With no loss of generality, we may suppose $\overline{y} > y_0$ (otherwise a subsequence of the y_n 's will do). Thus for every $n < \omega, \overline{y} \neq \overline{y}_n$. Let λ_n be the least element of λ such that

$$
\overline{y}(\lambda_n) \neq \overline{y}_n(\lambda_n)
$$

Then $\left(\leftarrow \lambda_n\right]$ is the domain of $\overline{y} \wedge \overline{y}_n$.

Since $y_0 > y_1 > \cdots > y_n > \cdots$ we have $\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots$. Let $A := \bigcup \{ |\lambda_{n+1}, \lambda_n| : n < \omega \}, B := \bigcap \{ (\leftarrow, \lambda_n| : n < \omega \},$

Since $Supp(\overline{y})$ is finite, we may suppose that it is disjoint from A (otherwise a subsequence of the \overline{y}_n 's will do). In particular $\overline{y}(\lambda_n) = 0$. From [\(5\)](#page-10-1) we have $\overline{y}(\lambda_n) \neq \overline{y}_n(\lambda_n)$. Hence $\overline{y}_n(\lambda_n) > 0$.

Since $\overline{y}_n \in Y \subseteq X^*$, this is a finite sum of members of X^* . We may choose such a member \overline{z}_n such that $\overline{z}_n(\lambda_n) \neq 0$. Since \overline{z}_n is a term in a finite sum which is equal to \overline{y}_n , we have $\overline{z}_n \leq \overline{y}_n$, meaning that

(6)
$$
\overline{z}_n(\mu) \leq \overline{y}_n(\mu) \text{ for every } \mu < \lambda.
$$

Claim 2. \overline{z}_n is 0 on ($\leftarrow \lambda_n \cap A$.

Proof of Claim 2.

Combine [\(5\)](#page-10-1) and inequality [\(6\)](#page-11-1) for $\mu \in (\leftarrow \lambda_n \cap A$.

$$
\Box
$$

Claim 3. There are infinitely many \overline{z}_n 's whose restrictions to B are the same.

Proof of Claim 3. Since $B \subseteq (\leftarrow \lambda_n])$, \overline{y}_n and \overline{y} coincide on B. Hence from inequality [\(6\)](#page-11-1), $z_n(\mu) \leq \overline{y}(\mu)$ for $\mu \in B$. We also have $Supp(\overline{y}) \cap B = Supp(\overline{y}_n)$. If K denote this set, we have $\overline{z}_n(\mu) = 0$ if $\mu \in B \setminus K$. Now, since K is finite and \overline{y} takes only non-negative integer values, there are infinitely many \overline{z}_n 's which coincide on K. These \overline{z}_n 's coincide on B.

Let $\overline{z}_{n_0}, \ldots, \overline{z}_{n_k}, \ldots$ be an infinite subsequence of \overline{z}_n 's such that the \overline{z}_{n_k} 's coincide on B. For each $k < \omega$ set $x_k := \overline{z}_{n_0} \wedge \overline{z}_{n_k}$.

Claim 4. For $k \geq 1$, x_k is the restriction of z_{n_0} to $\left(\leftarrow \lambda_{n_k}$.

Proof of Claim 4.

First, from our construction z_{n_0} and z_{n_k} coincide on B. Next, from Claim 2, z_{n_0} and z_{n_k} are 0 on ($\leftarrow \lambda_{n_0}$ $\cap A$ and ($\leftarrow \lambda_{n_k}$ $\cap A$ respectively. Since $n_0 < n_k$, we have $\lambda_{n_k} < \lambda_{n_0}$. It follows that z_{n_0} and z_{n_k} coincide on $\left(\leftarrow \lambda_{n_k}$ and $z_{n_0}(\lambda_{n_k}) = 0 \neq 0$ $z_{n_k}(\lambda_{n_k})$. The result follows. $\mathbf{\Pi}$

From Claim 3, we immediately have:

Claim 4. The sequence x_0, \ldots, x_k, \ldots is strictly decreasing.

Consequently, $T(X)$ contains an infinite strictly decreasing sequence. With that, the proof of the lemma is complete. \Box

Theorem 8. *Every countable ultrametric space* M *extends to an ultrametric commutative monoid* M[∗] *having the same spectrum. Moreover, if* M *has a well founded nerve,* M[∗] *too.*

Proof. Let M be countable. Let $\lambda := Spec(M) \setminus \{0\}$ dually ordered. Then M isometrically embeds into $\omega^{[\lambda]}$. Let X be its image. Set $\mathbb{M}^* := X^*$.

We may note that if T has at least two elements then M^* has infinitely many sons.

3.2. The path extension of an ultrametric space. We define the path extension $\mathbb{P}ath(\mathbb{M})$ of an ultrametric space M. Its elements are finite unions of chains in $(Nerv(M) \supseteq).$

Notations 11. Let \mathbb{M} be an ultrametric space. Let $r \in \mathbb{R}_+$, set $Nerv_r(\mathbb{M}) := \{B \in$ $Nerv(\mathbb{M}) : \delta(B) = r\}$. Let $\beta < \alpha \in \mathbb{R}_+^* \cup \{+\infty\}$. Set $Nerv_{[\beta,\alpha]}(\mathbb{M}) := \bigcup \{Nerv_r(\mathbb{M}) :$ $\beta \leq r < \alpha\}$ and set $Nerv_{\leq \alpha}(\mathbb{M}) := Nerv_{[0,\alpha]}(\mathbb{M})$. For $B \in Nerv_{\leq \alpha}(\mathbb{M})$ set

$$
]_{\alpha} \leftarrow B] := \{ C \in Nerv_{< \alpha}(\mathbb{M}) : C \supseteq B \}
$$

Definition 12. *A subset* B *of* Nerv(M) *is* slim *if*

(7)
$$
\delta(B) = \delta(B') \Rightarrow B = B'
$$

for all $B, B' \in \mathcal{B}$

.

