

MINIMAL SURFACES IN THE THREE-SPHERE BY DOUBLING THE CLIFFORD TORUS

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ABSTRACT. We construct embedded closed minimal surfaces in the round three-sphere $\mathbb{S}^3(1)$, resembling two parallel copies of the Clifford torus, joined by m^2 small catenoidal bridges symmetrically arranged along a square lattice of points on the torus.

1. INTRODUCTION

Historical background and the general idea.

An interesting general construction for minimal surfaces is motivated by examples of minimal surfaces which resemble two copies of a minimal surface joined together with many catenoidal bridges. Karcher, Pinkall, and Sterling have constructed [13] minimal surfaces resembling roughly an equatorial sphere in $\mathbb{S}^3(1)$ which has been “doubled”, and the two sheets have been connected by necks arranged at the vertices of a Platonic solid, with the corresponding symmetry imposed. The examples constructed this way are finitely many, because the Platonic solids are finitely many and the size of the neck is determined by the neck configuration (their number and positions). Pitts and Rubinstein have discussed [18] constructions for families of minimal surfaces, where the size of the catenoidal bridges used can be arbitrarily small and the genus then tends to infinity, while the surfaces tend to a limit varifold. These constructions are highly symmetrical. Some of the constructions have a limit varifold which is a minimal surface counted with multiplicity two.

We call such constructions “doubling constructions” as suggested in [12]. The ingredients for such a construction would be a minimal surface Σ in a Riemannian three-manifold, two nearby copies of Σ , Σ_1 and Σ_2 , and a set of points $L \subset \Sigma$. Σ_1 and Σ_2 can be thought of as the graphs of two functions ϕ_1 and ϕ_2 on Σ . ϕ_1 and ϕ_2 are assumed to be small and with small derivatives. The minimal surface constructed would consist of a region M_Σ which approximates Σ_1 and Σ_2 minus small discs, and a collection of regions which approximate small truncated catenoids. The discs removed are centered at the points on Σ_1 and Σ_2 corresponding to the points of L . The catenoidal regions serve as bridges connecting to M_Σ at the boundaries of the removed discs. We call directions perpendicular to Σ “vertical”, and directions along Σ “horizontal”. The axes of the catenoidal regions would be approximately vertical.

Since a Riemannian manifold at small scale is approximately Euclidean, we can use horizontal and vertical (approximate) translations to find balancing obstructions to the existence of such surfaces. More precisely we can consider the force F exerted by the region close to Σ_1 to a catenoidal bridge, and the force F_c exerted through the waist of the bridge to the part of the catenoidal bridge closer to Σ_1 , by the other part. The vertical component of F_c is approximately equal to the length of its waist. (Balancing for minimal surfaces is based simply

Date: September 6, 2018.

Key words and phrases. Differential geometry, minimal surfaces, partial differential equations, perturbation methods.

on the first variation formula [16, 20]. For a general discussion see [12].) If F is intercepted at a suitable curve which can be approximated by a curve on Σ_1 enclosing a domain $\Omega \subset \Sigma_1$, then the vertical component F can be approximated by the integral of the mean curvature of Σ_1 on Ω . Because of the smallness assumptions for ϕ_1 , we can ignore the nonlinear terms and the derivatives, and then the mean curvature is approximated by $(|A|^2 + Ric(\nu, \nu))\phi_1$, and the vertical component of F by the area of Ω times the value of $(|A|^2 + Ric(\nu, \nu))\phi_1$ at the corresponding point of L .

The above heuristic argument suggests that a necessary condition for a doubling construction is that the mean curvature of the parallel surfaces to Σ points away from Σ , which in general amounts to

$$(1.1) \quad |A|^2 + Ric(\nu, \nu) > 0 \quad \text{on} \quad \Sigma.$$

This condition then ensures that the vertical components of F and F_c point in opposite directions. Moreover, vertical component balancing considerations as above, relate the size of ϕ_1 and ϕ_2 with the size of the catenoidal bridge and the area of Ω . Since the matching of the catenoidal bridge to Σ_1 and Σ_2 gives further relations between ϕ_1 , ϕ_2 , and the size of the bridge, and the area of Ω can be guessed from L , it would appear that L determines completely the construction. Horizontal force considerations should further restrict the possible neck configurations L and the sizes of the catenoidal bridges.

Developing in detail such a general construction is beyond the scope of this paper. Instead we present a particular doubling construction where $\Sigma = \mathbb{T}$, the Clifford torus in the unit three-sphere, and the neck configuration L is a square lattice of points on \mathbb{T} . Because of the high symmetry involved the construction simplifies significantly, in particular we do not need to consider horizontal forces. This construction has been outlined in [12]. The method used is a gluing Partial Differential Equations method. The particular kind of methods used relates most closely to the methods developed in [4, 19], especially as they evolved and were systematized in [8]. We refer the reader to [12] for a general discussion.

Another motivation for the construction in this paper is that it is nontrivial to obtain new examples of minimal surfaces in the round three-sphere, and the list of known examples is limited [13–15, 17]. Moreover, desingularization methods can be used to combine the new surfaces produced here, to construct a more varied class of further examples [12, Theorem G]. The desingularization constructions will appear elsewhere.

Outline of the construction.

It is convenient that there is a simple coordinate system which is well adjusted to our purposes. We study this coordinate system in appendix A. We call the corresponding coordinates (x, y, z) . The surface $\{z = 0\}$ in $\mathbb{S}^3(1)$ is the Clifford torus \mathbb{T} on which the doubling construction is based. The surfaces parallel to \mathbb{T} are the surfaces of constant z .

In section 2 we construct the initial surfaces M . The construction is based on a square lattice $L \subset \mathbb{T}$ (see 2.2) which consists of m^2 points. The construction of the minimal surfaces in the main theorem works when m is large. The surfaces constructed have genus $m^2 + 1$ because they amount to two tori connected by m^2 handles. The size of the catenoidal bridges τ can not be predicted precisely, but up to a factor which is uniformly controlled independently of m is given by $\underline{\tau} := m^{-1}e^{-m^2/4\pi}$ (see 2.4). This formula can be guessed from balancing considerations as outlined above (or see [12]). It allows us to prove that we can choose τ so that the construction works (see 4.3 and the proof of the main theorem 4.4).

The construction of M is carried out in parallel with a similar construction of a surface \widehat{M} which would give a doubling of the plane in three-dimensional Euclidean space. By the

maximum principle, \widehat{M} can not be perturbed to minimality, in contrast with M which by the main theorem of the paper 4.4 can (for a certain τ). This is consistent with 1.1, since $|A|^2 + Ric(\nu, \nu) = 4 > 0$ on \mathbb{T} , while $|A|^2 + Ric(\nu, \nu) = 0$ on the plane and the mean curvature vanishes on its parallel surfaces which are planes themselves. Actually the conormal on a perturbed \widehat{M} on the vertical planes of reflectional symmetry (that is on $\partial\widehat{\mathbb{D}}$, see 2.1) is horizontal, so the force F in the discussion above vanishes providing an alternative proof that \widehat{M} can not be corrected.

As it is often the case in such constructions [3–12, 21, 22], it is convenient to define two more metrics on the initial surfaces M , h and χ , besides the induced metric g . h and χ are conformal to g . h allows us to write the linearized equation with uniformly bounded coefficients. Moreover, it allows us to understand the spectrum and the approximate kernel. In the usual terminology M modulo the symmetries has two standard regions, which when viewed with respect to h tend to a planar square and a unit sphere. The square corresponds to a fundamental domain of \mathbb{T} and the unit sphere to the catenoidal bridge. There is only one (modulo the symmetries) transition region Λ connecting the standard regions. (Λ, χ) is approximately isometric to a standard cylinder of length $m^2/4\pi$ up to lower order terms. The geometric quantities of M are discussed in 2.13. These estimates are important because they allow us to ensure that we can perturb to minimality with an appropriately small perturbation. Finally in 2.20 we quantify the limiting behavior of the standard regions in the h metric as $m \rightarrow \infty$.

In section 3 we develop the linear theory needed. All we need from this section is 3.26 and 3.25. In 3.25 we simply extract from the information we have on the mean curvature from 2.13 the relevant estimate we can use according to the linear theory. In 3.26 we provide a solution modulo the substitute kernel for the linear problem with appropriate decay estimates. The construction leading to 3.26 follows the general methodology of [8]. It is simpler than usual however, because of the small number of standard and transition regions, and the one-dimensionality of the substitute kernel, which can serve also as extended substitute kernel (see [12] for a general discussion). The one-dimensionality of the approximate and (hence) the substitute kernel follows from the fact that the symmetries kill the first harmonics of the Laplacian on the spherical standard region corresponding to the catenoidal bridge, and therefore the only eigenfunctions allowed in the kernel in the limiting configuration as $m \rightarrow \infty$, are the constants on the square (see 3.13). It turns out that the substitute kernel is enough for arranging the decay we need (see 3.20 and 3.21), and hence there is no need for extra “extended substitute kernel”.

Finally in section 4 we prove the main theorem. To do so we first provide in 4.1 an estimate of the nonlinear terms consistent with the decay estimates we have. This estimate is based on a general estimate which can be derived from general principles (see B.3) and which we present in appendix B. Next we calculate in detail the forces in the spirit of the discussion earlier (see 4.3), and use that information to ensure that there is some initial surface M which can be perturbed to minimality. This is consistent with the Geometric Principle (see [8, 12]) because effectively creation of substitute kernel is achieved by repositioning the copies of \mathbb{T} used in the construction at varying distances $a\tau$ from \mathbb{T} . Finally we state and prove the main theorem 4.4 by using as usual the Schauder fixed point theorem [2, Theorem 11.1] to minimize the required estimates. We remark that the minimal surfaces we find are consistent with the description of the surfaces in Example 12 in [18, page 306].

Notation and conventions.

In this paper we use weighted Hölder norms. A definition which works for our purposes in this paper is the following:

$$(1.2) \quad \|\phi : C^{k,\beta}(\Omega, g, f)\| := \sup_{x \in \Omega} \frac{\|\phi : C^{k,\beta}(\Omega \cap B_x, g)\|}{f(x)},$$

where Ω is a domain inside a Riemannian manifold (M, g) , f is a weight function on Ω , B_x is a geodesic ball centered at x and of radius the minimum of 1 and half the injectivity radius at x .

We will be using extensively cut-off functions and for this reason we adopt the following notation: We fix a smooth function $\Psi : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

- (i). Ψ is nondecreasing.
- (ii). $\Psi \equiv 1$ on $[1, \infty]$ and $\Psi \equiv 0$ on $(-\infty, -1]$.
- (iii). $\Psi - \frac{1}{2}$ is an odd function.

Given then $a, b \in \mathbb{R}$ with $a \neq b$, we define a smooth function $\psi[a, b] : \mathbb{R} \rightarrow [0, 1]$ by

$$(1.3) \quad \psi[a, b] = \Psi \circ L_{a,b},$$

where $L_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$ is the linear function defined by the requirements $L(a) = -3$ and $L(b) = 3$.

Clearly then $\psi[a, b]$ has the following properties:

- (i). $\psi[a, b]$ is weakly monotone.
- (ii). $\psi[a, b] = 1$ on a neighborhood of b and $\psi[a, b] = 0$ on a neighborhood of a .
- (iii). $\psi[a, b] + \psi[b, a] = 1$ on \mathbb{R} .

