

OPERATIONS ON THE A-THEORETIC NIL-TERMS

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ABSTRACT. For a space X , we define Frobenius and Verschiebung operations on the nil-terms $\mathrm{NA}_{\pm}^{\mathrm{fd}}(X)$ in the algebraic K -theory of spaces, in three different ways. Two applications are included. Firstly, we show that the homotopy groups of $\mathrm{NA}_{\pm}^{\mathrm{fd}}(X)$ are either trivial or not finitely generated as abelian groups. Secondly, the Verschiebung defines a $\mathbb{Z}[\mathbb{N}_{\times}]$ -module structure on the homotopy groups of $\mathrm{NA}_{\pm}^{\mathrm{fd}}(X)$, with \mathbb{N}_{\times} the multiplicative monoid.

We also give a calculation of the homotopy type of the nil-terms $\mathrm{NA}_{\pm}^{\mathrm{fd}}(*)$ after p -completion for an odd prime p and their homotopy groups as $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -modules up to dimension $4p - 7$. We obtain non-trivial groups only in dimension $2p - 2$, where it is finitely generated as a $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -module, and in dimension $2p - 1$, where it is not finitely generated as a $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -module.

1. INTRODUCTION

The fundamental theorem of algebraic K -theory of spaces which is proved in [HKV⁺01] states that for any space X there is a splitting

$$(1.1) \quad A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times BA^{\mathrm{fd}}(X) \times \mathrm{NA}_{+}^{\mathrm{fd}}(X) \times \mathrm{NA}_{-}^{\mathrm{fd}}(X),$$

where $A^{\mathrm{fd}}(X)$ denotes the finitely-dominated version of the algebraic K -theory of the space X , by $BA^{\mathrm{fd}}(X)$ is denoted certain canonical non-connective delooping of $A^{\mathrm{fd}}(X)$ and $\mathrm{NA}_{+}^{\mathrm{fd}}(X)$, $\mathrm{NA}_{-}^{\mathrm{fd}}(X)$ are two homeomorphic nil-terms. Thus the study of $A^{\mathrm{fd}}(X \times S^1)$ splits naturally into studying $A^{\mathrm{fd}}(X)$ and the nil-terms. Over time there has been steady progress in understanding $A^{\mathrm{fd}}(X)$ for some X , mostly for $X = *$ [KR97], [Rog02], [Rog03], but not much is known about the nil-terms. These are the subject of the present paper.

Recall that the splitting (1.1) is analogous to the splitting of the fundamental theorem of algebraic K -theory of rings [Gra76] for any ring R

$$(1.2) \quad K(R[\mathbb{Z}]) \simeq K(R) \times BK(R) \times \mathrm{NK}_{+}(R) \times \mathrm{NK}_{-}(R),$$

where $K(R)$ denotes the (-1) -connective K -theory space of the ring R and $BK(R)$ is a certain canonical non-connective delooping of $K(R)$. In fact, for any space X there is a *linearization map* $l: A^{\mathrm{fd}}(X) \rightarrow K(\mathbb{Z}[\pi_1 X])$ and the two splittings are natural with respect to this map.

We pursue the analogy between the algebraic K -theory of spaces (the non-linear situation) and the algebraic K -theory of rings (the linear situation) further. We define the Frobenius and Verschiebung operations on the nil-terms $\mathrm{NA}_{\pm}^{\mathrm{fd}}(X)$, which are analogs of such operations defined in the linear case. As a consequence we obtain the following result (compare with [Far77]):

Theorem 1.1. *The homotopy groups $\pi_* \mathrm{NA}_{\pm}^{\mathrm{fd}}(X)$ and all of their p -primary subgroups are either trivial or not finitely generated as abelian groups.*

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This result suggests that it is very hard to determine the homotopy groups of $\mathrm{NA}_{\pm}^{\mathrm{fd}}(X)$ and that it is important to have more structure on them. We achieve this in Corollary 5.2 which says that the Verschiebung operations define a structure of a $\mathbb{Z}[\mathbb{N}_{\times}]$ -module on the homotopy groups of the nil-terms. Here $\mathbb{N}_{\times} = \{1, 2, \dots\}$ is a monoid with respect to multiplication. In the algebraic K -theory of rings the corresponding structure was studied in [CdS95], [Lod81].

It is standard knowledge that $\mathrm{NK}_{\pm}(\mathbb{Z}) \simeq *$, since \mathbb{Z} is a regular ring [Ros94]. However, not much is known about the spaces $\mathrm{NA}_{\pm}^{\mathrm{fd}}(*)$. One approach is to compare them with the linear nil-terms via the linearization map $\mathrm{NA}_{\pm}^{\mathrm{fd}}(*) \rightarrow \mathrm{NK}_{\pm}(\mathbb{Z})$. This is known to be a rational equivalence. Moreover, after p -completion at a prime p it is $(2p - 3)$ -connected. Hence the nil-terms $\mathrm{NA}_{\pm}^{\mathrm{fd}}(*)$ are rationally trivial. It seems that the only other known results in this direction are that they are trivial in dimensions 0, 1 and non-trivial in dimension 2 [Wal78], [Igu82].