Let $\mathcal{B} \subseteq Nerv(\mathbb{M})$. We set $Spec(\mathcal{B}) := {\delta(B) : B \in \mathcal{B}}$. If $Spec(\mathcal{B})$ has a least element (w.r.t the order on the reals), we denote it $\delta(\mathcal{B})$. If moreover, \mathcal{B} is slim, we denote by $end(\mathcal{B})$ the unique $B \in \mathcal{B}$ such that $\delta(B) = \delta(\mathcal{B})$. Let \mathcal{B} be a finite non-empty slim subset of $Nerv_{\leq \alpha}(\mathbb{M})$. Let $n := |\mathcal{B}|$. We denote by $\overline{\beta} := (\beta_i)_{i \leq n}$ the unique enumeration of $Spec(\mathcal{B})$ into a decreasing order $\beta_0 > \cdots > \beta_{n-1}$. The *enumeration of* \mathcal{B} is the sequence $\mathcal{B} := (B_i)_{i \leq n}$ of elements of \mathcal{B} such that $\delta(B_i) = \beta_i$. We set $\beta_{-1} := \alpha$ and we set:

$$
]_{\alpha} \leftarrow \mathcal{B}]:= \bigcup \{]_{\beta_{i-1}} \leftarrow B_i]: i < n \}
$$

Definition 13. An α -path *is any subset* \mathcal{I} *of* $Nerv_{\alpha}(\mathbb{M})$ *of the form* $\mathcal{I} =]_{\alpha} \leftarrow \mathcal{B}].$ *We denote by* L_{α} *the set of* α *-paths.* A finite set B generates an α *-path* I *if* I = $]_{\alpha} \leftarrow \mathcal{B}].$

Fact 1. If B is slim, $\mathcal{B}' \subseteq \mathcal{B}$ and $Spec(\mathcal{B}) = Spec(\mathcal{B}')$ then $\mathcal{B} = \mathcal{B}'$.

Fact 2. Let $B, B' \in Nerv_{\leq \alpha}(\mathbb{M})$ *such that* $B' \supseteq B$ *. Then:*

$$
]_{\alpha} \leftarrow B] =]_{\alpha} \leftarrow B'] \cup]_{\delta(B')} \leftarrow B]
$$

Proof of Fact [2.](#page-12-0) Observe that two balls containing B are comparable w.r.t. inclusion. \Box

Fact 3. *Every* α*-path is slim.*

Fact 4. If B generates the α -path I, then every finite subset \mathcal{B}' of I which contains B *generates* I*.*

Fact 5. If B is a finite non-empty slim subset of $Nerv_{\leq \alpha}(\mathbb{M})$ then end(B) = $end([\alpha \leftarrow \mathcal{B}]).$

Fact 6. A set B generates an α -path I if and only if it satisfies the following con*ditions:*

- *(1)* \mathcal{B} *is a finite subset of* \mathcal{I} *.*
- (2) end(\mathcal{B}) = end(\mathcal{I}).
- *(3)* If \overline{B} *is the enumeration of* \overline{B} *then for every* $B \in \mathcal{I}$, *if* $\beta_{i-1} > \delta(B) \geq \delta(B'_i)$ *for some* $i < n$ *then* $B \supseteq B'_i$ *.*

Proof of Fact [6.](#page-12-1) The three conditions stated are obviously necessary. Suppose that they hold. According to Fact [3,](#page-12-2) $\mathcal{J} :=]_{\alpha} \leftarrow \mathcal{B}$ is well-defined. Condition (3) yields that $\mathcal{I} \subseteq \mathcal{J}$. For the converse, let $i < n$ and $\mathcal{A}_i :=]_{\beta_{i-1}} \leftarrow B_i$. We prove that $\mathcal{A}_i \subset \mathcal{I}$. For that, let \mathcal{B}' such that $\mathcal{I} =]_{\alpha} \leftarrow \mathcal{B}'$ and let $\overline{\mathcal{B}'} := (B'_i)_{i \leq n'}$ be the corresponding enumeration of B'. Since $B_i \in \mathcal{I}$ there is some B'_j such that $\delta(B'_{j-1}) > \delta(B_i)$ and $B_i \supseteq B'_j$. It follows that $\big|\delta(B'_{j-1}) \big| \leftarrow B_i \big| \subseteq I$. If $\delta(B'_{j-1}) \geq \delta(B_{i-1})$ we are done. If not, let $B' \in \mathcal{A}_i$ and B'_k such that $\delta(B'_{k-1}) > \delta(B') \ge B'_k$. If $k = j$, $B' \supseteq B'_j$, hence $B' \in \mathcal{I}$. If $k < j$ then since \mathcal{B} satisfies Condition (3), we have $B'_{k-1} \supseteq B_i$. From Fact [2](#page-12-0) this yields $B' \ge B'_k$, hence $B' \in \mathcal{I}$. \Box **Fact 7.** Let $\mathcal I$ be an α -path. Then $\mathcal I \cap Nerv_{\leq \beta}(\mathbb{M})$ is a β -path provided that it is *non empty and* $\alpha > \beta$.

Fact 8. Let *I* be an α -path. Then $I \setminus Nerv_{\leq \beta}(\mathbb{M})$ is an α -path whenever $\beta \in$ $Spec(\mathcal{I}).$

Proof of Fact [3.2.](#page-13-0) Let $B \in \mathcal{I}$ such that $\delta(B) = \beta$. Let B which generates \mathcal{I} . According to Fact [4](#page-12-3) we may suppose that $B \in \mathcal{B}$. Use the definition of $]_{\alpha} \leftarrow \mathcal{B}$ to conclude. \Box

Fact 9. Let $\mathcal I$ be an α -path and $\mathcal J$ be a $\delta(\mathcal I)$ -path. Then $\mathcal I \cup \mathcal J$ is an α -path.

Definition 14. *A finite slim subset* \mathcal{B} *of* $Nerv_{\leq \beta}(\mathbb{M})$ *is* pure *if two consecutive terms in the enumeration of* B *are incomparable w.r.t inclusion.*

Lemma 15. *Every* α -path $\mathcal I$ *is generated by a unique pure set.*

Proof. Let B a generating subset of I with minimum size and let $\overline{\mathcal{B}} := (B_i)_{i \le n}$ be its enumeration. Then β is pure. Indeed, suppose that B_i and B_{i+1} are comparable. Then $B_i \supset B_{i+1}$. It follows that $\mathcal{B} \setminus \{B_i\}$ satisfies the conditions of Fact [6.](#page-12-1) Hence it generates I . This contradicts the minimality of the size of B . We show the uniqueness of \mathcal{B} by induction on $n := |\mathcal{B}|$.