We will denote the span of vectors e_1, \dots, e_k with coefficients in a field \mathbb{F} by $\langle e_1, \dots, e_k \rangle_{\mathbb{F}}$.

Acknowledgments. The authors would like to thank Rick Schoen for his constant interest and support and insightful discussions and suggestions. N.K. would like to thank the Mathematics Department and the MRC at Stanford University for providing a stimulating mathematical environment and generous financial support during Fall 2006.

2. THE INITIAL SURFACES

In this section we define and discuss the initial surfaces. The genus and the geometry of the initial surfaces depend on $m \in \mathbb{N}$ which we fix now and is assumed to be as large as needed. The number of catenoidal bridges used to connect the two parallel copies of the Clifford torus is m^2 and the genus of the resulting surface $m^2 + 1$. These bridges are arranged with maximal symmetry at the points of a square lattice. To describe the symmetry involved we have the following (recall Appendix A):

Definition 2.1. We denote by $\widehat{\mathcal{G}}$ the group of diffeomorphisms of Dom_{Φ} generated by the reflections $\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_{\pi/\sqrt{2}m}, \widehat{\mathbf{Y}}, \widehat{\mathbf{Y}}_{\pi/\sqrt{2}m}$, and $\widehat{\mathbf{Z}}$. We denote by \mathcal{G} the group of isometries of $\mathbb{S}^3(1)$ generated by the reflections $\mathbf{X}, \mathbf{X}_{\pi/\sqrt{2}m}, \mathbf{Y}, \mathbf{Y}_{\pi/\sqrt{2}m}$, and \mathbf{Z} . We also define $\mathbb{D} \subset \mathbb{S}^3(1)$ and $\widehat{\mathbb{D}} \subset Dom_{\Phi}$ by $\mathbb{D} := \Phi(\widehat{\mathbb{D}})$ and

$$\widehat{\mathbb{D}} := \left\{ (x, y, z) \in Dom_{\Phi} : |x| \leq \frac{\pi}{\sqrt{2}m}, |y| \leq \frac{\pi}{\sqrt{2}m} \right\}.$$

The reflections $\widehat{\mathbf{X}}, \widehat{\mathbf{X}}_{\pi/\sqrt{2}m}, \widehat{\mathbf{Y}}, \widehat{\mathbf{Y}}_{\pi/\sqrt{2}m}$, and $\widehat{\mathbf{Z}}$ generating $\widehat{\mathcal{G}}$ are with respect to the planes $\{x = 0\}$, $\{y = 0\}$, $\{x = \frac{\pi}{\sqrt{2}m}\}$, $\{y = \frac{\pi}{\sqrt{2}m}\}$, and the line $\{x = y, z = 0\}$ respectively. Clearly $\widehat{\mathbf{X}}_{\sqrt{2}\pi/m}, \widehat{\mathbf{Y}}_{\sqrt{2}\pi/m} \in \widehat{\mathcal{G}}$ and $\mathbf{X}_{\sqrt{2}\pi/m}, \mathbf{Y}_{\sqrt{2}\pi/m} \in \mathcal{G}$. $\widehat{\mathbb{D}}$ is a fundamental domain for the action of

the translations in $\widehat{\mathcal{G}}$ and is invariant under the action of $\widehat{\underline{X}}$, $\widehat{\underline{Y}}$ and $\widehat{\underline{Z}}$. Similarly (recall A.4) \mathbb{D} is a fundamental domain for the action of the rotations in \mathcal{G} and is invariant under the action of \underline{X} , \underline{Y} and \underline{Z} .

We define square lattices \widehat{L} on the plane $\{z = 0\}$ and L on \mathbb{T} (recall A.4 and A.8) by

$$(2.2) \quad \widehat{L} := \widehat{\mathcal{G}}(0, 0, 0), \quad L := \Phi(\widehat{L}) = \mathcal{G}\Phi(0, 0, 0).$$

L consists of m^2 points which will be the centers of the catenoidal bridges we use.

The size of the catenoidal bridges depends on m and on $\zeta \in \mathbb{R}$ which is a parameter of the construction. ζ is assumed to satisfy

$$(2.3) \quad |\zeta| \leq \underline{c},$$

where \underline{c} is a constant which will be chosen later. We define then

$$(2.4) \quad \underline{\tau} := m^{-1}e^{-m^2/4\pi}, \quad \tau := e^\zeta \underline{\tau}.$$

We define now a constant $a > 0$, a map $\widehat{X} : [-a, a] \times \mathbb{S}^1 \rightarrow \widehat{\mathbb{D}}$, and a truncated catenoidal bridge \widehat{M}_{cat} of size τ , by the following:

$$(2.5) \quad \begin{aligned} \widehat{M}_{cat} &:= \widehat{X}([-a, a] \times \mathbb{S}^1), & \widehat{X}(t, \theta) &:= (r(t) \cos \theta, r(t) \sin \theta, z(t)), \\ \text{where } r(t) &:= \tau \cosh t, & z(t) &:= \tau t, & r(a) &= \frac{1}{m}. \end{aligned}$$

Note that the definition of a just given, together with 2.4 and 2.3 (see also 2.8), implies that

$$(2.6) \quad \left| a + \zeta - \frac{m^2}{4\pi} - \log 2 \right| < \underline{\tau}.$$

We also define a region of a horizontal plane (corresponding under Φ to a parallel surface to \mathbb{T}) together with a gluing region by

$$(2.7) \quad \begin{aligned} \widehat{M}_{tor} &:= \left\{ (x, y, z) \in \widehat{\mathbb{D}} : z = \varphi(\sqrt{x^2 + y^2}), \frac{1}{m} \leq \sqrt{x^2 + y^2} \right\}, \\ \text{where } \varphi(r) &:= \varphi_{cat}(r) + \psi[m^{-1}, 2m^{-1}](r) (\varphi_{cat}(m^{-1}) - \varphi_{cat}(r)), \\ \text{where } \varphi_{cat}(r) &:= \tau \operatorname{arccosh} \frac{r}{\tau} = \tau \left(\log r - \log \tau + \log \left(1 + \sqrt{1 - \frac{\tau^2}{r^2}} \right) \right). \end{aligned}$$

Notice that φ transits smoothly from being φ_{cat} in a neighborhood of $r = 1/m$, to being the constant

$$(2.8) \quad \varphi_{cat}(1/m) = \tau a$$

for $r \geq 2/m$ (note that $2 < \pi/\sqrt{2}$). Correspondingly \widehat{M}_{tor} extends smoothly \widehat{M}_{cat} close to its inner boundary circle and transits to the plane $z = \varphi_{cat}(1/m)$ close to its outer boundary. We define then smooth embedded surfaces $\widehat{M} \subset \operatorname{Dom}_\Phi$ and $M_{cat}, M_{tor}, M \subset \mathbb{S}^3(1)$ by

$$(2.9) \quad \begin{aligned} \widehat{M} &:= \widehat{\mathcal{G}}(\widehat{M}_{cat} \cup \widehat{M}_{tor}), & M_{cat} &:= \Phi(\widehat{M}_{cat}), & M_{tor} &:= \Phi(\widehat{M}_{tor}), \\ M &:= \Phi(\widehat{M}) = \mathcal{G}(M_{cat} \cup M_{tor}). \end{aligned}$$

$\Phi|_{\widehat{M}} : \widehat{M} \rightarrow M$ is clearly a covering map and M is a closed embedded surface of genus $m^2 + 1$. We take M to be our initial surface and we will prove in the Main Theorem that for some value of ζ it can be perturbed to a nearby minimal surface.

Geometric quantities on the initial surfaces.

We start by discussing some of the metrics we use. We denote by \widehat{g} the standard Euclidean metric on Dom_Φ and by g the standard metric on the round sphere $\mathbb{S}^3(1)$. Since Φ is a covering map, these metrics induce metrics on the range and the domain of Φ respectively, which we denote by slight abuse of notation by the same symbols. We also use the same symbols to denote the metrics induced on M , \widehat{M} and (by using \widehat{X}) on the cylinder $\mathbb{S}^1 \times [-a, a]$. We also define cylindrical coordinates (r, θ, t) on $\widehat{\mathbb{D}}$ and \mathbb{D} by

$$(2.10) \quad (x, y, z) = (r \cos \theta, r \sin \theta, \tau t).$$

We define a smooth function ρ on M (or \widehat{M}) by requiring it is invariant under the action of \mathcal{G} (or $\widehat{\mathcal{G}}$) and on $\mathbb{D} \cap M$ (or $\widehat{\mathbb{D}} \cap \widehat{M}$) it satisfies

$$(2.11) \quad \rho = \frac{1}{r} + \psi[m^{-1}, 2m^{-1}](r) \left(\frac{2}{m} - \frac{1}{r} \right).$$

We define then smooth metrics χ and $\widehat{\chi}$ on our surfaces by

$$(2.12) \quad \chi := \rho^2 g, \quad \widehat{\chi} := \rho^2 \widehat{g}.$$

We denote by ν the unit normal which satisfies $\langle \nu, \partial_z \rangle > 0$ on \widehat{M}_{tor} , A the second fundamental form induced by g , by $|A|^2$ its square length, and by H the mean curvature. We use a hat to denote the corresponding geometric quantities induced by \widehat{g} . Note that z is constant on M_{tor} close to $\partial \mathbb{D} \cap \partial M_{tor}$, and so we will consider it extended to M as a smooth function, by requesting invariance under the action of \mathcal{G} . We have the following:

Lemma 2.13. *Assuming that m is large enough in terms of $k \in \mathbb{N}$ the following hold:*

- (i). $\|\rho^{\pm 1} : C^k(M, \widehat{\chi}, \rho^{\pm 1})\| \leq C(k)$.
- (ii). $\|z : C^k(M, \widehat{\chi}, |z| + \tau)\| \leq C(k)$.
- (iii). $\|\chi - \widehat{\chi} : C^k(M, \widehat{\chi}, |z| + \tau)\| \leq C(k)$. On \widehat{M}_{cat} we have $\widehat{\chi} = dt^2 + d\theta^2$.
- (iv). $\|\rho^{-2} H : C^k(M, \chi, (\tau + \rho^{-2})(|z| + \tau))\| \leq C(k)$.
- (v). $\||A|^2 - 2\tau^2 \rho^4 : C^k(M, \chi, 1 + \tau \rho^2)\| \leq C(k)$. Moreover on \widehat{M}_{cat} we have $|\widehat{A}|^2 = 2\tau^2 \rho^4$.