We give a calculation of the homotopy type of the nil-terms $\mathrm{NA}_{\pm}^{\mathrm{fd}}(*)$ and of the module structure on their homotopy groups in a certain range. For this we first p -complete $\mathrm{NA}_{\pm}^{\mathrm{fd}}(*)$ at an odd prime p so that the homotopy groups become $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -modules. Below $\mathbb{H}\mathbb{F}_p$ denotes the Eilenberg-Mac Lane spectrum of \mathbb{F}_p and $S^1(n)$ is just a copy of S^1 indexed by $n \in \mathbb{N}_{\times}$. The result is:

Theorem 1.2. *If p is an odd prime the following holds.*

- (1) *There is a $(4p - 7)$ -connected map*

$$\bigvee_{\pm n \in \mathbb{N}_{\times}} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(\pm n)_+) \longrightarrow \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge},$$

hence

$$\begin{aligned} \pi_{2p-2} \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge} &\cong \bigoplus_{n \in \mathbb{N}_{\times}} \mathbb{F}_p \{\beta_{\pm n}\} \\ \pi_{2p-1} \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge} &\cong \bigoplus_{n \in \mathbb{N}_{\times}} \mathbb{F}_p \{\gamma_{\pm n}\} \\ \pi_i \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge} &\cong 0 \text{ for } i < 2p - 2, 2p - 1 < i \leq 4p - 7, \end{aligned}$$

where

$$\begin{aligned} \beta_{\pm n} &\in \pi_{2p-2} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(\pm n)_+) \cong \mathbb{F}_p \\ \gamma_{\pm n} &\in \pi_{2p-1} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(\pm n)_+) \cong \mathbb{F}_p \end{aligned}$$

represent a certain choice of generators of these \mathbb{Z}_p -modules.

- (2) *The $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -module structure on $\pi_* \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge}$ is given by*

$$\begin{aligned} (n, \beta_m) &\mapsto \beta_{nm} \\ (n, \gamma_m) &\mapsto n \cdot \gamma_{nm} \end{aligned}$$

The proof of Theorem 1.2 follows the general strategy suggested by Madsen in [Mad94] to study the linearization map $A^{\mathrm{fd}}(S^1) \rightarrow K(\mathbb{Z}[\mathbb{Z}])$. Certain homotopy theoretic constructions and relations between $A^{\mathrm{fd}}(X)$, the trace invariants of K -theory, and the free loop space of X are used.

In the linear situation there are partial results pointing in the direction that the homotopy groups of the nil-terms could be finitely-generated as $\mathbb{Z}[\mathbb{N}_{\times}]$ -modules, see [CdS95], [Lod81]. However, Theorem 1.2 shows that this is not the case in the non-linear situation. We have

Corollary 1.3. *The $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -module $\pi_{2p-2} \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge}$ is (finitely) generated by $\beta_{\pm 1}$, and the $\mathbb{Z}_p[\mathbb{N}_{\times}]$ -module $\pi_{2p-1} \mathrm{NA}_{\pm}^{\mathrm{fd}}(*)_p^{\wedge}$ is not finitely generated.*

Combining the Frobenius and Verschiebung operations we can define also a structure of a $\mathbb{Z}_p[\mathbb{N}_{\times} * \mathbb{N}_{\times}]$ -module on the homotopy groups of $\mathrm{NA}_{\pm}^{\mathrm{fd}}(X)_p^{\wedge}$, where $\mathbb{N}_{\times} * \mathbb{N}_{\times}$

denotes the free product of monoids. In Corollary 6.5 we show that $\pi_{2p-1} \text{NA}_{\pm}^{\text{fd}}(*)^{\wedge}_p$ is also not finitely-generated as a $\mathbb{Z}_p[\mathbb{N}_{\times} * \mathbb{N}_{\times}]$ -module.

For a geometric application of our results recall that algebraic K -theory of spaces is related via the assembly maps to automorphisms of manifolds. For a general scheme we refer the reader to the survey article [WW01]. This general scheme can be applied for example in the case of smooth manifolds with negative sectional curvature. For such a manifold M of dimension n the work of Farrell and Jones [FJ91] gives the following results about the space $\text{TOP}(M)$ of self-homeomorphisms of M :

$$\pi_j \text{TOP}(M) \cong \bigoplus_T \pi_{j+2} \text{NA}_{+}^{\text{fd}}(*)$$

for $1 < j \leq \phi(n)$, where T runs through simple closed geodesics in M and $\phi(n)$ denotes the concordance stable range which is approximately $n/3$, and there is an exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_T \pi_3 \text{NA}_{+}^{\text{fd}}(*) \rightarrow \pi_1 \text{TOP}(M) \rightarrow Z(\pi_1 M) \rightarrow \\ \rightarrow \bigoplus_T \pi_2 \text{NA}_{+}^{\text{fd}}(*) \rightarrow \pi_0 \text{TOP}(M) \rightarrow \text{Out}(\pi_1 M). \end{aligned}$$

So our results describe new non-trivial families of automorphisms of negatively curved smooth manifolds.

Organization. Section 2 contains the background about algebraic K -theory and nil-terms. In section 3 we define the Frobenius and Verschiebung operations in three different ways. In section 4 we prove that the three definitions coincide. In section 5 we prove the certain identities satisfied by these operations and derive some consequences out of them. Section 6 contains the proof of Theorem 1.2.

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2. PRELIMINARIES

Waldhausen defined algebraic K -theory for categories with cofibrations and weak equivalences [Wal85]. For a topological space X its algebraic K -theory $A(X)$ can be defined as the algebraic K -theory of the category of based spaces with an action of the geometric realization of the Kan loop group of X or using equivariant stable homotopy theory, in terms of certain spaces of matrices. The modern approach to the latter alternative uses the language of \mathbb{S} -algebras and is presented in [DGM04]. We briefly review the definitions in both cases. Further we recall two definitions of the nil-terms which are the principal objects of study in this paper.