Claim 5. Let \mathcal{I}_0 be the subset of \mathcal{I} made of the elements B such that: *(a)* \vert_{α} ← *B* \vert \subseteq *I (b)* for every $B' \in \mathcal{I}$, if $\alpha > \delta(B') \geq \delta(B)$ then $B' \supseteq B$. *Then* $\mathcal{I}_0 :=]_\alpha \leftarrow B_0$

The proof is immediate and we omit it. Now, set $\mathcal{I}' := \mathcal{I} \setminus \mathcal{I}_0$. If $\mathcal{I}' = \emptyset$ we are done. If not, then \mathcal{I}' is a β_0 -path (Fact [7\)](#page-13-0) and $\mathcal{I}' =]_{\beta_0} \leftarrow \mathcal{B}'$ where $\mathcal{B}' := \mathcal{B} \setminus \{B_0\}.$ Clearly \mathcal{B}' has minimum size. Hence induction applies. The result follows. \Box

Notation 16. Let *I* be a slim set, let a with end(*I*) $\lt a$. Let $P_a(\mathcal{I})$ be the set of $x \in Spec(\mathcal{I})$ *such that:*

(8)
$$
x \leq \delta(\mathcal{B}) \leq \delta(\mathcal{B}') < a \Longrightarrow B \subseteq B'
$$

for all $B, B' \in \mathcal{I}$ *. If this set has a least element, we denote it by* $\mu_a(\mathcal{I})$ *.*

For an example, if $\mathcal I$ *is an* α *-path,* $\mu_\alpha(\mathcal I) = Max(Spec(\mathcal B))$ *where* $\mathcal B$ *is the pure set generating* \mathcal{I} *. In this case, we set* $l(\mathcal{I}) := |\mathcal{B}|$ *,* $\mu(\mathcal{I}) := \mu_{\alpha}(\mathcal{I})$ *, init*(\mathcal{I}) := B_0 *such that* $B_0 \in \mathcal{I}$ *and* $\delta(B_0) = \mu(I)$ *.*

Let $\mathcal{I}',\mathcal{I}''$ be two α -paths. We set $\mathcal{I}' \leq_{\alpha} \mathcal{I}''$ if there is some $\beta \in Spec(\mathcal{I}'')$ such that $\mathcal{I}' = \mathcal{I}'' \setminus \mathbb{M}_{\leq \beta}$. Let \mathcal{B}' and \mathcal{B}'' be the pure generating subsets of \mathcal{I}' and \mathcal{I}'' respectively and let $\overline{\mathcal{B}'} := (B_i')_{i \leq n'}$ and $\overline{\mathcal{B}''} := (B_i'')_{i \leq n''}$ be the corresponding sequences. Set $\overline{\mathcal{B}'}_* = (B_i')_{i \leq n'-1}$ and $\overline{\mathcal{B}''}_* = (B_i'')_{i \leq n''-1}$.

Set $\overline{\mathcal{B}'} \leq_{\alpha} \overline{\mathcal{B}''}$ if $\overline{\mathcal{B}'}_*$ is a prefix of $\overline{\mathcal{B}''}$ and $B'_{n-1} \supseteq B''_{n-1}$.

Fact 10. We have $\overline{\mathcal{B}'} \leq_{\alpha} \overline{\mathcal{B}''}$ if and only if $\mathcal{I}' \leq_{\alpha} \mathcal{I}''$.

Fact 11. $\mathcal{I}' \leq_{\alpha} \mathcal{I}''$ if and only if there is some $\delta(\mathcal{I}')$ -path \mathcal{J}'' such that $I'' = \mathcal{I}' \cup \mathcal{J}''$ **Remark 17.** If $\mathcal{I}' \leq_{\alpha} \mathcal{I}''$ then $\mathcal{I}' \subseteq \mathcal{I}''$. But the converse does not necessarily holds. For $\mathcal{J} \in L_{\alpha}$, set $(\leftarrow \mathcal{J}]_{L_{\alpha}} := \{ \mathcal{I} \in L_{\alpha} : \mathcal{I} \leq_{\alpha} \mathcal{I} \}.$

Fact 12. Let $\mathcal{I}', \mathcal{I}'' \in (\leftarrow \mathcal{J}]_{L_{\alpha}}$. Then: $\mathcal{I}' \leq_{\alpha} \mathcal{I}''$ if and only if $\delta(\mathcal{I}') \geq \delta(\mathcal{I}'')$.

Fact 13. The relation \leq_{α} is an order on the set L_{α} of α -paths. For every $\mathcal{J} \in L_{\alpha}$, *the set* $\left(\leftarrow \mathcal{J}\right]_{L_{\alpha}}$ *is linearly ordered.*

Fact 14. *If* $(Nerv_{\leq \alpha}(\mathbb{M}), \supseteq)$ *is well-founded, then* $(L_{\alpha}, \leq_{\alpha})$ *too.*

Proof of Fact [14.](#page-14-1) Let $\mathcal{J} \in Nerv_{\leq \alpha}(\mathbb{M})$. Observe that $Spec(\mathcal{J})$ is a finite union of dually well ordered chains and apply Fact [12.](#page-14-2)

Let \perp_{α} be a set not belonging to L_{α} . Extend the order \leq_{α} to $\overline{L}_{\alpha} := L_{\alpha} \cup \{\perp_{\alpha}\},\$ with the requirement that $\perp_{\alpha} \leq \mathcal{I}$ for all $\mathcal{I} \in L_{\alpha}$.

Lemma 18. *Two elements* $\mathcal{I}', \mathcal{I}''$ *of* \overline{L}_{α} *have an infimum in* \overline{L}_{α} *that we will denote* $\mathcal{I}' \wedge_\alpha \mathcal{I}''$.

Proof. If \mathcal{I}' and \mathcal{I}'' are comparable, we have $\mathcal{I}' \wedge_{\alpha} \mathcal{I}'' = Min{\{\mathcal{I}', \mathcal{I}''\}}$. Otherwise, proceed by induction on $n := l(\mathcal{I}') + l(\mathcal{I}'')$. Set $B'_0 := init(\mathcal{I}')$, $B''_0 := init(\mathcal{I}'')$, $\beta := \delta(B_0' \cup B_0''), B := B'[B_0' \cup B_0'', \beta] \text{ and } A :=]_{\alpha} \leftarrow B_0' \cap]_{\alpha} \leftarrow B_0''].$ Hence $\mathcal{A} =]_{\alpha} \leftarrow B$] if $\alpha > \beta$.

Case 1. $B'_0 = B''_0$. In this case $B = B_0$. Set $\mathcal{I}'_1 := \mathcal{I}' \setminus \mathcal{A}$ and $\mathcal{I}''_1 := \mathcal{I}'' \setminus \mathcal{A}$. Since \mathcal{I}' and \mathcal{I}'' are incomparable, \mathcal{I}'_1 and \mathcal{I}''_1 are non-empty. Hence $\mathcal{I}'_1, \mathcal{I}''_1 \in L_\beta$ and $l(\mathcal{I}'_1) + l(\mathcal{I}''_1) = n-2$. Hence induction applies. Let $I_1 := \mathcal{I}'_1 \wedge_{\beta} \mathcal{I}''_1$ in \overline{L}_{β} . If $I_1 = \perp_{\beta}$ then $\mathcal{I}' \wedge_\alpha \mathcal{I}'' = \mathcal{A}$. If $I_1 \neq \bot_\beta$ then $\mathcal{I}' \wedge_\alpha \mathcal{I}'' = \mathcal{A} \cup I_1$.