Proof. We first check these estimates on M_{tor} . M_{tor} is the graph of φ (recall 2.7) and by the definition of φ and 2.8 we have

$$(a). \|\varphi - \tau a : C^k(M_{tor}, m^2(dx^2 + dy^2))\| \leq C(k) \tau.$$

By 2.6 we conclude

$$(b). \tau a \leq m^2 \tau.$$

By 2.11 we conclude that

$$(c). \|\rho^{\pm 1} : C^k(M_{tor}, m^2(dx^2 + dy^2))\| \leq C(k) m^{\pm 1}.$$

By straightforward calculation we have

$$\begin{aligned} \widehat{g} - (dx^2 + dy^2) &= \varphi_x^2 dx^2 + 2\varphi_x \varphi_y dx dy + \varphi_y^2 dy^2, \\ g - \widehat{g} &= \sin 2\varphi (dx^2 - dy^2). \end{aligned}$$

(a), (b), and (c) imply then (i), (ii), (iii), and also

$$(d). \|g - (dx^2 + dy^2) : C^k(M_{tor}, \chi, |z|)\| \leq C(k).$$

Using A.7 and calculating further we conclude

$$\begin{aligned} \|A - (\Gamma_{11}^3 dx^2 + \Gamma_{22}^3 dy^2) : C^k(M_{tor}, m^2(dx^2 + dy^2))\| &\leq \\ &C \|(\varphi_x, \varphi_y) : C^{k+1}(M_{tor}, m^2(dx^2 + dy^2))\|, \end{aligned}$$

which by (a), (b) and A.7 implies that

$$(e). \|A + dx^2 - dy^2 : C^k(M_{tor}, m^2(dx^2 + dy^2))\| \leq C(k)\tau.$$

(d) and (e) imply then (iv) and (v).

It remains to check that the estimates hold on M_{cat} . For convenience we adopt the notation $O(f)$ to denote a function (or tensor field) which satisfies for each disc $D \subset M_{cat}$ of radius 1 with respect to the $\widehat{\chi}$ metric the inequality

$$\|O(f) : C^k(D, \widehat{\chi})\| \leq C(k) \|f : C^k(D, \widehat{\chi})\|.$$

By a straightforward calculation we have then

$$\begin{aligned}\widehat{X}_t &= \tau(\sinh t \cos \theta, \sinh t \sin \theta, 1), \\ \widehat{X}_\theta &= \tau \cosh t(-\sin \theta, \cos \theta, 1),\end{aligned}$$

which implies

$$(f). \widehat{g} = r^2(dt^2 + d\theta^2) \text{ and } g = r^2(dt^2 + d\theta^2 + O(z)).$$

Using (f) and the definitions, (i), (ii), and (iii) follow. Using A.7 and calculating further we find that

$$(g). \widehat{A} = \tau(-dt^2 + d\theta^2), \text{ and } A = \widehat{A} + \widetilde{A} + O((\tau + r^2)z), \text{ where}$$

$$\widetilde{A} = r^2(\cos 2\theta(-dt^2 + d\theta^2) + 2\sin 2\theta dt d\theta).$$

Using (f) and (g) it is straightforward to check (iv) and (v) and complete the proof (notice also that $\tau^2 z < C\tau r^2$). \square

Standard and transition regions.

We proceed to define carefully the various regions on the initial surface M in the usual fashion of [3–12]. Modulo the symmetries imposed, there are only two standard regions which we denote by $S[0]$ (corresponding to the catenoidal bridge) and $S[1]$ (corresponding to the torus), and only one transition region we denote by Λ . The extended standard regions $\widetilde{S}[0]$ and $\widetilde{S}[1]$ are the standard regions augmented by the transition region.

In order to ensure uniformity with respect to different values of the parameter ζ , we define and use a variant \underline{t} of the parameter t by

$$(2.14) \quad \underline{t} = \frac{a}{a} t,$$

where a is defined as in 2.5 for the current value of ζ and \underline{a} is defined in the same way when $\zeta = 0$ and hence $\tau = \underline{\tau}$ (recall 2.4). This way the range of values of \underline{t} on M_{cat} is $[-\underline{a}, \underline{a}]$ and it does not depend on ζ or τ . Note also that by 2.6 we have

$$(2.15) \quad |t - \underline{t}| \leq C\underline{c} \text{ on } \widetilde{S}[0], \quad \left| \frac{a}{a} - 1 \right| \leq C\underline{c} m^{-2}.$$

We use a constant b to control the exact size of the standard and transition regions. b will be determined later so that the linearized equation and its spectrum behave appropriately. We use subscripts x and y to modify the usual sizes and boundary circles. In particular each $S_x[n]$ is a neighborhood of $S[n]$, while $\widetilde{S}_x[n]$ is $\widetilde{S}[n]$ with an appropriate neighborhood of its boundary excised.

Definition 2.16. We define the following:

$$(2.17a) \quad S_x[0] := M \cap \mathbb{D} \cap \{\underline{t} \in [-b-x, b+x]\},$$

$$(2.17b) \quad S_x[1] := M \cap \mathbb{D} \cap \{\underline{t} \geq \underline{a} - b - x\},$$

$$(2.17c) \quad \tilde{S}_x[0] := M \cap \mathbb{D} \cap \{\underline{t} \in [-\underline{a} + b + x, \underline{a} - b - x]\},$$

$$(2.17d) \quad \tilde{S}_x[1] := M \cap \mathbb{D} \cap \{\underline{t} \geq b + x\},$$

$$(2.17e) \quad \Lambda_{x,y} := M \cap \mathbb{D} \cap \{\underline{t} \in [b+x, a-b-y]\},$$

$$(2.17f) \quad C_x[0] := M \cap \mathbb{D} \cap \{\underline{t} = b+x\},$$

$$(2.17g) \quad C_x[1] := M \cap \mathbb{D} \cap \{\underline{t} = \underline{a} - b - x\},$$

$$(2.17h) \quad C_{\partial} := \partial \mathbb{D} \cap \partial M_{tor}$$

where $b > 5$ is a constant chosen finally in the proof of 3.21 independently of m , and $0 \leq x, y < \frac{1}{3}a - b$. When $x = y = 0$ we drop the subscripts. We also write Λ_x for $\Lambda_{x,x}$.

The limiting behavior of the standard regions, and the linearized operator on them, as $m \rightarrow \infty$, is best understood in the h metric which is defined on our surfaces by

$$(2.18) \quad h := \frac{|A|^2 + m^2}{2}g.$$

We define the map $\varpi : \mathbb{D} \rightarrow \mathbb{R}^2$ by

$$(2.19) \quad \varpi(x, y, z) := \frac{m}{\sqrt{2}}(x, y).$$

The following lemma describes the limiting behavior as $m \rightarrow \infty$:

Lemma 2.20. *If m is large enough in terms of $b+x$, then the following hold:*

(i). $\|h - \hat{\nu}^*g : C^5(S_x[0], \hat{\nu}^*g)\| \leq C(b+x)\tau$, where $\hat{\nu}^*g$ is the pullback of the standard metric of the unit sphere $\mathbb{S}^2(1)$ by $\hat{\nu}$ and satisfies $\hat{\nu}^*g = \frac{1}{2}|A|^2\hat{g} = \tau^2r^{-4}\hat{g} = \tau^2r^{-2}\hat{\chi}$. Moreover $\hat{\nu}(S_x[0]) = \{(x, y, z) \in \mathbb{S}^2(1) : x^2 + y^2 \geq \check{R}_x^2\}$, where $\check{R}_x = 1/\cosh[(b+x)a/\underline{a}]$.

(ii). $\|h - \varpi^*g : C^5(S_x[0], \varpi^*g)\| \leq C(b+x)/m^2$, where ϖ^*g is the pullback of the standard Euclidean metric on \mathbb{R}^2 by ϖ (restricted to $S_x[1]$). Moreover there is \tilde{R}_x such that $|\tilde{R}_x - 2^{-1/2}e^{-(b+x)a/\underline{a}}| \leq \tau$ and $\varpi(S_x[1]) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : |\tilde{x}| \leq \frac{\pi}{2}, |\tilde{y}| \leq \frac{\pi}{2}, \tilde{x}^2 + \tilde{y}^2 \geq \tilde{R}_x^2\}$.

Proof. Since the catenoid is a minimal surface it follows from standard theory that $\hat{\nu}^*g = \frac{1}{2}|A|^2\hat{g}$, and the expressions in terms of r follow from 2.13.v and the definitions. This implies that the length of $\hat{\nu}(C_x[0])$ is $2\pi\tau/r(t) = 2\pi/\cosh[(b+x)a/\underline{a}]$, which implies that $\hat{\nu}(S_x[0])$ is as stated. Since

$$h - \hat{\nu}^*g = \frac{1}{2}(|A|^2 + m^2 - 2\tau^2\rho^4)\rho^{-2}\chi + \tau^2\rho^2(\chi - \hat{\chi}),$$

we conclude by using 2.13 that

$$\|h - \hat{\nu}^*g : C^5(S_x[0], \hat{\chi})\| \leq C(m^2r^2 + \tau) \leq C\tau.$$

This implies the desired estimate and completes the proof of (i).

The second part of (ii) follows easily from the definitions and the observation that $\tilde{R}_x = (m/\sqrt{2})r(a - (b+x)a/\underline{a})$. By writing

$$h - \varpi^*g = \frac{|A|^2 + m^2}{2}(g - (dx^2 + dy^2)) + \frac{|A|^2}{m^2}\varpi^*g,$$

using 2.13.i to establish the analogue of (c) in the proof of 2.13, estimating $g - (dx^2 + dy^2)$ as for (d) in the proof of 2.13, and estimating $|A|^2$ by 2.13.v, we conclude the proof. \square

3. THE LINEARIZED EQUATION

Introduction.

In this section we study the linearized equation on M which can be stated in any of the following equivalent formulations,

$$(3.1) \quad \mathcal{L}_\chi u = E, \quad \text{or} \quad \mathcal{L}u = \rho^2 E, \quad \text{or} \quad \mathcal{L}_h u = \frac{2\rho^2}{|A|^2 + m^2} E,$$

where the corresponding linear operators are given by

$$(3.2) \quad \begin{aligned} \mathcal{L}_\chi &:= \Delta_\chi + \rho^{-2}(|A|^2 + 2), & \mathcal{L}_h &:= \Delta_h + 2\frac{|A|^2 + 2}{|A|^2 + m^2}, \\ \mathcal{L} &:= \Delta_g + |A|^2 + 2 = \rho^2 \mathcal{L}_\chi = \frac{|A|^2 + m^2}{2} \mathcal{L}_h. \end{aligned}$$

The linearized equation on the transition region.

In this subsection we consider the linearized equation on the transition region $\Lambda_{x,y}$ defined as in 2.17e, where we assume that $x, y \in [0, 4]$. For simplicity in this subsection we will denote the neck under consideration by Λ , and its boundary circles $C_x[0]$ and $C_y[1]$ by \underline{C} and \overline{C} respectively. We next define $\underline{x}, \overline{x}, \underline{\underline{x}} : \Lambda \rightarrow \mathbb{R}$ to measure the \underline{t} -coordinate distance from \underline{C} , \overline{C} , and $\partial\Lambda = \underline{C} \cup \overline{C}$ respectively:

$$(3.3) \quad b + x + \underline{x} = \underline{t}, \quad \underline{a} - b - y - \overline{x} = \underline{t}, \quad \underline{\underline{x}} := \min(\underline{x}, \overline{x}).$$

Note that we can use $\Phi \circ \widehat{X}$ to identify Λ with the cylinder $[(b+x)a/\underline{a}, a - (b+y)a/\underline{a}] \times \mathbb{S}^1$. We define $\underline{\ell}$ to be the \underline{t} -coordinate length of the cylinder and ℓ to be the t -coordinate length of the cylinder, so that

$$(3.4) \quad \underline{\ell} = \underline{a} - 2b - x - y, \quad \ell = a - (2b + x + y)a/\underline{a}.$$

Using 2.6 and our assumption that $x, y \in [0, 4]$, we estimate

$$(3.5) \quad \left| \ell + 2b + \zeta - \frac{m^2}{4\pi} \right| < 10.$$

Our understanding of the linear equations on the transition region is based on the comparison with Δ_χ , which is based on the following lemma:

Lemma 3.6. *The following hold on Λ :*

- (i). $\|\chi - \widehat{\chi} : C^5(M, \widehat{\chi})\| \leq Cm^2\tau$.
- (ii). $\|\rho^{-2}(|A|^2, m^2) : C^5(\Lambda, \chi, e^{-3\underline{\underline{x}}/2})\| \leq Ce^{-3b/2}$.