2.1. K-theory. We follow the notation of [HKV⁺01], [HKV⁺02], [KW95]. Recall that a *Waldhausen category* is a category with cofibrations and weak equivalences satisfying axioms of [Wal85, 1.2]. If \mathcal{C} is such a category there is defined a connected based space $|w\mathcal{S}_{\bullet}\mathcal{C}|$ (in fact an infinite loop space), the \mathcal{S}_{\bullet} -construction of \mathcal{C} whose loop space is taken to be the definition of the algebraic K -theory of \mathcal{C} .

We recall the examples of Waldhausen categories relevant for us. Let M_{\bullet} be a simplicial monoid whose realization is denoted by M . Let $\mathbb{T}(M)$ be the category whose objects are based M -spaces, i.e. based spaces Y equipped with a based left action $M \times Y \rightarrow Y$. Morphisms of $\mathbb{T}(M)$ are based equivariant maps. The cell of dimension n is defined to be

$$D^n \times M.$$

If Z is an object in $\mathbb{T}(M)$, and $\alpha: S^{n-1} \times M \rightarrow Z$ is an equivariant map we can form the pushout

$$Z \cup_{\alpha} (D^n \times M).$$

This operation is called the effect of *attaching a cell*. A morphism $Y \rightarrow Z$ of $\mathbb{T}(M)$ is a *cofibration* if either Z is obtained from Y by a sequence of cell attachments or it is a retract of the foregoing. An object is said to be *cofibrant* if the inclusion of the base point is a cofibration.

Let $\mathbb{C}(M) \subset \mathbb{T}(M)$ be the full subcategory of cofibrant objects. An object of $\mathbb{C}(M)$ is called *finite* if it is isomorphic to a finite free based M -CW-complex, i.e. it is built from a point by a finite number of cell attachments. An object Y is called *homotopy finite* if there is a finite object X and a morphism $Y \rightarrow X$ which considered as a map of ordinary spaces is a weak homotopy equivalence. An object is called *finitely dominated* if it is a retract of a homotopy finite object. The full subcategory of finite (finitely dominated) objects, is denoted $\mathbb{C}_f(M)$ ($\mathbb{C}_{fd}(M)$). Both categories $\mathbb{C}_f(M)$ and $\mathbb{C}_{fd}(M)$ have the structure of a Waldhausen category by defining a morphism to be a cofibration if it is a cofibration in $\mathbb{T}(M)$ and a weak homotopy equivalence if it is a weak homotopy equivalence of the underlying spaces.

Notation 2.1. Denote

$$A^{fd}(*, M) := \Omega |h\mathcal{S}_{\bullet}\mathbb{C}_{fd}(M)|,$$

$$A^f(*, M) := \Omega |h\mathcal{S}_{\bullet}\mathbb{C}_f(M)|.$$

Remark 2.2. Recall that if X is a connected and based topological space and M is the geometric realization of its Kan loop group, then $A^f(*, M)$ is one of the definitions of $A(X)$ given in [Wal85]. In analogy we regard $A^{fd}(*, M)$ as the definition of $A^{fd}(X)$.

Remark 2.3. The relationship between the two versions is

$$A^{fd}(*, M) \simeq \tilde{K}_0(\mathbb{Z}[\pi_0 M]) \times A^f(*, M),$$

where $\tilde{K}_0(\mathbb{Z}[\pi_0 M])$ is the reduced class group of the monoid ring $\mathbb{Z}[\pi_0 M]$, [HKV⁺01, Lemma 1.7.(3)]. It is necessary to use the finitely dominated version in order for the fundamental theorem to hold.

For the second definition of $A^{fd}(X)$ we need the language of \mathbb{S} -algebras, which we briefly recall following [DGM04]. By Γ^{op} is denoted the category of finite pointed sets and pointed maps, more precisely the skeletal subcategory consisting of the sets $k_+ = \{0, \dots, k\}$, where 0 is the base point. By \mathcal{S}_* is denoted the category of pointed simplicial sets. A Γ -space \mathbb{B} is a functor $\mathbb{B}: \Gamma^{op} \rightarrow \mathcal{S}_*$. The category of Γ -spaces is denoted $\Gamma\mathcal{S}_*$, the morphisms are natural transformations of functors from Γ^{op} to \mathcal{S}_* . Γ -spaces give rise to spectra, cf. [DGM04, II.2.1.12]. There are several relevant examples for us. The *sphere spectrum* is $\mathbb{S}: k_+ \mapsto k_+$, where k_+ in the target is a constant pointed simplicial set. For any $X \in \mathcal{S}_*$ the *suspension spectrum* is given by $\Sigma^\infty X: k_+ \mapsto X \wedge k_+$ and for an abelian group G the *Eilenberg-MacLane spectrum* $\mathbb{H}G$ is given by $k_+ \mapsto G^{\times k}$, cf. [DGM04, II.1.2]. The homotopy groups of a Γ -space are defined as the homotopy groups of the corresponding spectrum.

There is a notion of a smash product of Γ -spaces $\mathbb{B} \wedge \mathbb{C}$, cf [DGM04, II.1.2.3], [Lyd99], and a unit of the smash product, which is the already mentioned sphere spectrum \mathbb{S} . The triple $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$ turns out to be a symmetric monoidal category. An \mathbb{S} -algebra \mathbb{A} is a monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbb{S})$, i.e. a Γ -space with a multiplication map $\mu: \mathbb{A} \wedge \mathbb{A} \rightarrow \mathbb{A}$ and a unit map $1: \mathbb{S} \rightarrow \mathbb{A}$ satisfying certain relations, cf [DGM04, II.1.4.1]. \mathbb{S} -algebras give rise to ring spectra with a well-behaved smash product.