Case 2. $B'_0 \neq B''_0$. In this case, $\mathcal{I}' \wedge \mathcal{I}'' = \mathcal{A}$ if $\alpha > \beta$. Otherwise $\mathcal{I}' \wedge_{\alpha} \mathcal{I}'' = \perp_{\alpha}$.

Notations 19. Let $\alpha \in \mathbb{R}_+ \cup \{+\infty\}$. Set $\delta(\perp_\alpha) = \mu(\perp_\alpha) := \alpha$. For $\mathcal{I}', \mathcal{I}'' \in L_\alpha$, *set* $d_{\alpha}(\mathcal{I}', \mathcal{I}'') := \delta(\mathcal{I}' \wedge_{\alpha} \mathcal{I}'')$ *. Let* $\beta < \alpha$ *, an* (α, β) -path *is any* α -path \mathcal{I} *such that* $\delta(\mathcal{I}) = \beta$ *. We denote* $L_{\alpha,\beta}$ *the set of* (α,β) *-paths. We set* $L_{\alpha,\beta} := L_{\alpha,\beta} \cup \{\perp_{\alpha}\}.$

Lemma 20. *Let* $\mathcal{I}', \mathcal{I}'', \mathcal{J} \in L_{\alpha}$. *Then:*

- (1) $d_{\alpha}(\mathcal{I}', \mathcal{I}'') \leq \mu(\mathcal{I}' \wedge_{\alpha} \mathcal{I}'').$
- (2) $d_{\alpha}(\mathcal{I}', \mathcal{I}'') \leq Max\{d_{\alpha}(\mathcal{I}', \mathcal{J}), d_{\alpha}(\mathcal{I}'', \mathcal{J})\}.$ *Moreover, if* $\mathcal{I}', \mathcal{I}'', \mathcal{J} \in L_{\alpha,\beta}$ then:
- (3) $d_{\alpha}(\mathcal{I}', \mathcal{I}'') = \beta$ *if and only if* $\mathcal{I}' = \mathcal{I}''$ *.*

Proof. Item (1). According to our definitions of δ and μ , we have $\delta(\mathcal{J}) \leq \mu(\mathcal{J})$ for all $\mathcal{J} \in L_{\alpha}$.

Item (2). We have $\mathcal{I}' \wedge_\alpha \mathcal{J} \leq_\alpha \mathcal{I}'$ and $\mathcal{I}'' \wedge_\alpha \mathcal{J} \leq_\alpha \mathcal{I}''$. Hence $\mathcal{I}' \wedge_\alpha \mathcal{I}'' \wedge \mathcal{J} \leq_\alpha \mathcal{I}' \wedge_\alpha \mathcal{I}''$ (*). We have $\mathcal{I}' \wedge_{\alpha} \mathcal{J} \leq_{\alpha} \mathcal{J}$ and $\mathcal{I}'' \wedge_{\alpha} \mathcal{J} \leq_{\alpha} \mathcal{J}$. Hence, $\mathcal{I}' \wedge_{\alpha} \mathcal{J}$ and $\mathcal{I}'' \wedge_{\alpha} \mathcal{J}$ are comparable (Fact [13\)](#page-14-3). Suppose $\mathcal{I}' \wedge_{\alpha} \mathcal{J} \leq_{\alpha} \mathcal{I}'' \wedge_{\alpha} \mathcal{J}$. In this case, (*) yields $\mathcal{I}' \wedge_\alpha \mathcal{J} \leq_\alpha \mathcal{I}' \wedge_\alpha \mathcal{I}''$. Hence, with Fact [12,](#page-14-2) $d_\alpha(\mathcal{I}', \mathcal{J}) \geq d_\alpha(\mathcal{I}', \mathcal{I}'').$ Item (3). Apply Fact [12.](#page-14-2) \Box

Notation 21. We set $\mathbb{M}_{\alpha} := (M, d \wedge \alpha)$, where $d \wedge \alpha$ is defined by $d \wedge \alpha(x, y) :=$ $Min({d(x,y), \alpha})$ *. For* $x \in M$ *, we set* $\varphi_{\alpha}(x) :=]_{\alpha} \leftarrow \{x\}$ *. We denote by d be the restriction of* d_{α} *to* $L_{\alpha,0}$ *, we set* $Path_{\alpha}(\mathbb{M}) := L_{\alpha,0}$ *and* $Path_{\alpha}(\mathbb{M}) := (Path_{\alpha}(\mathbb{M}), d)$ *.*

Theorem 9. Let M be an ultrametric space and $\alpha \in \mathbb{R}_+^* \cup \{+\infty\}$. Then:

- *(1)* $\mathbb{P}ath_{\alpha}(\mathbb{M})$ *is an ultrametric space.*
- *(2) The map* φ_{α} *is an isometric embedding of* \mathbb{M}_{α} *into* $\mathbb{P}ath_{\alpha}(\mathbb{M})$ *. Moreover* $Spec(\mathbb{M}_{\alpha},x) = Spec(\mathbb{P}ath_{\alpha}(\mathbb{M}),\varphi_{\alpha}(x))$ *for every* $x \in \mathbb{M}$ *.*
- *(3)* $Spec(\mathbb{M}_{\alpha}) = Spec(\mathbb{P}ath_{\alpha}(\mathbb{M}))$.

Proof. Item (1). Apply item (2) and Item (3) of Lemma [20.](#page-14-4) Item (2).

Let $x, y \in M$. Set $r := d_{\alpha}(x, y) = Min{d(x, y), \alpha}$, $X := \varphi_{\alpha}(x) \wedge_{\alpha} \varphi_{\alpha}(y)$. According to Lemma [20,](#page-14-4) we have $d(\varphi_\alpha(x), \varphi_\alpha(y)) = \delta(X)$. Let B be the least ball in Nerv(M) containing x and y. If $d(x, y) \ge \alpha$, that is $r = \alpha$, then $X := \perp_{\alpha}$, hence $\delta(X) = \alpha$. If not, $X = \alpha$ + B, in which case $\delta(X) = r$. In both cases $d(\varphi_\alpha(x), \varphi_\alpha(y)) = r$. This proves that φ_α is an embedding. Let $\mathcal{I} \in Path_\alpha(\mathbb{M})$ such that $d(\varphi_\alpha(x), \mathcal{I}) = r$. Hence for $Y := \mathcal{I} \wedge_\alpha \varphi_\alpha(x)$ we have $\delta(Y) = r$. Pick $y \in \mathbb{M}$ such that $\{y\} \in \mathcal{I}$. We have $d(x, y) \geq \delta(Y)$. Thus $d(x, y) \geq \alpha$ if $\delta(Y) = \alpha$. If not, there is $B \in Nerv(M)$ such that $Y =]\alpha \leftarrow B]$. In this case $\delta(B) = r$. Thus $r \in Spec(\mathbb{M}_{\alpha}, x)$.