Proof. This is a straightforward consequence of 2.13, 2.6, and the various definitions. \square

Proposition 3.7. *If m is large enough then the lowest eigenvalue of the Dirichlet problem for \mathcal{L}_χ on Λ is $> C\ell^{-2}$.*

Proof. The proof is similar to the arguments leading to Proposition 2.28 in [8]. It is easy to prove that for $\phi \in L^2(\Lambda)$ with L^2 derivatives and $\phi = 0$ on $\partial\Lambda$ we have

$$\int_\Lambda e^{-3\underline{\underline{x}}/2} \phi^2 d\widehat{\chi} \leq C \int_\Lambda |\nabla \phi|_\chi^2 d\widehat{\chi},$$

which together with 3.6 implies

$$\int_{\Lambda} |\nabla \phi|_{\chi}^2 d\chi - \int_{\Lambda} \rho^{-2}(|A|^2 + 2)\phi^2 d\chi \geq \left(\frac{2}{3} - Ce^{-3b/2}\right) \int_{\Lambda} |\nabla \phi|_{\widehat{\chi}}^2 d\widehat{\chi}.$$

Using the variational characterization of eigenvalues and assuming b large enough the result follows since the smallest eigenvalue for $\Delta_{\widehat{\chi}}$ is $> C\ell^{-2}$. \square

Corollary 3.8. (i). *The Dirichlet problem for \mathcal{L}_{χ} on Λ for given $C^{2,\beta}$ Dirichlet data has a unique solution.*

(ii). *For $E \in C^{0,\beta}(\Lambda)$ there is a unique $\varphi \in C^{2,\beta}(\Lambda)$ such that $\mathcal{L}_{\chi}\varphi = E$ on Λ and $\varphi = 0$ on $\partial\Lambda$. Moreover $\|\varphi : C^{2,\beta}(\Lambda, \chi)\| \leq C(\beta)\ell^2 \|E : C^{0,\beta}(\Lambda, \chi)\|$.*

Proof. (i) follows trivially and (ii) by using standard linear theory. \square

All our constructions have to respect the symmetries imposed, in particular we only consider functions on M which are invariant under the action of \mathcal{G} . Λ is not invariant under \mathcal{G} but it is invariant under \underline{X} and \underline{Y} . Under the identification of Λ with a cylinder as discussed above, \underline{X} corresponds to $\theta \rightarrow \pi - \theta$, and \underline{Y} corresponds to $\theta \rightarrow -\theta$. We use the subscript ‘‘S’’ to specify subspaces of functions on Λ which are invariant under these symmetries. In the next Proposition and its Corollary, we study the Dirichlet problem when we are allowed to modify the lowest harmonic on the boundary data in order to have decay estimates appropriate for our purposes:

Proposition 3.9. *Assuming b large enough in terms of given $\beta, \gamma \in (0, 1)$, there is a linear map $\mathcal{R}_{\Lambda} : C_S^{0,\beta}(\Lambda) \rightarrow C_S^{2,\beta}(\Lambda)$ such that the following hold for $E \in C_S^{0,\beta}(\Lambda)$ and $V := \mathcal{R}_{\Lambda} E$:*

(i). $\mathcal{L}_{\chi} V = E$ on Λ .

(ii). V is constant on \overline{C} and vanishes on \underline{C} .

(iii). $\|V : C_S^{2,\beta}(\Lambda, \chi, e^{-\gamma\overline{x}})\| \leq C(\beta, \gamma) \|E : C_S^{0,\beta}(\Lambda, \chi, e^{-\gamma\overline{x}})\|$.

(iv). \mathcal{R}_{Λ} depends continuously on τ .

The proposition still holds if the roles of \underline{C} and \overline{C} are exchanged in (ii) and \overline{x} is replaced by \underline{x} in (iii). Another possibility is to allow V to be constant on each of \overline{C} and \underline{C} in (ii), while \overline{x} is replaced by \underline{x} in (iii).

Proof. The proposition follows by standard theory if \mathcal{L}_{χ} is replaced by $\Delta_{\widehat{\chi}}$. We denote the corresponding linear map and solution in the Δ_{χ} case by $\widetilde{\mathcal{R}}_{\Lambda}$ and \widetilde{V} respectively. Using then 3.6 we have

$$\|\mathcal{L}_{\chi} \widetilde{V} : C^{0,\beta}(\Lambda, \widehat{\chi}, e^{-\gamma\overline{x}})\| \leq C(\beta, \gamma) (m^2\tau + e^{-3b/2}) \|E : C^{0,\beta}(\Lambda, \widehat{\chi}, e^{-\gamma\overline{x}})\|,$$

and the proposition then follows by an iteration where we treat \mathcal{L}_{χ} and \mathcal{R}_{Λ} as small perturbations of $\Delta_{\widehat{\chi}}$ and $\widetilde{\mathcal{R}}_{\Lambda}$ and assuming b and m large enough. \square

We will only need the next statement with $\varepsilon_1 = 1$:

Corollary 3.10. *Assuming b large enough in terms of given $\beta, \gamma \in (0, 1)$ and $\varepsilon_1 > 0$, there is a linear map*

$$\mathcal{R}_{\partial} : \{u \in C_S^{2,\beta}(\overline{C}) : \int_{\overline{C}} u d\theta = 0\} \rightarrow C_S^{2,\beta}(\Lambda)$$

such that the following hold for u in the domain of \mathcal{R}_{∂} and $V := \mathcal{R}_{\partial} u$:

(i). $\mathcal{L}_{\chi} V = 0$ on Λ .

(ii). $V - u$ is constant on \overline{C} and V vanishes on \underline{C} .

(iii). $\|V - u\| \leq \varepsilon_1 \|u : C_S^{2,\beta}(\overline{C}, d\theta^2)\|$.

(iv). $\|V : C_S^{2,\beta}(\Lambda, \chi, e^{-\gamma\bar{x}})\| \leq C(\beta, \gamma) \|u : C_S^{2,\beta}(\bar{C}, d\theta^2)\|$.

(v). \mathcal{R}_∂ depends continuously on τ .

The Proposition still holds if the roles of \bar{C} and \underline{C} are exchanged and \bar{x} is replaced by \underline{x} .

Proof. By standard theory there is a linear map

$$\tilde{\mathcal{R}}_\partial : \{u \in C_S^{2,\beta}(\bar{C}) : \int_{\bar{C}} u d\theta = 0\} \rightarrow C_S^{2,\beta}(\Lambda)$$

such that for u in the domain and $\tilde{V} = \tilde{\mathcal{R}}_\partial u$ the following hold:

(a). $\Delta_{\tilde{\chi}} \tilde{V} = 0$ on Λ .

(b). $\tilde{V} = u$ on \bar{C} and \tilde{V} vanishes on \underline{C} .

(c). $\|\tilde{V} : C_S^{2,\beta}(\Lambda, \chi, e^{-\gamma\bar{x}})\| \leq C(\beta, \gamma) \|u : C_S^{2,\beta}(\bar{C}, d\theta^2)\|$.

The corollary then follows by defining

$$\mathcal{R}_\partial u := \tilde{\mathcal{R}}_\partial u - \mathcal{R}_\Lambda \mathcal{L}_\chi \tilde{\mathcal{R}}_\partial u,$$

applying the Proposition, and using 3.6. □

Corollary 3.11. *If $u \in C_S^{2,\beta}(\Lambda)$ satisfies $\mathcal{L}_\chi u = 0$ on Λ , then*

$$\|u : C_S^{2,\beta}(\Lambda, \chi)\| \leq C(\beta) \|u : C_S^{2,\beta}(\partial\Lambda, \chi)\|.$$

Proof. Because of 3.10 and 3.8 it is enough to prove the Corollary when u is constant on each boundary circle. Let \tilde{V} be the solution of

$$\Delta_{\tilde{\chi}} \tilde{V} = 0 \text{ on } \Lambda, \quad \tilde{V} = 1 \text{ on } \bar{C}, \quad \tilde{V} = 0 \text{ on } \underline{C}.$$

By 3.6 we can write $\mathcal{L}_\chi \tilde{V} = E_1 + E_2$ where $\|E_1 : C_S^{2,\beta}(\Lambda, \chi, e^{-\gamma\bar{x}})\| \leq C e^{-3b/2}$ and $\|E_2 : C_S^{2,\beta}(\Lambda, \chi, e^{-\gamma\underline{x}})\| \leq C e^{-3b/2}/\ell$. By applying twice 3.9 and assuming b large enough we obtain $\bar{V} \in C_S^{2,\beta}$ such that $\mathcal{L}_\chi \bar{V} = 0$ on Λ , \bar{V} is constant on each boundary circle of Λ , $\|\bar{V} : C_S^{2,\beta}(\Lambda, \chi)\| \leq C(\beta)$, $|\bar{V} - 1| \leq 1/9$ on \bar{C} , and $|\bar{V}| \leq 1/9$ on \underline{C} . By exchanging \bar{C} with \underline{C} we obtain \underline{V} instead of \bar{V} . By considering linear combinations of \bar{V} and \underline{V} we complete the proof. □

The approximate kernel.

We proceed now to discuss the approximate kernel of \mathcal{L}_h on the extended standard regions, cf. [8, Prop. 2.22]. By approximate kernel we mean the span of eigenfunctions whose eigenvalues are close to 0. Since we have to take into account the symmetries imposed, note that the stabilizer of $\tilde{S}[0]$ with respect to the action of \mathcal{G} is generated by the reflections \underline{X} , \underline{Y} , and \underline{Z} , and the stabilizer of $\tilde{S}[1]$ by \underline{X} and \underline{Y} . Therefore we have to restrict our attention to functions on the extended standard regions which are invariant under the action of these subgroups. Moreover the functions on $\tilde{S}[1]$ should extend smoothly to $\mathcal{G}\tilde{S}[1]$.

Definition 3.12. *We call functions which satisfy the above conditions appropriately symmetric and we use the subscript “sym” to denote subspaces of appropriately symmetric functions.*

We understand the approximate kernel in the next proposition by comparing it to the kernel of the operator $\Delta + 2$ on the round sphere $\mathbb{S}^2(1)$, and Δ on the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with Neumann boundary conditions on the boundary. Because of the symmetries the former is trivial and the latter one-dimensional:

Proposition 3.13. *Assuming b large enough in absolute terms, and $\underline{\tau}$ small enough (equivalently m large enough) in terms of a given $\varepsilon > 0$, the following hold:*

(i). \mathcal{L}_h acting on appropriately symmetric functions on $\tilde{S}[0]$ with vanishing Dirichlet conditions, has no eigenvalues in $[-1, 1]$ and the corresponding approximate kernel is trivial.