The relevant examples here start again with the sphere spectrum \mathbb{S} with the multiplication map given by the action of \mathbb{S} on itself. For a simplicial monoid M there is the *monoid \mathbb{S} -algebra* $\mathbb{S}[M] = \Sigma^\infty(M_+): k_+ \rightarrow M_+ \wedge k_+$ with the multiplication map induced by the multiplication in \mathbb{S} and the monoid operation of M . For a ring R there is the Eilenberg-MacLane \mathbb{S} -algebra $\mathbb{H}R$, with the multiplication given by the multiplication in R . The homotopy groups of an \mathbb{S} -algebra form a simplicial ring.

For an \mathbb{S} -algebra \mathbb{A} the \mathbb{S} -algebra $M_m(\mathbb{A})$ of $m \times m$ matrices is

$$M_m(\mathbb{A}): k_+ \mapsto \text{Map}(m_+, m_+ \wedge \mathbb{A}(k_+)) = \prod_0^{m-1} \bigvee_0^{m-1} \mathbb{A}(k_+),$$

with the multiplication $M_m(\mathbb{A}) \wedge M_m(\mathbb{A}) \rightarrow M_m(\mathbb{A})$ given in [DGM04, Example 1.4.4.(6)]. The “points” in $M_m(\mathbb{A})(k_+)$ are matrices with entries in $\mathbb{A}(k_+)$ with at most one non-zero entry in every column. By $\widehat{M}_m(\mathbb{A})$ is denoted the simplicial ring $\Omega^\infty M_m(\mathbb{A})$ whose group of components is $\pi_0 \widehat{M}_m(\mathbb{A}) = M_m(\pi_0(\mathbb{A}))$. The simplicial ring of “invertible matrices” $\widehat{GL}_m(\mathbb{A})$ is defined as the pullback

$$\begin{array}{ccc} \widehat{GL}_m(\mathbb{A}) & \longrightarrow & \widehat{M}_m(\mathbb{A}) \\ \downarrow & & \downarrow \\ GL_m(\pi_0(\mathbb{A})) & \longrightarrow & M_m(\pi_0(\mathbb{A})). \end{array}$$

Definition 2.4. [DGM04, III.2.3.2] The K -theory of an \mathbb{S} -algebra \mathbb{A} is defined as

$$K(\mathbb{A}) = K_0(\mathbb{Z}[\pi_0 \mathbb{A}]) \times \text{colim}_{m \rightarrow \infty} B\widehat{GL}_m(\mathbb{A})^+,$$

where the superscript $+$ denotes the Quillen $+$ -construction.

It can be shown that $K(\mathbb{A})$ is an infinite loop space, so we think of $\mathbb{A} \mapsto K(\mathbb{A})$ as of a functor from \mathbb{S} -algebras to spectra.

Remark 2.5. The examples of the \mathbb{S} -algebras mentioned above yield for a based space X with the Kan loop group ΩX and for a ring R

$$\begin{aligned} K(\mathbb{S}[\Omega X]) &\simeq A^{\text{fd}}(X) = A^{\text{fd}}(*, \Omega X), \\ K(\mathbb{H}R) &\simeq K(R), \end{aligned}$$

where $K(R)$ denotes the Quillen K -theory of a ring R (more precisely its (-1)-connective version), cf [DGM04, Chapter III].

Finally we recall the *linearization map*. This is for any space X with the Kan loop group ΩX the map of \mathbb{S} -algebras $l: \mathbb{S}[\Omega X] \rightarrow \mathbb{H}\mathbb{Z}[\pi_1 X]$ given by sending a loop in ΩX to its equivalence class in $\pi_1 X$. This map induces the map $l: A^{\text{fd}}(X) \rightarrow K(\mathbb{Z}[\pi_1 X])$ which we also call the linearization map.

2.2. Nil-terms. We closely follow [HKV⁺01]. For the definition of the nil-terms we need to recall the definition of the mapping telescope from [HKV⁺01, page 27]. The symbols \mathbb{N}_+ , \mathbb{N}_- or \mathbb{Z} will always denote monoids with the addition and with the generators t , t^{-1} , or t . For an object $Y_+ \in \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_+)$ the object $Y_+(t^{-1}) \in \mathbb{C}_{\text{fd}}(M \times \mathbb{Z})$ is defined to be the categorical colimit of the sequence

$$\cdots \xrightarrow{t} Y_+ \xrightarrow{t} Y_+ \xrightarrow{t} \cdots.$$

Explicitly it is the quotient space of $Y_+ \times \mathbb{Z}$ in which a pair (y, n) is identified with the pair $(t \cdot y, n + 1)$. The action of t^k for $k \in \mathbb{Z}$ on a pair (y, n) yields the pair $(y, n - k)$. This assignment defines an exact functor [HKV⁺01, page 26].

Similarly, for $Y_- \in \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_-)$, we have the mapping telescope $Y_-(t) \in \mathbb{C}_{\text{fd}}(M \times \mathbb{Z})$ given by the colimit of the sequence

$$\cdots \xrightarrow{t^{-1}} Y_- \xrightarrow{t^{-1}} Y_- \xrightarrow{t^{-1}} \cdots .$$

As above we use the quotient of $Y_- \times \mathbb{Z}$ in which a pair (y, n) is identified with the pair $(t^{-1} \cdot y, n + 1)$. The action of t^k for $k \in \mathbb{Z}$ on a pair (y, n) yields the pair $(y, n - k)$.

Recall from [HKV⁺01, page 36] that $\mathbb{D}_{\text{fd}}(M \times L)$ is defined to be the category whose objects are diagrams

$$Y_- \longrightarrow Y \longleftarrow Y_+$$

with $Y_- \in \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_-)$, $Y \in \mathbb{C}_{\text{fd}}(M \times \mathbb{Z})$ and $Y_+ \in \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_+)$, and where the maps $Y_- \rightarrow Y$ and $Y_+ \rightarrow Y$ are assumed to be cofibrations. We take the liberty of specifying an object as a diagram or as a triple (Y_-, Y, Y_+) .