Item(3). $\delta(\mathcal{I}) \in Spec(\mathbb{M}_{\alpha})$ for every $\mathcal{I} \in L_{\alpha,0}$ hence $Spec(\mathbb{P}ath_{\alpha}(\mathbb{M})) \subseteq Spec_{\alpha}(\mathbb{M})$. The reverse inclusion follows from Item 2. \Box

Notation 22. *We denote by* $\mathbb{P}ath(\mathbb{M})$ *the space* $\mathbb{P}ath_{+\infty}(\mathbb{M})$ *. We call it the path* extension *of* M*.* For $r \in \mathbb{R}_+^*$ and $\mathcal{I} \in L_{+\infty,r}$, we set $\mathcal{I} * L_{r,0} := \{I \cup J : J \in L_{r,0}\}.$

Lemma 23. Let $B \subseteq Path(\mathbb{M})$. Then B is a non trivial member of $Nerv(\mathbb{P}ath(\mathbb{M}))$ *if and only if there are* $r \in \mathbb{R}_+ \setminus \{0\}$ *and* $\mathcal{I} \in L_{+\infty,r}$ *such that* $B = \mathcal{I} * L_{r,0}$ *.*

Proof. Let $\mathbb{P} := \mathbb{P}ath(\mathbb{M})$. Suppose that $B = \mathcal{I} * L_{r,0}$. Let $X \in B$ and $Y \in Path(\mathbb{M})$ such that $d(X, Y) \leq r$. We have $\delta(X \wedge_{+\infty} Y) \leq r$. Thus $\mathcal{I} \leq_{+\infty} Y$, or equivalently $Y = X \cup Y'$ for some $Y' \in L_{r,0}$, that is $Y \in B$. Hence $B \in Nerv_r(\mathbb{P})$. Conversely, let $B \in Nerv_r(\mathbb{P})$. Then there are $X, Y \in B$ such that $d(X, Y) = r$. Set $\mathcal{I} := X \wedge_{+\infty} Y$ and $B' := \mathcal{I} * L_{r,0}$. Since, as we have just seen, B' is a ball or radius r, and since B' contains B, we have $B = B'$. \Box

Corollary 2. *Two members of* Nerv(Path(M)) *are isometric if and only if they have the same diameter.*

Proof. Let $B, B' \in Nerv_r(\mathbb{P}ath(\mathbb{M}))$. We may suppose $r \neq 0$. From Lemma [23,](#page-15-0) $B=\mathcal{I}*L_{r,0}$ and $B'=\mathcal{I}'*L_{r,0}$. For $\mathcal{J}\in B$ set $f(\mathcal{J}):=\mathcal{I}'\cup(\mathcal{J}\setminus\mathcal{I})$. Then f defines an isometry of B onto B' . \Box

Lemma 24. Let $r \in Spec(\mathbb{M}) \setminus \{0\}$. If some $B \in Nerv_r(\mathbb{M})$ has infinitely many *sons, then every* $B' \in Nerv_r(\mathbb{P}ath(\mathbb{M}))$ *has infinitely many sons.*

Proof. Let $B' \in Nerv_r(\mathbb{P}ath(\mathbb{M}))$. According to Lemma [23,](#page-15-0) $B = \mathcal{I} * L_{r,0}$ for some $\mathcal{I} \in L_{+\infty,r}$. For $x \in M$, set $\varphi_r(x) := |r| \leftarrow \{x\}$ and $\theta(x) := \mathcal{I} \cup \varphi_r(x)$. Then, as one can readily see, θ is an isometry from B into B'. Since these two balls have the same diameter, B' has as many sons as B . \Box

3.3. Extension of a compatible operation to the path extension of an ultrametric space. In this section we extend a compatible operation on an ultrametric space M to its path extension $\mathbb{P}ath(\mathbb{M})$. The path we follow is motivated by the following observation. If the operation on M is a kind of addition, then viewing members of $\mathbb{P}ath(\mathbb{M})$ as kind of piecewise linear maps each defined on a subdivision of an interval $[\beta, \alpha]$, the natural idea to add two maps, f and g is to take a common refinement, and add the maps on the intervals of the refinement. But, as in our case, it is possible that one map, say f , is undefined on some interval I of the refinement, we are forced to look at the values of f outside I , and this makes the definition of sum a bit more complicated.

Our result is this:

Theorem 10. Let \mathbb{M} be an ultrametric space and $\alpha \in \mathbb{R}_+^* \cup \{+\infty\}$. Suppose that *there is a compatible binary operation* + *on* M *. Then there is a compatible operation* $+_{\alpha}$ *on* $\mathbb{P}ath_{\alpha}(\mathbb{M})$ *such that:*

(9)
$$
\varphi_{\alpha}(x+y) = \varphi_{\alpha}(x) +_{\alpha} \varphi_{\alpha}(y)
$$

for every $x, y \in M$ *. If moreover* $+$ *is associative, resp. commutative, resp. has a neutral element then* $+_{\alpha}$ *too.* And *if* 0 *is the neutral element for* + *then* $\varphi_{\alpha}(0)$ *is the neutral element for* $+_{\alpha}$ *.*

The proof will occupy the rest of this section.

We extend successively the operation + to $Nerv_{\leq \alpha}(\mathbb{M})$, to $\check{S}_{\alpha,\beta}$ and to $Path_{\alpha,\beta}(\mathbb{M})$. The extension to $Nerv_{\leq \alpha}(\mathbb{M})$ is immediate.