(ii). \mathcal{L}_h acting on appropriately symmetric functions on $\tilde{S}[1]$ has exactly one eigenvalue λ_0 in $[-\varepsilon, \varepsilon]$, and no other eigenvalues in $[-1/2, 1/2]$, and therefore the corresponding approximate kernel is one-dimensional. Moreover the approximate kernel is spanned by a function $f_0 \in C_{sym}^\infty(\tilde{S}[1])$ which depends continuously on ζ and satisfies

$$\|f_0 - 1 : C^{2,\beta}(S_5[1])\| < \varepsilon, \quad \|f_0 : C^{2,\beta}(\tilde{S}[1], \chi)\| < C.$$

Proof. The proof is based on the results of [4, Appendix B] which are based on basic facts about eigenvalues and eigenfunctions [1]. Before using those results we remark the following: First, the first inequality in [4, B.1.6] should read

$$\|F_i f\|_\infty \leq 2\|f\|_\infty$$

instead. Second, the spaces of functions can be constrained to satisfy appropriate symmetries, as indeed was the case in some of the constructions in [4], and will be the case here. Third, the only use of the Sobolev inequality [4, B.1.5] is to establish supremum bounds for the eigenfunctions. These in our case can be alternatively established by using the uniformity of geometry of $S_5[n]$ to obtain interior estimates on $S_1[n]$, and then using a variant of 3.11 to obtain estimates on the transition regions. More precisely the eigenvalue equation under consideration is $\mathcal{L}_h u + \lambda u = 0$, which is equivalent to

$$(3.14) \quad \mathcal{L}_{\chi,\lambda} u = 0 \quad \text{where} \quad \mathcal{L}_{\chi,\lambda} = \Delta_\chi + \frac{|A|^2 + m^2}{2\rho^2} \lambda.$$

Since the modified part of the operator, $\frac{|A|^2 + m^2}{2\rho^2} \lambda$, satisfies the same estimates by 3.6 (assume $|\lambda| < 9$) as $\rho^{-2}(|A|^2 + 2)$, we can repeat the arguments leading to 3.11 to establish the same estimate under the modified assumption that $\mathcal{L}_{\chi,\lambda} u = 0$ on Λ .

For (i) we compare with the following:

$$N[0] = \mathbb{S}^2(1) \bigcup \left(\{1, -1\} \times \tilde{D}(\tilde{R}_0) \right) \quad \text{where} \quad \tilde{D}(\tilde{R}_0) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : \tilde{x}^2 + \tilde{y}^2 \leq \tilde{R}_0^2\},$$

where \tilde{R}_x was defined in 2.20. The action of \underline{X} , \underline{Y} , and \underline{Z} , on $N[0]$ should be consistent with their action on M (recall A.2) and the maps $\tilde{\nu}$ and ϖ : We define for $(x, y, z) \in \mathbb{S}^2(1)$ and $(i, \tilde{x}, \tilde{y}) \in \{1, -1\} \times \tilde{D}(\tilde{R}_0)$

$$(3.15) \quad \begin{aligned} \underline{X}(x, y, z) &= (-x, y, z), & \underline{X}(i, \tilde{x}, \tilde{y}) &= (i, -\tilde{x}, \tilde{y}), \\ \underline{Y}(x, y, z) &= (x, -y, z), & \underline{Y}(i, \tilde{x}, \tilde{y}) &= (i, \tilde{x}, -\tilde{y}), \\ \underline{Z}(x, y, z) &= (y, x, -z), & \underline{Z}(i, \tilde{x}, \tilde{y}) &= (-i, \tilde{y}, \tilde{x}). \end{aligned}$$

We consider the Dirichlet problem on $N[0]$ where the operator is $\Delta + 2$ on $\mathbb{S}^2(1)$, and the standard Laplacian Δ on $\{1, -1\} \times \tilde{D}(\tilde{R}_0)$. By standard theory then there are no eigenvalues in $[-1, 1]$ because the symmetries do not allow the first harmonics on $\mathbb{S}^2(1)$, and \tilde{R}_0 is small enough so that the smallest eigenvalue on the discs is > 2 .

For (ii) we compare with the following:

$$N[1] = \tilde{D} \bigcup \left([-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \right),$$

where $\tilde{D} = \{(x, y, z) \in \mathbb{S}^2(1) : x^2 + y^2 \leq \tilde{R}_0^2, \quad z \geq 0\}$,

where $\tilde{R}_0 = 1/\cosh(ab/\underline{a})$ (recall 2.20). The action of \underline{X} and \underline{Y} on $N[1]$ should be consistent again with their action on M (recall A.2) and the maps $\hat{\nu}$ and ϖ : We define for $(x, y, z) \in \check{D}$ and $(\tilde{x}, \tilde{y}) \in [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$

$$(3.16) \quad \begin{aligned} \underline{X}(x, y, z) &= (-x, y, z), & \underline{X}(\tilde{x}, \tilde{y}) &= (-\tilde{x}, \tilde{y}), \\ \underline{Y}(x, y, z) &= (x, -y, z), & \underline{Y}(\tilde{x}, \tilde{y}) &= (\tilde{x}, -\tilde{y}), \end{aligned}$$

As before the operator on $\check{D} \subset \mathbb{S}^2(1)$ is $\Delta + 2$ and on $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \subset \mathbb{R}^2$ is the standard Laplacian Δ . The boundary conditions are the Dirichlet condition on $\partial\check{D}$ and the Neumann condition—more precisely extendibility to \mathbb{R}^2 by reflections across the lines $\{\tilde{x} = n\pi/2\}$ and $\{\tilde{y} = n\pi/2\}$ ($n \in \mathbb{N}$)—for the boundary of the square $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2] \subset \mathbb{R}^2$. The smallness of \tilde{R}_0 and our knowledge of the eigenvalues on the square imply the only eigenvalue in $[-2/3, 2/3]$ is 0, with corresponding eigenfunctions the functions which are constant on the square and vanish on \check{D} .

To complete the proof we use $\hat{\nu}$, ϖ , and the logarithmic cut-off function $\psi[2d, d] \circ \bar{x}$ on Λ to define the maps F_1 and F_2 required by [4, B.1.4] as usual. d is taken to be large enough in terms of ε . It is straightforward then to check the required assumptions by using 3.6, and then the results of [4, Appendix B] apply. We upgrade the L^2 estimates for $f_0 - 1$ to $C^{2,\beta}$ estimates on $S_5[1]$ by using the uniformity of the geometry of $S_6[1]$ (see 2.20) and standard linear theory interior estimates. Applying then the variant of 3.11 we discussed earlier, we estimate f_0 on Λ and complete the proof. \square

The (extended) substitute kernel.

As we have already mentioned in the introduction, the extended substitute kernel in this case is particularly simple since it is one-dimensional. This reflects the fact that the approximate kernel, and hence the substitute kernel also, are one-dimensional. Moreover decay can be ensured by using the substitute kernel and so no further extended substitute kernel is required. Motivated by proposition 3.13 above we define a function $w \in C_{sym}^\infty(M)$ by requiring that on $M \cap \mathbb{D}$ it satisfies

$$(3.17) \quad w := \psi[m^{-1}, 2m^{-1}](r).$$

For future reference we record the following:

Lemma 3.18. *Given $E \in C_{sym}^0(\tilde{S}[1])$ there is a unique $\mu \in \mathbb{R}$ such that $\frac{2\rho^2}{|A|^2+m^2}(E + \mu w)$ is $L^2(\tilde{S}[1], h)$ -orthogonal to f_0 , where f_0 is the eigenfunction in 3.13. Moreover*

$$|\mu| \leq C \left\| \frac{2\rho^2}{|A|^2+m^2} E : L_{sym}^2(\tilde{S}[1], h) \right\|.$$

Proof. Using 3.6 and 2.11 we conclude that $\frac{1}{C} \leq \frac{2\rho^2}{|A|^2+m^2} \leq C$ on the support of w which together with 2.20 implies the result. \square

To arrange the decay we define $v \in C_{sym}^\infty(\tilde{S}[1])$ by

$$(3.19) \quad v := f_0 + u,$$

where u is the solution to $\mathcal{L}_\chi u = -\mathcal{L}_\chi f_0 + \mu' w$ on $\tilde{S}[1]$ with vanishing Dirichlet data on $C[0] \subset \partial\tilde{S}[1]$, where $\mu' \in \mathbb{R}$ is determined by the requirement (recall 3.18) that

$$E' := \frac{2\rho^2}{|A|^2+m^2}(-\mathcal{L}_\chi f_0 + \mu' w) = \lambda_0 f_0 + \mu' \frac{2\rho^2}{|A|^2+m^2} w$$

is $L^2(\tilde{S}[1], h)$ -orthogonal to f_0 . Note that the equation on $\tilde{S}[1]$ is equivalent to $\mathcal{L}_h u = E'$, and hence the orthogonality condition together with 3.18 implies the existence of a unique u . We record now the properties of v :

Lemma 3.20. *v satisfies the following:*

- (i). $\mathcal{L}_\chi v = \mu_v w$ on $\tilde{S}[1]$ for some $\mu_v \in \mathbb{R}$, and therefore $\mathcal{L}_\chi v = 0$ on Λ .
- (ii). $v = 0$ on $C[0] \subset \partial\tilde{S}[1]$.
- (iii). $\|v : C^{2,\beta}(\tilde{S}[1], \chi)\| \leq C$.
- (iv). $|\mu_v| \leq C\varepsilon$.
- (v). $\|v - 1 : C^{2,\beta}(C_1[1], \chi)\| \leq C(b)\varepsilon$, where ε is as in 3.13.

Proof. (i) and (ii) follow from the definitions. Using 3.13 and 3.18 we have that

$$\left\| \lambda_0 f_0 + \mu' \frac{2\rho^2}{|A|^2 + m^2} w : L^2(\tilde{S}, h) \right\| \leq C\varepsilon,$$

which together with interior $C^{2,\beta}$ estimates on $S_5[1]$ allows us to conclude (iii), (iv), and (v). \square

Solving the linearized equation semi-locally.

In this subsection we solve and estimate the linear equation on the extended standard regions. We can assume the inhomogeneous term E to vanish on Λ_1 , because in the proof of 3.26 we use 3.9 to solve for the part of the inhomogeneous term which is supported there. In the case of $\tilde{S}[1]$ we have nontrivial approximate kernel and therefore we have to adjust the inhomogeneous term appropriately by using w . w can also be used so that appropriate exponential decay can be arranged for the solution:

Lemma 3.21. *There is a linear map*

$$\mathcal{R}_{\tilde{S}[1]} : \{E \in C_{sym}^{0,\beta}(\tilde{S}[1]) : E \text{ is supported on } S_1[1]\} \rightarrow C_{sym}^{2,\beta}(\tilde{S}[1]) \times \mathbb{R},$$

such that the following hold for E in the domain of $\mathcal{R}_{\tilde{S}[1]}$ above and $(\varphi, \mu) = \mathcal{R}_{\tilde{S}[1]}(E)$:

- (i). $\mathcal{L}_\chi \varphi = E + \mu w$ on $\tilde{S}[1]$.
- (ii). φ vanishes on $C[0] \subset \partial\tilde{S}[1]$ and satisfies appropriate Neumann boundary conditions on $C_\partial \subset \partial\tilde{S}[1]$ (recall 2.17h).
- (iii). $|\mu| + \|\varphi : C_{sym}^{2,\beta}(\tilde{S}[1], \chi)\| \leq C(b, \beta) \|E : C_{sym}^{0,\beta}(S_1[1], \chi)\|$.
- (iv). $\|\varphi : C_{sym}^{2,\beta}(\Lambda, \chi, e^{-\gamma\bar{x}})\| \leq C(b, \beta, \gamma) \|E : C_{sym}^{0,\beta}(S_1[1], \chi)\|$.
- (v). $\mathcal{R}_{\tilde{S}[1]}$ depends continuously on ζ .