A morphism $(Y_-, Y, Y_+) \rightarrow (Z_-, Z, Z_+)$ of $\mathbb{D}_{\text{fd}}(M \times \mathbb{Z})$ is a morphism $Y_- \rightarrow Z_-$, a morphism $Y \rightarrow Z$ and a morphism $Y_+ \rightarrow Z_+$ so that the evident diagram commutes. A cofibration in $\mathbb{D}_{\text{fd}}(M \times \mathbb{Z})$ is a morphism such that each of the maps $Y_- \rightarrow Z_-$, $Y_+ \rightarrow Z_+$ and $Y \rightarrow Z$ is a cofibration and the induced maps

$$\begin{aligned} Y \cup_{Y_-(t)} Z_-(t) &\rightarrow Z \\ Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) &\rightarrow Z \end{aligned}$$

are cofibrations.

Further $\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_{\pm}) \subset \mathbb{D}_{\text{fd}}(M \times \mathbb{Z})$ is defined to be the full subcategory consisting of objects with the additional property that the induced map

$$Y_{\mp}(t) \rightarrow Y$$

is a weak equivalence.

For $L = \mathbb{N}_+$, \mathbb{N}_- or \mathbb{Z} recall from [HKV⁺01, page 48] an augmentation functor

$$\begin{aligned} \epsilon: \mathbb{D}_{\text{fd}}(M \times L) &\rightarrow \mathbb{C}_{\text{fd}}(M) \\ (Y_-, Y, Y_+) &\mapsto Y/\mathbb{Z}. \end{aligned}$$

The homotopy fiber of the induced map

$$\epsilon: |h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times L)| \rightarrow |h\mathcal{S}_{\bullet}\mathbb{C}_{\text{fd}}(M)|$$

is denoted by $|h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times L)|^{\epsilon}$. The canonical map

$$|h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times L)|^{\epsilon} \rightarrow |h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times L)|$$

admits a left homotopy inverse [HKV⁺01, page 48] and there is a homotopy equivalence

$$|h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times L)| \simeq |h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times L)|^{\epsilon} \times |h\mathcal{S}_{\bullet}\mathbb{C}_{\text{fd}}(M)|.$$

We have

Definition 2.6. [HKV⁺01, page 51]

$$\begin{aligned} \text{NA}_{+}^{\text{fd}}(*, M) &:= \Omega |h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)|^{\epsilon} \\ \text{NA}_{-}^{\text{fd}}(*, M) &:= \Omega |h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_-)|^{\epsilon}. \end{aligned}$$

Similarly as in the linear case the nil-terms can be identified with a subgroup of the K -theory of a certain nil-category. Here we follow [KW95]. As before let M be the geometric realization of a simplicial monoid M_{\bullet} . The nil-category $\text{NIL}_{\text{fd}}(*, M)$ has as its objects pairs (Y, f) where Y is an object in $\mathbb{C}_{\text{fd}}(M)$ and f is an M -map from Y to Y which is homotopically nilpotent, i.e. there exist a non negative integer k such that $f^{\circ k}$ is equivariantly null homotopic. A morphism $(Y, f) \rightarrow (Y', f')$ is a map $e: Y \rightarrow Y'$ in $\mathbb{C}_{\text{fd}}(M)$ such that $f' \circ e = e \circ f$. A *cofibration* in $\text{NIL}_{\text{fd}}(*, M)$ is

a morphism whose underlying map in $\mathbb{C}_{\text{fd}}(M)$ is a cofibration. A *weak homotopy equivalence* is a morphism whose underlying map in $\mathbb{C}_{\text{fd}}(M)$ is a weak equivalence. This gives $\text{NIL}_{\text{fd}}(*, M)$ the structure of a Waldhausen category [KW95, Lemma 1.2] and therefore it has a K -theory $K(\text{NIL}_{\text{fd}}(*, M))$. The forgetful functor

$$\begin{aligned} \text{NIL}_{\text{fd}}(*, M) &\rightarrow \mathbb{C}_{\text{fd}}(M) \\ (Y, f) &\mapsto Y \end{aligned}$$

induces a map in K -theory whose homotopy fiber is denoted $\text{Nil}^{\text{fd}}(*, M)$. The main theorem of [KW95] says

Theorem 2.7 ([KW95]). *There are natural homotopy equivalences*

$$\delta_{\pm} : \text{Nil}^{\text{fd}}(*, M) \xrightarrow{\simeq} \Omega \text{NA}_{\pm}^{\text{fd}}(*, M).$$

Remark 2.8. Another possibility is to define the nil-terms in the language of \mathbb{S} -algebras as the fibers of the map in K -theory induced by the augmentation map of \mathbb{S} -algebras $\mathbb{S}[M \times L] \rightarrow \mathbb{S}[M]$ where $L = \mathbb{N}_+$ or \mathbb{N}_- . We will not use this definition in this paper, so we omit the details.

3. FROBENIUS AND VERSCHIEBUNG OPERATIONS

In this section we define for a natural number $n \in \mathbb{N}_{\times}$ the operations Frobenius F_n and Verschiebung V_n in the algebraic K -theory of spaces using all three definitions from the previous section. We also show that they restrict to operations on the nil-terms.