Lemma 25. *Let* M *be an ultrametric space endowed with a compatible binary oper* $ation +$ *. Then for all* $x, x', y, y' \in M$ *:*

(10)
$$
d(x + x', y + y') \le Max\{d(x, y), d(x', y')\}
$$

Inequality 10 asserts that $+$ is a non-expansive map from $\mathbb{M} \times \mathbb{M}$ equipped with the ℓ^{∞} metric. Let $B, B' \subseteq M$. Set $B \check{+} B' := \{x + x' : x \in B, x' \in B'\}.$

Fact 15. *If* M *is an ultrametric space, and* B, B′ *are non-empty then:*

(11)
$$
\delta(B \check{+} B') = Max\{\delta(B), \delta(B')\}
$$

Moreover, $\delta(B \check{+} B')$ *is attained whenever* $\delta(B)$ *and* $\delta(B')$ *are attained.*

Notation 26. If B , B' are two bounded subsets of M , we denote by $B + B'$ the least *member of Nerv*(M) *containing* $B \H + B'$ *.*

Ordered by reverse of the inclusion, $Nerv(M)$ is a meet-lattice. In lattice terms, $B + B' = \bigwedge \{ \{x + x'\} : x \in B, x' \in B' \}.$ This extend to $Nerv_{\leq \alpha}(\mathbb{M})$ provided that a least element \perp_{α} is added.

The following relationship between the operation $+$, the meet and δ is the clue for a proof of the theorem.

Lemma 27. *Let* M *be an ultrametric space endowed with a compatible binary oper* $ation + and \alpha \in \mathbb{R}_+^* \cup \{+\infty\}$ *. Then*

(1)
$$
C + (B \wedge_{\alpha} B') = (C + B) \wedge_{\alpha} (C + B')
$$

\n(2) $(B \wedge_{\alpha} B') + C = (B + C) \wedge_{\alpha} (B + C)$
\n(3) $\delta(B + B') = Max{\delta(B), \delta(B')}$
\nfor all $B, B' \in Nerv_{\leq \alpha}(\mathbb{M})$.

One may note that, conversely, an operation on $Nerv_{< \alpha}(\mathbb{M})$ satisfying the three conditions of the lemma come from a compatible operation on M. To go further we need the following:

Notations 28. Let $\beta \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+ \cup \{+\infty\}$ *such that* $\beta < \alpha > \beta$. Let $S_{\alpha,\beta}$ be the set of slim subsets $\mathcal B$ of $Nerv_{[\beta,\alpha]}(\mathbb M)$ such that $\beta \in Spec(\mathcal B)$ and let $\check{S}_{\alpha,\beta}$ the subset *of those which are finite.*

For a finite subset X of β, α *containing* β *and* $a \in [\beta, \alpha]$ *, set* $X(a) := Max\{x :$ $\beta \leq x \leq a$ }. For $\mathcal{B} \in \check{S}_{\alpha,\beta}$, let $\mathcal{B}(a)$ be the unique $B \in \mathcal{B}$ such that $\delta(B) :=$ $Spec(\mathcal{B})(a)$.

Lemma 29. Let *n* be a positive integer. Let \oplus : $Nerv_{[\beta,\alpha]}(\mathbb{M})^n \to Nerv_{[\beta,\alpha]}(\mathbb{M})$ be *an n*-ary operation on $Nerv_{[\beta,\alpha]}(\mathbb{M})$. For every $(\mathcal{B}_i)_{i \leq n} \in \check{S}^n_{\alpha,\beta}$, set

(12)
$$
\overline{\oplus}_{i
$$

Suppose that:

(13)
$$
\delta(\oplus_{i
$$

for all $(B_i)_{i < n} \in Nerv_{[\beta,\alpha]}(\mathbb{M})^n$. (1) Let $(\mathcal{B}_i)_{i \le n} \in \check{S}^n_{\alpha,\beta}$ then: *(a)* $Spec(\overline{\oplus}_{i$ $\overline{\oplus}_{i\leq n}$ $\mathcal{B}_i \in \check{S}_{\alpha,\beta}$. *(c)* $(\overline{\oplus}_{i\leq n} \mathcal{B}_i)(a) = \oplus_{i\leq n} \mathcal{B}_i(a)$ *for all* $a \in [\beta, \alpha]$ *. Suppose moreover that :*

$$
(14) \qquad \qquad \oplus_{i
$$

for all $(B_i)_{i \leq n}, (B'_i)_{i \leq n} \in Nerv_{[\beta,\alpha]}(\mathbb{M})^n$ such that $B_i \subseteq B'_i$ for every $i < n$. (2) Let $(\mathcal{J}_i)_{i \leq n} \in L^n_{\alpha,\beta}$ and $(\mathcal{B}_i)_{i \leq n}, (\mathcal{B}'_i)_{i \leq n} \in \check{S}^n_{\alpha,\beta}$.

If B_i and B'_i generates \mathcal{J}_i for every $i < n$ then $\overline{\oplus}_{i < n} B_i$ and $\overline{\oplus}_{i < n} B'_i$ generate the *same member of* $L_{\alpha,\beta}$ *.*

Proof. Let $a \in [\beta, \alpha]$, From inequality [\(13\)](#page-17-0) and the definition of $\mathcal{B}_i(a)$ we have

(15)
$$
\delta(\oplus \mathcal{B}_i(a)) = Max\{\delta(\mathcal{B}_i(a)) : i < n\} \le a
$$

(1) Item (a) follows immediately. Item (b). Suppose that $\delta(\oplus \mathcal{B}_i(a)) = \delta(\oplus \mathcal{B}_i(a'))$. Suppose that $a \neq a'$. W.l.o.g. we may suppose $a < a'$. Let $i < n$. From inequality [\(15\)](#page-17-1) we have $\delta(\mathcal{B}_i(a)) \leq \delta(\oplus \mathcal{B}_i(a)) \leq a$, hence $\delta(\mathcal{B}_i(a)) = \delta(\mathcal{B}_i(a'))$. Since \mathcal{B}_i is slim, this yields $\mathcal{B}_i(a) = \mathcal{B}_i(a')$. From which we get $\oplus \mathcal{B}_i(a) = \oplus \mathcal{B}_i(a')$, proving that $\overline{\oplus}_{i is slim.$

Item (c). From inequality [\(15\)](#page-17-1) we have $\delta(\oplus_{i\leq n}B_i(a)) = a$ if and only if $a \in$ $\cup_{i\leq n} Spec(\mathcal{B}_i)$. Since $\overline{\oplus}_{i\leq n} \mathcal{B}_i$ is slim, this yields $(\overline{\oplus}_{i\leq n} \mathcal{B}_i)(a) = \oplus_{i\leq n} \mathcal{B}_i(a)$. (2)To simplify notations, set $\mathcal{B} := \overline{\oplus}_{i \leq n} \mathcal{B}_i$, $A := Spec(\mathcal{B}), A_i := Spec(\mathcal{B}_i), \mathcal{J} :=]_{\alpha} \leftarrow$ \mathcal{B} , and define similarly $\mathcal{B}', A', A'_i, \mathcal{J}'$. We prove our affirmation in two steps.