Proof. We fix b to be large enough so that 3.13, 3.18, and 3.10 with $\varepsilon_1 = 1$ apply. By applying 3.18 and using that E is supported on $S_1[1]$ we have μ_1 such that $|\mu_1| \leq C(b) \|E : C_{sym}^{0,\beta}(S_1[1], \chi)\|$ and $\frac{2\rho^2}{|A|^2 + m^2} (E + \mu_1 w)$ is $L^2(\tilde{S}[1], h)$ -orthogonal to f_0 . There is a unique solution $\varphi_1 \in C_{sym}^{2,\beta}(\tilde{S}[1])$ which is $L^2(\tilde{S}[1], h)$ -orthogonal to f_0 , vanishes on $C[0] \subset \partial\tilde{S}[1]$, and satisfies $\mathcal{L}_\chi \varphi_1 = E + \mu_1 w$ on $\tilde{S}[1]$. Using interior estimates on $S_2[1]$ for φ_1 and applying 3.10 on $\Lambda_{0,1}$ with $u = v - \text{avg } v$ on $C_1[1] \subset \partial\Lambda_{0,1}$, and once more with $u = \varphi_1 - \text{avg } \varphi_1$ on $C_1[1] \subset \partial\Lambda_{0,1}$, we determine μ_2 such that by taking $\varphi := \varphi_1 + \mu_2 v$, and $\mu := \mu_1 + \mu_2 \mu_v$ and using 3.8 and the available estimates from 3.10 and 3.20 we complete the proof. \square

The corresponding statement for $\tilde{S}[0]$ is simpler, reflecting the triviality of the approximate kernel there and that we do not need exponential decay either:

Lemma 3.22. *There is a linear map*

$$\mathcal{R}_{\tilde{S}[0]} : \{E \in C_{sym}^{0,\beta}(\tilde{S}[0]) : E \text{ is supported on } S_1[0]\} \rightarrow C_{sym}^{2,\beta}(\tilde{S}[0]),$$

such that the following hold for E in the domain of $\mathcal{R}_{\tilde{S}[0]}$ above and $\varphi = \mathcal{R}_{\tilde{S}[0]}(E)$:

- (i). $\mathcal{L}_\chi \varphi = E$ on $\tilde{S}[0]$.
- (ii). φ vanishes on $\partial\tilde{S}[0]$.
- (iii). $\|\varphi : C_{sym}^{2,\beta}(\tilde{S}[0], \chi)\| \leq C(b, \beta) \|E : C_{sym}^{0,\beta}(S_1[0], \chi)\|$.
- (iv). $\mathcal{R}_{\tilde{S}[0]}$ depends continuously on ζ .

Proof. By 3.13 there are no small eigenvalues and so we can solve and obtain $L^2(h)$ estimates which together with interior estimates on $S_2[0]$ and 3.11 imply the result. \square

Solving the linearized equation globally.

In order to solve the linearized equation 3.1 globally on M and provide estimates for the solutions, we paste together the semi-local solutions provided by 3.9, 3.21, and 3.22 to obtain a global solution in the proof of 3.26. Before we state the Proposition we define appropriate norms:

Definition 3.23. *For $k \in \mathbb{N}$ and $\beta, \gamma \in (0, 1)$ we define a norm $\|\cdot\|_{k,\beta,\gamma}$ on $C_{sym}^{k,\beta}(M)$ by*

$$\|\phi\|_{k,\beta,\gamma} := \|\phi : C_{sym}^{k,\beta}(M, \chi, \tilde{f})\|,$$

where the weight function \tilde{f} is defined by requesting that it is invariant under the action of \mathcal{G} , $\tilde{f} = 1$ on $S[1]$, $\tilde{f} = e^{-\gamma\bar{x}}$ on Λ , and $\tilde{f} = e^{(a-2b)\gamma} = e^{-\gamma\bar{x}}|_{C[0]}$ on $S[0]$ (recall 3.4).

Note that \tilde{f} is continuous and its minimum as well the maximum of $\rho\tilde{f}$ are attained on $S[0]$, and therefore using 2.6 we have

$$(3.24) \quad \underline{\tau}^{\frac{8}{9}\gamma + \frac{1}{9}} \leq \tilde{f} \quad \text{and} \quad \rho\tilde{f} \leq \underline{\tau}^{\frac{8}{9}\gamma - 1} \quad \text{on } M.$$

Before we proceed to state and prove the main Proposition of this section, we give an estimate of the inhomogeneous term $E = \rho^{-2}H$ of the main linearized equation in this paper:

Lemma 3.25. *If m is large enough in terms of γ we have on M the estimate*

$$\|\rho^{-2}H\|_{2,\beta,\gamma} \leq C\tau.$$

Proof. Using 2.6, 2.7, 2.8, 2.11, and 3.24, we easily check that $|z| + \tau \leq m^2\tau$, $m^2\tau^2 \leq \tilde{f}\tau$, and $\rho^{-2}m^2\tau \leq C\tilde{f}\tau$. These imply that $(\tau + \rho^{-2})(|z| + \tau) \leq C\tilde{f}\tau$, which by 2.13 implies the result. \square

Proposition 3.26. *There is a linear map $\mathcal{R}_M : C_{sym}^{0,\beta}(M) \rightarrow C_{sym}^{2,\beta}(M) \times \mathbb{R}$ such that for $E \in C_{sym}^{0,\beta}(M)$ and $(\varphi, \mu) = \mathcal{R}_M E$ the following hold:*

- (i). $\mathcal{L}_\chi \varphi = E + \mu w$ on M .
- (ii). $|\mu| + \|\varphi\|_{2,\beta,\gamma} \leq C(b, \beta, \gamma) \|E\|_{0,\beta,\gamma}$.
- (iii). \mathcal{R}_M depends continuously on ζ .

Proof. We decompose $E = E_{S[0]} + E_{S[1]} + E_\Lambda$ by requesting that $E_{S[0]}$, $E_{S[1]}$, and E_Λ , are invariant under \mathcal{G} and satisfy

$$\begin{aligned} E_{S[0]} &:= E \psi[1, 0] \circ \underline{x}, \\ E_{S[1]} &:= E \psi[1, 0] \circ \bar{x}, \\ E_\Lambda &:= E \psi[0, 1] \circ \underline{x}, \end{aligned}$$

on Λ , $E_{S[0]} := E$, $E_{S[1]} := 0$, $E_\Lambda := 0$ on $S[0]$, and $E_{S[0]} := 0$, $E_{S[1]} := E$, $E_\Lambda := 0$ on $S[1]$. Using 3.9 we define $V_\Lambda \in C_{sym}^{2,\beta}(M)$ by $V_\Lambda = 0$ on $S[0] \cup S[1]$ and $V_\Lambda = \psi[0, 1] \circ \underline{x} \mathcal{R}_\Lambda E_\Lambda$ on Λ . $\mathcal{L}_\chi V_\Lambda - E_\Lambda$ is supported on $\Lambda \setminus \Lambda_1$, and can be decomposed as $\mathcal{L}_\chi V_\Lambda - E_\Lambda = \underline{E} + \overline{E}$ where \underline{E} is supported on $\{\underline{x} \leq 1\}$ and \overline{E} is supported on $\{\overline{x} \leq 1\}$.

Using 3.21 and 3.22 we define $V_{S[0]} \in C_{sym}^{2,\beta}(M)$ and $V_{S[1]} \in C_{sym}^{2,\beta}(M)$ by requesting the following: $V_{S[1]} = 0$ on $S[1]$ and $V_{S[1]} = \psi[0, 1] \circ \underline{x} V'_{S[1]}$ on $\widetilde{S}[1]$, where

$$(V'_{S[1]}, \mu_1) = \mathcal{R}_{\widetilde{S}[1]}(E_{S[1]} - \overline{E}).$$

$V_{S[0]} = 0$ on $S[1]$ and $V_{S[0]} = \psi[0, 1] \circ \overline{x} \mathcal{R}_{\widetilde{S}[0]}(E_{S[0]} - \underline{E})$ on $\widetilde{S}[0]$. We define then $\varphi_1 := V_\Lambda + V_{S[0]} + V_{S[1]}$ and E_1 by $\mathcal{L}_\chi \varphi_1 + E_1 = E + \mu_1 w$. We iterate with E_1 instead of E and so on. We define then $\varphi := \sum_{n=1}^{\infty} \varphi_n$ and $\mu := \sum_{n=1}^{\infty} \mu_n$ and complete the proof by using the estimates and results of 3.9, 3.21, and 3.22, where 3.21 is applied with $\gamma' = \frac{\gamma+1}{2}$ in place of γ . \square

4. THE MAIN RESULTS

The nonlinear terms.

If $\phi \in C_{sym}^1(M)$ is appropriately small, we denote by M_ϕ the perturbation of M by ϕ , defined as $I_\phi(M)$ in the notation of Appendix B, where $I : M \rightarrow \mathbb{S}^3(1)$ is the inclusion map of M . Clearly then M_ϕ is invariant under the action of \mathcal{G} on the sphere $\mathbb{S}^3(1)$. Using then rescaling and Proposition B.3 we prove a global estimate of the nonlinear terms for the mean curvature of M_ϕ as follows:

Lemma 4.1. *If $\phi \in C_{sym}^{2,\beta}(M)$ satisfies $\|\phi\|_{2,\beta,\gamma} < \underline{\tau}^{1-\frac{3\gamma}{4}}$, then M_ϕ is well defined as above and satisfies*

$$\|\rho^{-2} H_\phi - \rho^{-2} H - \mathcal{L}_\chi \phi\|_{0,\beta,\gamma} \leq \underline{\tau}^{\frac{3\gamma}{4}-1} \|\phi\|_{2,\beta,\gamma}^2,$$

where H_ϕ is the mean curvature of M_ϕ (pulled back to M by I_ϕ), and H is the mean curvature of M .

Proof. Let D be a disc of radius 1 and center at some point $p \in M$ with respect to the χ metric. If we magnify the metric of the sphere $\mathbb{S}^3(1)$ by a factor $\rho(p)$ it is easy to arrange for the hypothesis B.1 to be satisfied so that we can apply B.3 with some universal c_1 to conclude

$$\|(\rho(p))^{-1}(H_\phi - H - \mathcal{L}\phi) : C^{0,\beta}(D, \chi)\| \leq \frac{1}{\epsilon(c_1)} \|\rho(p)\phi : C^{2,\beta}(D, \chi)\|^2,$$

where the factors $\rho(p)$ correspond to the scaling of the quantities involved. By the multiplicative properties of the Holder norms we conclude

$$\|\rho^{-2}(H_\phi - H - \mathcal{L}\phi) : C^{0,\beta}(D, \chi)\| \leq \frac{\rho(p)}{\epsilon(c_1)} \|\phi : C^{2,\beta}(D, \chi)\|^2.$$

By 3.23 we conclude

$$\frac{1}{\widetilde{f}(p)} \|\rho^{-2}(H_\phi - H - \mathcal{L}\phi) : C^{0,\beta}(D, \chi)\| \leq \frac{\rho(p)\widetilde{f}(p)}{\epsilon(c_1)} \|\phi\|_{2,\beta,\gamma}^2.$$

This implies the result by using 3.24. \square

The vertical force and balancing.