Throughout the symbol L will denote either \mathbb{N}_+ , \mathbb{N}_- or \mathbb{Z} as an additive monoid with the generator t , t^{-1} or t . For $n \in \mathbb{N}_{\times}$ we will use the monoid homomorphism

$$\begin{aligned} \varphi_n : L &\rightarrow L \\ l &\mapsto n \cdot l. \end{aligned}$$

Informally, in all three cases, the Verschiebung operation V_n will be induced by φ_n and the Frobenius operation F_n will be the corresponding transfer.

3.1. Operations on $A^{\text{fd}}(*, M \times L)$. The definition is on the categorical level and goes in three steps. First we define the operations on $\mathbb{C}_{\text{fd}}(M \times L)$, then on $\mathbb{D}_{\text{fd}}(M \times L)$ and then we show that the operations restrict to the nil-terms.

Definition 3.1 (Verschiebung). Define an exact functor

$$\begin{aligned} V_n : \mathbb{C}_{\text{fd}}(M \times L) &\rightarrow \mathbb{C}_{\text{fd}}(M \times L), \\ Y &\mapsto Y \otimes_{\varphi_n} L, \end{aligned}$$

where $Y \otimes_{\varphi_n} L = Y \times L / \sim$ such that $(*, k) \sim (*, l)$ and $(y, \varphi_n(k) + l) \sim (t^k \cdot y, l)$ for all k, l . The L -action on $Y \otimes_{\varphi_n} L$ is given by $(t^k, (y, l)) \mapsto (y, k + l)$. The induced endomorphisms of $A^{\text{fd}}(*, M \times L)$ are also denoted by V_n and are called the *Verschiebung operations*.

Definition 3.2 (Frobenius). Define an exact functor

$$\begin{aligned} F_n : \mathbb{C}_{\text{fd}}(M \times L) &\rightarrow \mathbb{C}_{\text{fd}}(M \times L) \\ Y &\mapsto Y_{\varphi_n} \end{aligned}$$

where Y_{φ_n} is the same space Y but with the L -action precomposed with φ_n , i.e. $(t^k, y) \mapsto t^{\varphi_n(k)} \cdot y = t^{nk} \cdot y$. The induced endomorphisms of $A^{\text{fd}}(*, M \times L)$ are also denoted by F_n and are called the *Frobenius operations*.

To obtain the operations on the nil-terms we need to pass to the operations on $\mathbb{D}_{\text{fd}}(M \times L)$.

Definition 3.3 (Verschiebung on $\mathbb{D}_{\text{fd}}(M \times L)$). Define an exact functor

$$\begin{aligned} V_n: \mathbb{D}_{\text{fd}}(M \times L) &\rightarrow \mathbb{D}_{\text{fd}}(M \times L) \\ (Y_-, Y, Y_+) &\mapsto (Y_- \otimes_{\varphi_n} \mathbb{N}_-, Y \otimes_{\varphi_n} \mathbb{Z}, Y_+ \otimes_{\varphi_n} \mathbb{N}). \end{aligned}$$

Definition 3.4 (Frobenius on $\mathbb{D}_{\text{fd}}(M \times L)$). Define an exact functor

$$\begin{aligned} F_n: \mathbb{D}_{\text{fd}}(M \times L) &\rightarrow \mathbb{D}_{\text{fd}}(M \times L) \\ (Y_-, Y, Y_+) &\mapsto (Y_- \varphi_n, Y \varphi_n, Y_+ \varphi_n). \end{aligned}$$

Recall the following [HKV⁺01, Lemma 4.5]

Lemma 3.5. *The forgetful functors*

$$\begin{aligned} \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+) &\rightarrow \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_+) \\ \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_-) &\rightarrow \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_-) \end{aligned}$$

induce homotopy equivalences $|h\mathcal{S}_\bullet \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_\pm)| \rightarrow |h\mathcal{S}_\bullet \mathbb{C}_{\text{fd}}(M \times \mathbb{N}_\pm)|$.

It is obvious that the map of Lemma 3.5 identifies the operations defined on $\mathbb{D}_{\text{fd}}(M \times L)$ with the operations given on $\mathbb{C}_{\text{fd}}(M \times L)$.

Proposition 3.6. *The Frobenius and Verschiebung operations restrict to operations on $\text{NA}_+^{\text{fd}}(*, M)$ and $\text{NA}_-^{\text{fd}}(*, M)$.*

Proof. We start with the Verschiebung operations. Recall that the augmentation map $\epsilon: \Omega|h\mathcal{S}_\bullet \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \rightarrow \Omega|h\mathcal{S}_\bullet \mathbb{C}_{\text{fd}}(M)|$ is induced by the functor which maps an object (Y_-, Y, Y_+) to Y/\mathbb{Z} . This implies that the following diagram commutes:

$$\begin{array}{ccc} \Omega|h\mathcal{S}_\bullet \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \\ \epsilon \downarrow & & \downarrow \epsilon \\ \Omega|h\mathcal{S}_\bullet \mathbb{C}_{\text{fd}}(M)| & \xrightarrow{\text{id}} & \Omega|h\mathcal{S}_\bullet \mathbb{C}_{\text{fd}}(M)|. \end{array}$$

Since $\text{NA}_+^{\text{fd}}(*, M)$ is the homotopy fiber of ϵ we obtain that the Verschiebung operation restricts to an operation on $\text{NA}_+^{\text{fd}}(*, M)$.