Claim 6. *If* $\mathcal{B}_i \subseteq \mathcal{B}'_i$ for all $i < n$. then $\mathcal{J}' = \mathcal{J}$.

Proof of Claim [6.](#page-17-2)

Subclaim 1. Let $a \in [\beta, \alpha]$. Then $\mathcal{B}(a) \subseteq \mathcal{B}'(a)$.

Proof. Since $\mathcal{B}_i \subseteq \mathcal{B}'_i$ we have $A_i \subseteq A'_i$, hence we have $A_i(a) \leq A'_i(a) \leq a$. That is $\delta(\mathcal{B}_i(a)) \leq \delta(\mathcal{B}'_i(a)) \leq a$. Since \mathcal{B}_i generates \mathcal{J}_i , $]_a \leftarrow \mathcal{B}_i(a) \subseteq \mathcal{J}_i$. Item 3 of Fact [6](#page-12-1) yields that $\mathcal{B}_i(a) \subseteq \mathcal{B}'_i(a)$. According to inequality [14,](#page-17-3) $\mathcal{B}(a) \subseteq \mathcal{B}'(a)$. \Box Subclaim 2. $\mathcal{B}' \subseteq J$.

Proof. Since B generates $\mathcal{J}, \,]_a \leftarrow \mathcal{B}(a) \subseteq \mathcal{J}$. Since $\mathcal{B}(a) \subseteq \mathcal{B}'(a)$ we have $\mathcal{B}'(a) \subseteq$ $]_a \leftarrow \mathcal{B}(a)$ and since \mathcal{B} generates $\mathcal{J}, \mathcal{B}'(a) \subseteq]_a \leftarrow \mathcal{B}(a)$. \Box Subclaim 3. $\mathcal{B} \subseteq \mathcal{B}'$.

Proof. Let $a \in A$. We have $\delta(\mathcal{B}(a)) = a$ and $\delta(\mathcal{B}'(a)) \leq a$. From Subclaim 1, we have $\mathcal{B}(a) \subseteq \mathcal{B}'(a)$. Hence $\mathcal{B}(a) = \mathcal{B}'(a)$, proving that $B(a) \in \mathcal{B}'$. \Box

Since $\mathcal{B} \subseteq \mathcal{B}' \subseteq J$ and \mathcal{B} generates \mathcal{J} , it follows from Fact [4](#page-12-3) that \mathcal{B}' generates \mathcal{J} . Since \mathcal{B}' generates \mathcal{J}' , we have $\mathcal{J}' = \mathcal{J}$ as claimed. \Box

Let $(\mathcal{B}'')_{i \leq n}$ be a family of finite slim sets, each \mathcal{B}'_i generating \mathcal{J}_i . Let \mathcal{J}'' be the corresponding (α, β) -path.

Claim 7. $\mathcal{J} = \mathcal{J}''$

Proof of Claim [7.](#page-18-0) Let $(\mathcal{B}')_{i, where $\mathcal{B}'_i := \mathcal{B}_i \cup \mathcal{B}''_i$ for all $i < n$. According to$ Claim [6](#page-17-2) we have $\mathcal{J} = \mathcal{J}'$ and $\mathcal{J} = \mathcal{J}''$. The result follows. \Box \Box

This completes the proof of the lemma.

Definition 30. Let \oplus : $Nerv_{[\beta,alpha]}(\mathbb{M})^n \to Nerv_{[\beta,alpha]}(\mathbb{M})$ be an *n*-ary operation *satisfying conditions* [\(13\)](#page-17-0) and [\(14\)](#page-17-3). Let $(\mathcal{J}_i)_{i \leq n} \in L_{\alpha,\beta}^n$. We set $\overline{\oplus}(\mathcal{J}_i)_{i \leq n} :=]_{\alpha} \leftarrow$ $\overline{\bigoplus}(\mathcal{B}_i)_{i where $(\mathcal{B}_i)_{i is such that $\mathcal{J}_i =]_\alpha \leftarrow \mathcal{B}_i]$. According to Lemma [29,](#page-17-4)$$ *this does not depends upon the choice of* $(\mathcal{B}_i)_{i \leq n}$ *.*

Corollary 3. Let $+$ be a compatible operation on M. The operation $+$ defined on $\mathbb{P}ath_{\alpha}(\mathbb{M})$ *satifies*

(16)
$$
\varphi_{\alpha}(x+y) = \varphi_{\alpha}(x) +_{\alpha} \varphi_{\alpha}(y)
$$

Proof. $\varphi_{\alpha}(x) = \vert_{\alpha} \leftarrow \{x\}$ thus the pure slim set B generating $\varphi_{\alpha}(x)$ reduces to a singleton (namely $\{\{x\}\}\)$. The result follows. \Box

Corollary 4. Let $+$ *be a binary operation on* $Nerv_{[\beta,\alpha]}(\mathbb{M})$ *satisfying conditions [\(13\)](#page-17-0) and [\(14\)](#page-17-3). If this operation is associative, then the extensions of this operation to* $\check{S}_{\alpha,\beta}$ *and to* $L_{\alpha,\beta}$ *are associative.*

Proof. We set $\bigoplus_{i\leq 2} B_i := B_0 + B_1$ and $\bigoplus_{i\leq 3} B_i := B_0 + B_1 + B_3$. Applying the definitions given in Lemma [29,](#page-17-4) we have with obvious notations:

$$
(17) \qquad (\mathcal{B}_1 + \mathcal{B}_2) + \mathcal{B}_3 = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3
$$

The associativity of the extension of + to $\check{S}_{\alpha,\beta}$ follows. The definitions of the corresponding operations on $L_{\alpha,\beta}$ yield the same formula. \Box

Lemma 31. *Let* M *be a metric space and* + *be a compatible binary operation on* M*. Then its extension to* $L_{\alpha,\beta}$ *is compatible:*

(18)
$$
d_{\alpha}(\mathcal{J} + \mathcal{I}', \mathcal{J} + \mathcal{I}'') = d_{\alpha}(\mathcal{I}', \mathcal{I}'') = d_{\alpha}(\mathcal{I}' + \mathcal{J}, \mathcal{I}' + \mathcal{J})
$$

for all $\mathcal{I}', \mathcal{I}'', \mathcal{J}$ in $L_{\alpha,\beta}$.