If $\phi \in C_{sym}^1(M)$, M_ϕ , and H_ϕ are as in the previous subsection we define \mathcal{F} by

$$(4.2) \quad \mathcal{F} := \int_{M_\phi \cap \mathbb{D}_+} H_\phi \langle \nu, \vec{K} \rangle dg = \int_{M_\phi \cap \partial \mathbb{D}_+} \langle \vec{\eta}, \vec{K} \rangle dg,$$

where $\mathbb{D}_+ := \mathbb{D} \cap \{z \geq 0\}$, ν the unit normal chosen so that $\langle \nu, \partial_z \rangle > 0$ on \widehat{M}_{tor} , \vec{K} is the Killing field defined in A.9, and η the outward conormal to $\partial(M_\phi \cap \mathbb{D}_+) = M_\phi \cap \partial \mathbb{D}_+$ tangent to M_ϕ . Note that the second equality in 4.2 follows from the first variation formula [16, 20]. We have then the following, where we could be using $\|\phi\|_{1,0,\gamma}$ instead of $\|\phi\|_{2,\beta,\gamma}$ as well:

Lemma 4.3. *If $\|\phi\|_{2,\beta,\gamma} < \underline{\tau}^{1-\frac{\gamma}{4}}$, then there is a universal constant C such that*

$$\left| \frac{m^2}{8\tau\pi^2} \mathcal{F} + \zeta \right| \leq C \left(1 + \frac{1}{\tau} \|\phi\|_{2,\beta,\gamma} \right).$$

Proof. Let $d := \pi/\sqrt{2}m$ and decompose

$$\partial(M_\phi \cap \mathbb{D}_+) = M_\phi \cap \partial \mathbb{D}_+ = \partial_{+1} \cup \partial_{-1} \cup \partial_{+2} \cup \partial_{-2} \cup \partial_0,$$

where $\partial_{+1} \subset \{x = d\}$, $\partial_{-1} \subset \{x = -d\}$, $\partial_{+2} \subset \{y = d\}$, $\partial_{-2} \subset \{y = -d\}$, and $\partial_0 \subset \{z = 0\}$. We use the big- O notation to denote terms $O(A)$ which satisfy $|O(A)| \leq CA$ for some universal constant C . Using then A.6 and A.10 we calculate that on $\partial_{\pm 1}$

$$\begin{aligned} \vec{\eta} &= \pm(1 + \sin 2z)^{-1/2} \partial_x, \\ \langle \vec{\eta}, \vec{K} \rangle &= -\frac{1}{\sqrt{2}} \sqrt{1 + \sin 2z} \cot\left(z + \frac{\pi}{4}\right) \sin \sqrt{2}d \cos \sqrt{2}y, \\ dg &= \sqrt{1 - \sin 2z + \phi_y^2} dy. \end{aligned}$$

Combining the above we obtain

$$\int_{\partial_{\pm 1}} \langle \vec{\eta}, \vec{K} \rangle dg = -\frac{1}{\sqrt{2}} \int_{-d}^d (1 - \sin 2z + O(|z|^2 + |\phi_y|^2)) \sin \sqrt{2}d \cos \sqrt{2}y dy.$$

Similarly

$$\int_{\partial_{\pm 2}} \langle \vec{\eta}, \vec{K} \rangle dg = \frac{1}{\sqrt{2}} \int_{-d}^d (1 + \sin 2z + O(|z|^2 + |\phi_y|^2)) \cos \sqrt{2}x \sin \sqrt{2}d dx.$$

Combining the above and using that on $\partial_{+1} \cup \partial_{-1} \cup \partial_{+2} \cup \partial_{-2}$ we have $z = \tau a + \phi$, we conclude that

$$\int_{\partial_{+1} \cup \partial_{-1} \cup \partial_{+2} \cup \partial_{-2}} \langle \vec{\eta}, \vec{K} \rangle dg = (8a\tau + O(\|\phi\| + a^2\tau^2)) (2d^2 + O(d^3)),$$

where $\|\phi\| := \|\phi\|_{2,\beta,\gamma}$. Similarly

$$\int_{\partial_0} \langle \vec{\eta}, \vec{K} \rangle dg = -2\pi\tau(1 + O(\tau^{\frac{\gamma}{2}-1}\|\phi\|)).$$

Combining and substituting $a = \frac{m^2}{4\pi} - \zeta + O(1)$ by 2.6 we conclude

$$\mathcal{F} = -\frac{8\pi^2\tau}{m^2}\zeta + \frac{1}{m^2}O(\tau + \|\phi\|),$$

which implies the result. \square

The main theorem.

We have now all the information we need to state and prove the main theorem of the paper:

Theorem 4.4. *There are absolute constants $\underline{c}, C > 0$ such that if m is large enough, then there is $\zeta_1 \in [-\underline{c}, \underline{c}]$ such that on the corresponding initial surface M there is $\phi \in C_{sym}^\infty(M)$ with $\|\phi\|_{2,\beta,\gamma} \leq C\tau$ (with τ defined as in 2.4) such that M_ϕ is a genus $m^2 + 1$ embedded minimal surface in $\mathbb{S}^3(1)$ invariant under the action of \mathcal{G} .*

Proof. We will use a subscript ζ to specify the initial surface M_ζ which is constructed as in the discussion preceding 2.9. We also define the map $\underline{X}_\zeta : [-\underline{a}, \underline{a}] \times \mathbb{S}^1 \rightarrow M_\zeta$ by requesting that $\underline{X}_\zeta(\underline{t}, \theta) = \Phi \circ X(at/\underline{a}, \theta)$ where the X is the one defined for the given value of the parameter ζ , that is \underline{X}_ζ is the parametrization corresponding to coordinates (\underline{t}, θ) for $M_{cat} \subset M_\zeta$. As in the proof of 2.20 it is easy to check that there is $\tilde{t} : [\underline{a} - 2, \underline{a} - 1] \rightarrow [\underline{a} - 3, \underline{a}]$ close to the identity map, such that for $(\underline{t}, \theta) \in [\underline{a} - 2, \underline{a} - 1] \times \mathbb{S}^1$ we have $\varpi \circ \underline{X}_\zeta(\underline{t}, \theta) = \varpi \circ \underline{X}_0(\tilde{t}(\underline{t}), \theta)$.

We define now a diffeomorphism $F_\zeta : M_\zeta \rightarrow M_0$ by requiring that it is equivariant under the action of \mathcal{G} , it satisfies $\varpi \circ F_\zeta = \varpi$ on $S_1[1] \subset M_\zeta$, and that for $(\underline{t}, \theta) \in [-1, \underline{a}] \times \mathbb{S}^1$ we have

$$F_\zeta \circ \underline{X}_\zeta(\underline{t}, \theta) = \underline{X}_0(\underline{t} + \psi[\underline{a} - 2, \underline{a} - 1](\underline{t})(\tilde{t}(\underline{t}) - \underline{t}), \theta).$$

We define now a map $\mathcal{J} : B \rightarrow B$ where

$$B := \{u \in C_{sym}^{2,\beta}(M_0) : \|u\|_{2,\beta,\gamma} \leq \tau^{\frac{7}{2}+1}\} \times [-\underline{c}, \underline{c}]$$

as follows: We assume $(u, \zeta) \in B$ given. Let $\phi \in C^{2,\beta}(M_\zeta)$ be defined by $\phi := u \circ F_\zeta + \varphi$ where $(\varphi, \mu) = \mathcal{R}_{M_\zeta}(-\rho^2 H)$ as in 3.26. We have then

- (a). $\mathcal{L}_\chi \varphi + \rho^{-2} H = \mu w_0$, or equivalently $\mathcal{L}\varphi + H = \mu \rho^2 w_0$.
- (b). By 3.26 and 3.25 we have

$$|\mu| + \|\phi\|_{2,\beta,\gamma} \leq C(b, \beta, \gamma) \tau.$$

Applying 3.26 again and using 4.1 we obtain $(v, \mu') := \mathcal{R}_{M_\zeta}(-(\rho^{-2} H_\phi - \rho^{-2} H - \mathcal{L}_\chi \phi))$ which satisfies the following:

- (c). $\mathcal{L}_\chi v + \rho^{-2} H_\phi - \rho^{-2} H - \mathcal{L}_\chi \phi = \mu' w_0$.
- (d). $|\mu'| + \|v\|_{2,\beta,\gamma} \leq \tau^{\frac{3\gamma}{4}-1} \|\phi\|_{2,\beta,\gamma}^2$.

Combining (a) and (c) with the definition of ϕ we obtain

- (e). $\mathcal{L}_\chi(v - u \circ F_\zeta) + \rho^{-2} H_\phi = (\mu + \mu') w_0$.

This motivates us to define

$$\mathcal{J}(u, \zeta) = \left(v \circ (F_\zeta)^{-1}, \frac{m^2}{8\tau\pi^2} \mathcal{F} + \zeta \right),$$

where \mathcal{F} is defined as in 4.2. By using (b), (d), and 4.3, and by choosing \underline{c} large enough in terms of an absolute constant, it is straightforward to check that $\mathcal{J}(B) \subset B$. B is clearly a compact convex subset of $C_{sym}^{2,\beta'}(M_0) \times \mathbb{R}$ for $\beta' \in (0, \beta)$, and it is easy to check that \mathcal{J} is a continuous map in the induced topology. By Schauder's fixed point theorem [2, Theorem 11.1] then, there is a fixed point of \mathcal{J} . Using (e) then we conclude that for the corresponding ζ and ϕ we have

$$H_\phi = (\mu + \mu') \rho^2 w_0, \quad \mathcal{F} = 0.$$

Since $\langle \nu, \vec{K} \rangle > 0$ on the support of w_0 in $(M_\zeta)_\phi$ the second equation implies that $\mu + \mu' = 0$ and hence $(M_\zeta)_\phi$ is a minimal surface. The smoothness of ϕ follows then by standard

regularity theory. The embeddedness of $(M_c)_\phi$ follows from the smallness of $\|\varphi\|_{2,\beta,\gamma}$ and the size (by 2.6) of $a\tau$. \square

APPENDIX A. A COORDINATE SYSTEM ON $\mathbb{S}^3(1)$

The parametrization Φ .

It is very helpful that there is a coordinate system which is ideally suited to describing the Clifford torus and its parallel surfaces. We proceed to describe this coordinate system and the local parametrization which is its inverse. To simplify the notation we identify $\mathbb{R}^4 \simeq \mathbb{C}^2 \supset \mathbb{S}^3(1)$. We define the parametrization Φ , which covers the unit sphere with two orthogonal circles removed, that is $\mathbb{S}^3(1) \setminus \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0 \text{ or } z_2 = 0\}$, by the following:

$$(A.1) \quad \begin{aligned} \Phi : \text{Dom}_\Phi &\rightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4 \sim \mathbb{C}^2, \quad \text{where } \text{Dom}_\Phi := \mathbb{R} \times \mathbb{R} \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ \Phi(x, y, z) &= \cos\left(z + \frac{\pi}{4}\right) e^{\sqrt{2}yi} \vec{e}_1 + \sin\left(z + \frac{\pi}{4}\right) e^{\sqrt{2}xi} \vec{e}_2, \end{aligned}$$

where $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$ form the standard basis of \mathbb{C}^2 .