We continue with the Frobenius operations. Define an exact functor $I: \mathbb{C}_{\text{fd}}(M) \rightarrow \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)$ by mapping Y to $(Y \otimes \mathbb{N}_-, Y \otimes \mathbb{Z}, Y \otimes \mathbb{N}_+)$ where $Y \otimes L$ is defined to be $Y \times L / \sim$ where $(*, k) \sim (*, l)$ for the base point $*$ and all k, l . Let ι denote the map induced by I and notice that $\epsilon \circ \iota = \text{id}$. This implies that $\text{NA}_+^{\text{fd}}(*, M)$ is the homotopy cofiber of ι . We also define a functor $\vee^n: \mathbb{C}_{\text{fd}}(M) \rightarrow \mathbb{C}_{\text{fd}}(M)$ which maps Y to $\vee^n Y$. The Frobenius operation restricts to an operation on $\text{NA}_+^{\text{fd}}(*, M)$ since the following diagram commutes:

$$\begin{array}{ccc} \Omega|h\mathcal{S}_\bullet \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| & \xrightarrow{F_n} & \Omega|h\mathcal{S}_\bullet \mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \\ \iota \uparrow & & \uparrow \iota \\ \Omega|h\mathcal{S}_\bullet \mathbb{C}_{\text{fd}}(M)| & \xrightarrow{\vee^n} & \Omega|h\mathcal{S}_\bullet \mathbb{C}_{\text{fd}}(M)|. \end{array}$$

The proof that the operations restrict also to $\text{NA}_-^{\text{fd}}(*, M)$ is analogous. \square

3.2. Operations on the NIL-Category. In this section we define Verschiebung and Frobenius on the nil-category. This definition has the advantage that it is in general easier to prove identities satisfied by the operations in this setting.

Definition 3.7 (Verschiebung). Define an exact functor

$$V'_n: \text{NIL}_{\text{fd}}(*, M) \rightarrow \text{NIL}_{\text{fd}}(*, M)$$

$$(Y, f) \mapsto (\vee^n Y, \begin{pmatrix} 0 & & & f \\ \text{id} & \ddots & & \\ & \ddots & 0 & \\ & & & \text{id} & 0 \end{pmatrix}),$$

where the matrix describes the map which sends the n -tuple (y_0, \dots, y_{n-1}) to $(f(y_{n-1}), y_0, \dots, y_{n-2})$. The induced endomorphisms of $K(\text{NIL}_{\text{fd}}(*, M))$ and on $\text{Nil}^{\text{fd}}(*, M)$ are also denoted by V'_n .

Definition 3.8 (Frobenius). Define an exact functor

$$F'_n: \text{NIL}_{\text{fd}}(*, M) \rightarrow \text{NIL}_{\text{fd}}(*, M)$$

$$(Y, f) \mapsto (Y, f^{\circ n}).$$

The induced endomorphisms of $K(\text{NIL}_{\text{fd}}(*, M))$ and on $\text{Nil}^{\text{fd}}(*, M)$ are also denoted by F'_n .

3.3. Operations on $K(\mathbb{S}[M \times L])$. The definition of the Verschiebung operations on the \mathbb{S} -algebra definition of the algebraic K -theory of spaces is needed for the identification of the $\mathbb{Z}[\mathbb{N}_\times]$ -module structure on the homotopy groups of the nil-terms in section 6. The definition of the Frobenius operations in this setup is not used in the rest of the paper, but we include it for completeness.

Recall that we have the monoid \mathbb{S} -algebra

$$\mathbb{S}[M \times L]: k_+ \mapsto (M \times L)_+ \wedge k_+ = M_+ \wedge L_+ \wedge k_+,$$

with the multiplication map defined via the monoid operation on $M \times L$.

Definition 3.9 (Verschiebung). Define an \mathbb{S} -algebra map

$$V''_n: \mathbb{S}[M \times L] \rightarrow \mathbb{S}[M \times L],$$

$$(m, l, i) \mapsto (m, \varphi_n(l), i),$$

where $m \in M$, $l \in L$ and $i \in k_+$. The induced map on $K(\mathbb{S}[M \times L])$ is also denoted by V''_n and is called the *Verschiebung* operation.

The Frobenius operation is the corresponding transfer. Informally, the map φ_n makes the \mathbb{S} -algebra $\mathbb{S}[M \times L]$ into a free $\mathbb{S}[M \times L]$ -module on n generators. Using this any $m \times m$ matrix over $\mathbb{S}[M \times L]$ defines also an $mn \times mn$ matrix over $\mathbb{S}[M \times L]$. This assignment induces the Frobenius operation. Recall that an \mathbb{A} -module \mathbb{B} is a Γ -space with an action map $\alpha: \mathbb{A} \wedge \mathbb{B} \rightarrow \mathbb{B}$ satisfying certain relations, cf [DGM04, 1.5.1]. Since the role of the monoid M in the sequel is not very interesting we omit it from the notation and just talk about the \mathbb{S} -algebra $\mathbb{S}[L]$.

We denote by $\mathbb{S}[L]_{\varphi_n}$ the $\mathbb{S}[L]$ with the new $\mathbb{S}[L]$ -module structure given by φ_n , i.e. with the action map

$$\mathbb{S}[L] \wedge \mathbb{S}[L]_{\varphi_n} \rightarrow \mathbb{S}[L]_{\varphi_n}$$

$$L_+ \wedge k_+ \wedge L_+ \wedge l_+ \rightarrow L_+ \wedge k l_+$$

$$(z, i, u, j) \mapsto (zn + u, ij).$$

With this $\mathbb{S}[L]$ -module structure we have an isomorphism of $\mathbb{S}[L]$ -modules

$$n_+ \wedge \mathbb{S}[L] \rightarrow \mathbb{S}[L]_{\varphi_n}$$

$$n_+ \wedge L_+ \wedge k_+ \rightarrow L_+ \wedge k_+$$

$$(j, z, i) \rightarrow (j + zn, i),$$

where we call $n_+ \wedge \mathbb{S}[L]$ a free $\mathbb{S}[L]$ -module on n generators.