Proof. Let $\gamma := d_{\alpha}(\mathcal{I}', \mathcal{I}'')$ and $\gamma' := d_{\alpha}(\mathcal{J} + \mathcal{I}', \mathcal{J} + \mathcal{I}'')$. Let $\mathcal{B}', \mathcal{B}'', \mathcal{C}$ be pure sets belonging to $\check{S}_{\alpha,\beta}$ and generating respectively $\mathcal{I}',\mathcal{I}''$ and \mathcal{J} . Since \mathcal{B}' and \mathcal{B}'' are pure, $\mathcal{B}'(a) = \mathcal{B}''(a)$ for all $a \in [\gamma, \alpha]$. Thus $\mathcal{C}(a) + \mathcal{B}'(a) = \mathcal{C}(a) + \mathcal{B}''(a)$ for all $a \in [\gamma, \alpha]$. Since $\mathcal{J} + \mathcal{I}'$ and $\mathcal{J} + \mathcal{I}''$ are respectively generated by $\mathcal{C} + \mathcal{B}'$ and $\mathcal{C} + \mathcal{B}''$, this yields $\gamma' \leq \gamma$.

For the converse, suppose $\gamma' < \gamma$. Set $b := \mu_{\gamma}(\mathcal{B}), b' := \mu_{\gamma}(\mathcal{B}'), c := \mu_{\gamma}(\mathcal{C})$ and $d := \mu_\gamma(\mathcal{D}), d' := \mu_\gamma(\mathcal{D}')$ where $\mathcal{D}, \mathcal{D}'$ are the two pure sets generating $\mathcal{J} + \mathcal{I}$ and $\mathcal{J} + \mathcal{I}'$ respectively. Set $e := Max\{b, b', c, c', d, d', \gamma'\}$. Since $\gamma', d, d' \leq e$, we have

(19)
$$
(\mathcal{C}(e) + \mathcal{B}(e)) \wedge_{\alpha} (\mathcal{C}(e) + \mathcal{B}(e)) \in \mathcal{J} + \mathcal{I}' \wedge_{\alpha} \mathcal{J} + \mathcal{I}''
$$

Hence $\delta(\mathcal{C}(e) + \mathcal{B}(e)) \wedge_{\alpha} (\mathcal{C}(e) + \mathcal{B}(e)) \leq e$.

On an other hand, since $Max\{b, b'\} \le e \langle \gamma$, we have $\delta(\mathcal{B}(e) \wedge_{\alpha} \mathcal{B}(e)) = \gamma$. Thus $\delta(\mathcal{C}(e)+\mathcal{B}(e)\wedge_{\alpha}\mathcal{B}(e))= Max\{\delta(\mathcal{C}(e)),\delta(\mathcal{B}(e)\wedge_{\alpha}\mathcal{B}(e))\}=\gamma.$ From the distributivity property stated in Lemma [27,](#page-16-2) we have

(20)
$$
(\mathcal{C}(e) + \mathcal{B}(e)) \wedge_{\alpha} (\mathcal{C}(e) + \mathcal{B}(e)) = \mathcal{C}(e) + \mathcal{B}(e) \wedge_{\alpha} \mathcal{B}(e)
$$

This yields a contradiction.

 \Box

Proof of Theorem [10.](#page-16-0) Let $+$ be a compatible operation on M. According to Lemma [27](#page-16-2) it extends to an operation on $Nerv(\mathbb{M})$ satisfying conditions [13](#page-17-0) and 13 of Lemma [29.](#page-17-4) Hence it extends to an operation on $\mathbb{P}ath_\alpha(\mathbb{M})$, which according to Lemma [31](#page-18-1) is compatible. According to Corollary [4,](#page-18-2) this operation is associative provided that the original one is associative.

\Box

3.4. Indivisibility properties of the path extension of an ultrametric space.

Lemma 32. *If a metric space* M *can be endowed with a compatible binary operation* + *then*

- (1) For every $a \in M$, M embeds into $\mathbb{M}_{\upharpoonright M(a)}$.
- *(2) If* M *has infinitely many sons then for each son* B *of* M *there are infinitely many sons* B' *such that* $\mathbb{M}_{\restriction B}$ *embeds into* $\mathbb{M}_{\restriction B'}$ *.*

Proof.

Claim 8.

(21)
$$
Spec(\mathbb{M}, a) \subseteq Spec(\mathbb{M}, a+b)
$$

for all $a, b \in \mathbb{M}$

Proof of Claim [8.](#page-19-1) Let $r \in Spec(M, a)$. Let $x \in M$ such that $d(a, x) = r$. Then from [\(4\)](#page-10-2), $d(a + b, x + b) = r$ proving that $r \in Spec(\mathbb{M}, a + b)$.

Item 1. Let $a \in M$. Let $T_a : M \to M$ defined by setting $T_a(x) := a + x$. Since + is compatible, T is an embedding of M. From Claim [8,](#page-19-1) we have $T_a(x) \in M(a)$ for every $x \in M$, as required.

Item 2 Let B' be a son of M and $b \in B'$, select a''_B in each son B'' distinct from B'. Then the images of B' by the $T_{a''_B}$'s are in different sons. \Box

Theorem 11. *The path extension* Path(M) *of a countable ultrametric monoid* M *is spec-endogeneous and hereditary indivisible provided that* $(Nerv(M), 2)$ *is well founded and has infinitely many sons*

Proof. It suffices to prove that $\mathbb{P}ath(\mathbb{M})$ satisfies conditions (1), (3) and (4) of Theorem [6.](#page-8-0)

Since $\mathbb{P}ath(\mathbb{M})$ is an extension of M, it has infinitely many sons. Hence, from Lemma [32:](#page-19-2)

- (a) $\mathbb{P}ath(\mathbb{M})$ embeds into $\mathbb{P}ath(\mathbb{M})_{\upharpoonright Path(\mathbb{M})(x)}$ for every $x \in Path(\mathbb{M})$.
- (b) For each son B of Path(M) there are infinitely many sons B' such that $\mathbb{P}ath(\mathbb{M})_{\restriction B}$ embeds into $\mathbb{P}ath(\mathbb{M})_{\upharpoonright B'}$.

From Theorem [10](#page-16-0) these properties hold for every $B \in Nerv(\mathbb{P}ath(\mathbb{M}))$ replacing $\mathbb{P}ath(\mathbb{M}).$

In particular, every non trivial member of $Nerv(Path(M))$ has infinitely many sons, this is condition (3). Since two members of $Nerv(\mathbb{P}ath(\mathbb{M}))$ with the same diameter are isometric, property $(ii)(a)$ of Proposition [4](#page-6-1) holds. Since property (b) above is property $(ii)(b)$ of Proposition [4,](#page-6-1) we get that $(i)(a)$ and $(i)(b)$ holds, that is conditions (1) and (4) hold. \Box

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