Symmetries of Φ .

To study the symmetries of the parametrization Φ , we first define for $c \in \mathbb{R}$ translations $\widehat{X}_c, \widehat{Y}_c$, and reflections $\widehat{\underline{X}}_c, \widehat{\underline{Y}}_c, \widehat{\underline{X}} := \widehat{\underline{X}}_0, \widehat{\underline{Y}} := \widehat{\underline{Y}}_0$, and $\widehat{\underline{Z}}$, of its domain Dom_Φ , by

$$(A.2) \quad \begin{aligned} \widehat{X}_c(x, y, z) &= (x + c, y, z), & \widehat{Y}_c(x, y, z) &= (x, y + c, z), \\ \widehat{\underline{X}}_c(x, y, z) &= (2c - x, y, z), & \widehat{\underline{Y}}_c(x, y, z) &= (x, 2c - y, z), \\ \widehat{\underline{Z}}(x, y, z) &= (y, x, -z). \end{aligned}$$

We also define corresponding rotations X_c, Y_c , and reflections $\underline{X}_c, \underline{Y}_c, \underline{X} := \underline{X}_0, \underline{Y} := \underline{Y}_0$, and \underline{Z} of $\mathbb{S}^3(1) \subset \mathbb{C}^2$ by

$$(A.3) \quad \begin{aligned} X_c(z_1, z_2) &= (z_1, e^{\sqrt{2}ci} z_2), & Y_c(z_1, z_2) &= (e^{\sqrt{2}ci} z_1, z_2), \\ \underline{X}(z_1, z_2) &= (z_1, \overline{z_2}), & \underline{Y}(z_1, z_2) &= (\overline{z_1}, z_2), \\ \underline{X}_c &:= X_{2c} \circ \underline{X}, & \underline{Y}_c &:= Y_{2c} \circ \underline{Y}, \\ \underline{Z}(z_1, z_2) &= (z_2, z_1). \end{aligned}$$

Note that $\underline{X}_c, \underline{Y}_c$ and \underline{Z} are reflections with respect to the 3-planes $\langle \vec{e}_1, i\vec{e}_1, e^{\sqrt{2}ci} \vec{e}_2 \rangle_{\mathbb{R}}$, $\langle e^{\sqrt{2}ci} \vec{e}_1, \vec{e}_2, i\vec{e}_2 \rangle_{\mathbb{R}}$, and the 2-plane $\{z_1 = z_2\}$ respectively. \underline{Z} exchanges the two sides of the Clifford torus and also interchanges its parallels with its meridians. $X_{\sqrt{2}\pi}$ and $Y_{\sqrt{2}\pi}$ are the identity map. We record the symmetries of Φ in the following lemma:

Lemma A.4. *Φ is a covering map onto $\mathbb{S}^3(1) \setminus \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0 \text{ or } z_2 = 0\}$. Moreover the following hold:*

- (i). *The group of covering transformations is generated by $\widehat{X}_{\sqrt{2}\pi}$ and $\widehat{Y}_{\sqrt{2}\pi}$, in particular $\Phi = \Phi \circ \widehat{X}_{\sqrt{2}\pi} = \Phi \circ \widehat{Y}_{\sqrt{2}\pi}$.*
- (ii). *$\underline{X}_c \circ \Phi = \Phi \circ \widehat{\underline{X}}_c$, $\underline{Y}_c \circ \Phi = \Phi \circ \widehat{\underline{Y}}_c$, and $\underline{Z} \circ \Phi = \Phi \circ \widehat{\underline{Z}}$.*
- (iii). *$X_c \circ \Phi = \Phi \circ \widehat{X}_c$ and $Y_c \circ \Phi = \Phi \circ \widehat{Y}_c$.*

Proof. (ii) and (iii) follow from the definitions. (i) follows from (iii) and the observation that $X_{\sqrt{2}\pi}$ and $Y_{\sqrt{2}\pi}$ are the identity map. \square

The coordinates xyz.

The local inverses of Φ provide us with local coordinate systems. We denote the corresponding coordinates by x, y, z . A straightforward calculation shows that

$$(A.5) \quad \begin{aligned} \partial_x &= \sqrt{2} \sin(z + \frac{\pi}{4}) i e^{\sqrt{2}xi} \vec{e}_2, \\ \partial_y &= \sqrt{2} \cos(z + \frac{\pi}{4}) i e^{\sqrt{2}yi} \vec{e}_1, \\ \partial_z &= -\sin(z + \frac{\pi}{4}) e^{\sqrt{2}yi} \vec{e}_1 + \cos(z + \frac{\pi}{4}) e^{\sqrt{2}xi} \vec{e}_2. \end{aligned}$$

By calculating further we obtain

$$(A.6) \quad \Phi^*g = (1 + \sin 2z) dx^2 + (1 - \sin 2z) dy^2 + dz^2,$$

where g is the induced metric on the unit sphere $\mathbb{S}^3(1)$. Moreover the only non-vanishing Christoffel symbols for the (x, y, z) -coordinate system are given by

$$(A.7) \quad \begin{aligned} \Gamma_{13}^1 &= \frac{\cos 2z}{1 + \sin 2z}, & \Gamma_{23}^2 &= -\frac{\cos 2z}{1 - \sin 2z}, \\ \Gamma_{11}^3 &= -\cos 2z, & \Gamma_{22}^3 &= \cos 2z. \end{aligned}$$

The level surface with $z = 0$ is the Clifford torus

$$(A.8) \quad \mathbb{T} := \Phi(\{z = 0\}) = \{(z_1, z_2) \in \mathbb{S}^3(1) \subset \mathbb{C}^2 : |z_1| = |z_2| = 1/\sqrt{2}\}.$$

The level surfaces $\Phi(\{z = c\})$ ($z \in (-\frac{\pi}{4}, \frac{\pi}{4})$) are tori of constant mean curvare, parallel at distance c to the Clifford torus \mathbb{T} , with ∂_z as their unit normal vector field. Note also that for $c \in \mathbb{R}$ we have the level surfaces

$$\begin{aligned} \Phi(\{x = c\}) &= \{t_1 \vec{e}_1 + t_2 i \vec{e}_1 + t_3 e^{\sqrt{2}ci} \vec{e}_2 : t_1, t_2 \in \mathbb{R}, t_3 \in \mathbb{R}^+\} \cap \mathbb{S}^3(1), \\ \Phi(\{y = c\}) &= \{t_1 e^{\sqrt{2}ci} \vec{e}_1 + t_2 \vec{e}_2 + t_3 i \vec{e}_2 : t_1 \in \mathbb{R}^+, t_2, t_3 \in \mathbb{R}\} \cap \mathbb{S}^3(1), \end{aligned}$$

which are equatorial half-two-spheres orthogonal to the parallel tori. These three families of level surfaces are orthogonal. The intersections of the last two are great semicircles orthogonal to the tori. Finally a calculation shows that $\det[\Phi, \Phi_x, \Phi_y, \Phi_z] = \cos 2z > 0$.

Killing fields.

Clearly ∂_x and ∂_y are Killing fields generating the rotations in the $\langle \vec{e}_2, i\vec{e}_2 \rangle_{\mathbb{R}}$ and $\langle \vec{e}_1, i\vec{e}_1 \rangle_{\mathbb{R}}$ planes respectively. However ∂_z is not a Killing field. For this reason we consider the Killing field \vec{K} which agrees with ∂_z at $\Phi(0, 0, 0) = 2^{-1/2}(\vec{e}_1 + \vec{e}_2)$ and is defined by

$$(A.9) \quad \vec{K} \Big|_{(z_1, z_2)} := -\operatorname{Re} z_2 \vec{e}_1 + \operatorname{Re} z_1 \vec{e}_2.$$

\vec{K} generates the rotations in the $\langle \vec{e}_1, \vec{e}_2 \rangle_{\mathbb{R}}$ plane. A straightforward calculation shows that

$$(A.10) \quad \begin{aligned} \vec{K} &= -\frac{1}{\sqrt{2}} \cot(z + \pi/4) \sin \sqrt{2}x \cos \sqrt{2}y \partial_x \\ &\quad + \frac{1}{\sqrt{2}} \tan(z + \pi/4) \cos \sqrt{2}x \sin \sqrt{2}y \partial_y + \cos \sqrt{2}x \cos \sqrt{2}y \partial_z. \end{aligned}$$

APPENDIX B. THE MEAN CURVATURE OF A PERTURBED SURFACE

We assume given an immersion $X : D \rightarrow U$, where D is a disc of radius 1 in the Euclidean plane \mathbb{R}^2 , and U is an open cube in \mathbb{R}^3 equipped with a metric g whose components are functions $g_{ij} : U \rightarrow \mathbb{R}$. We assume that the following holds for some $c_1 > 0$:

$$(B.1) \quad \|\partial X : C^{2,\beta}(D, g_0)\| \leq c_1, \quad \|g_{ij} : C^{2,\beta}(U, g_0)\| \leq c_1, \quad g_0 \leq c_1 X^*g,$$

where ∂X are the partial derivatives of the coordinates of X , and g_0 denotes the standard Euclidean metric on U or D respectively. Note that B.1 can be arranged by first appropriately magnifying the target (see for example 4.1). We also choose a unit normal $\nu : D \rightarrow \mathbb{R}^3$ for the immersion X with respect to the g metric. Given a function $\phi : D \rightarrow \mathbb{R}$ which is small enough we define $X_\phi : D \rightarrow U$ by

$$(B.2) \quad X_\phi(p) := \exp_{X(p)}(\phi(p) \nu(p)),$$

where \exp is the exponential map with respect to the g metric. We have then the following:

Proposition B.3. *There exists a (small) constant $\epsilon(c_1) > 0$ such that if X is an immersion satisfying B.1 and the function $\phi : D \rightarrow \mathbb{R}$ satisfies*

$$\|\phi : C^{2,\beta}(D, g_0)\| < \epsilon(c_1),$$

then $X_\phi : D \rightarrow U$ is a well-defined immersion by B.2 and satisfies

$$\|H_\phi - H - (\Delta_g + |A|^2 + Ric(\nu, \nu))\phi : C^{0,\beta}(D, g_0)\| \leq \frac{1}{\epsilon(c_1)} \|\phi : C^{2,\beta}(D, g_0)\|^2,$$

*where $H = \text{tr}_g A$ is the mean curvature of X , defined as the trace with respect to X^*g of the second fundamental form A , H_ϕ is the mean curvature of X_ϕ , Δ_g is the Laplacian with respect to X^*g , and Ric is the Ricci curvature of (U, g) .*

Proof. That the linear terms are as stated is well known and follows by a straightforward calculation we omit. The nonlinear terms are given by expressions of monomials consisting of contractions of derivatives of X and derivatives of ϕ . This implies both the existence results and the estimate on the nonlinearity. \square

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