Let \mathbb{A} be an \mathbb{S} -algebra and let \mathbb{B}, \mathbb{C} be \mathbb{A} -modules. There is a Γ -space of \mathbb{A} -module maps from \mathbb{B} to \mathbb{C} defined as in [DGM04, Definition 1.5.4]. Its k -th space can be thought of as the space of natural transformations

$$\lambda \in \Gamma\mathcal{S}_*(\mathbb{B}, \mathbb{C})(k_+) = \text{Nat}_{\Gamma\mathcal{S}_*}(\mathbb{B}(-), \mathbb{C}(k_+ \wedge -)).$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{A} \wedge \mathbb{B}(-) & \longrightarrow & \mathbb{B}(-) \\ \text{id} \wedge \lambda \downarrow & & \downarrow \lambda \\ \mathbb{A} \wedge \mathbb{C}(k_+ \wedge -) & \longrightarrow & \mathbb{C}(k_+ \wedge -). \end{array}$$

In analogy with linear algebra there is an isomorphism $\mathbb{A} \cong \text{Map}_{\mathbb{S}}(\mathbb{S}, \mathbb{A})$ of \mathbb{S} -algebras, since an \mathbb{S} -algebra map $\mathbb{S} \rightarrow \mathbb{A}$ is determined by what it does on $\mathbb{S}(1_+) = 1_+$. It follows that there are \mathbb{S} -algebra isomorphisms

$$M_m(\mathbb{A}) \cong \text{Map}_{\mathbb{S}}(m_+ \wedge \mathbb{S}, m_+ \wedge \mathbb{A}) \cong \text{Map}_{\mathbb{A}}(m_+ \wedge \mathbb{A}, m_+ \wedge \mathbb{A}).$$

The second isomorphism is given using the isomorphism $\mathbb{S} \wedge \mathbb{A} \rightarrow \mathbb{A}$ and the multiplication $\mathbb{A} \wedge \mathbb{A} \rightarrow \mathbb{A}$ as the composition

$$\text{Map}_{\mathbb{S}}(m_+ \wedge \mathbb{S}, m_+ \wedge \mathbb{A}) \rightarrow \text{Map}_{\mathbb{A}}(m_+ \wedge \mathbb{S} \wedge \mathbb{A}, m_+ \wedge \mathbb{A} \wedge \mathbb{A}) \rightarrow \text{Map}_{\mathbb{A}}(m_+ \wedge \mathbb{A}, m_+ \wedge \mathbb{A}).$$

A natural transformation $\mathbb{S}[L](-) \rightarrow \mathbb{S}[L](k_+ \wedge -)$ which represents a point in $\text{Map}_{\mathbb{S}[L]}(\mathbb{S}[L], \mathbb{S}[L])(k_+)$ is determined by what it does on $- = 1_+$ where it is of the form

$$z \mapsto (z + a, i)$$

for some $a \in L$ and $i \in k_+$. It follows that it also represents a point in the space $\text{Map}_{\mathbb{S}[L]}(\mathbb{S}[L]_{\varphi_n}, \mathbb{S}[L]_{\varphi_n})(k_+)$. More generally we have an \mathbb{S} -algebra map

$$\varphi_n^* : \text{Map}_{\mathbb{S}[L]}(m_+ \wedge \mathbb{S}[L], m_+ \wedge \mathbb{S}[L]) \rightarrow \text{Map}_{\mathbb{S}[L]}(m_+ \wedge \mathbb{S}[L]_{\varphi_n}, m_+ \wedge \mathbb{S}[L]_{\varphi_n}).$$

Further we obtain an \mathbb{S} -algebra map

$$\begin{aligned} \varphi_n^* : M_m(\mathbb{S}[L]) &\rightarrow \text{Map}_{\mathbb{S}[L]}(m_+ \wedge \mathbb{S}[L], m_+ \wedge \mathbb{S}[L]) \xrightarrow{\varphi_n^*} \\ &\rightarrow \text{Map}_{\mathbb{S}[L]}(m_+ \wedge \mathbb{S}[L]_{\varphi_n}, m_+ \wedge \mathbb{S}[L]_{\varphi_n}) \rightarrow \\ &\rightarrow \text{Map}_{\mathbb{S}[L]}(mn_+ \wedge \mathbb{S}[L], mn_+ \wedge \mathbb{S}[L]) \rightarrow M_{mn}(\mathbb{S}[L]). \end{aligned}$$

This induces the maps of simplicial monoids $\widehat{M}_m(\mathbb{S}[L]) \rightarrow \widehat{M}_{mn}(\mathbb{S}[L])$ and

$$(3.1) \quad \varphi_n^* : \widehat{\text{GL}}_m(\mathbb{S}[L]) \rightarrow \widehat{\text{GL}}_{mn}(\mathbb{S}[L]).$$

Definition 3.10 (Frobenius). The map induced by the maps φ_n^* from (3.1) is denoted $F_n'' : K(\mathbb{S}[L]) \rightarrow K(\mathbb{S}[L])$ and called the *Frobenius* operation.

The formula for the map $\varphi_n^* : M_m(\mathbb{S}[L]) \rightarrow M_{mn}(\mathbb{S}[L])$ in case $m = 1$ is

$$(a) \quad \longmapsto \begin{pmatrix} & & & & b \\ & & & & \\ & & & \ddots & \\ & & & & b \\ b & & & & \\ & \ddots & & & \\ & & b & & \end{pmatrix}$$

where $a = bn + c$, $0 \leq c < n$ and the first non-zero entry in the first column is in the row $c + 1$. For $m > 1$ replace any non-zero entry a by the matrix as above.

Remark 3.11. In view of the definition of $K(\mathbb{A})$ we have defined the operations V_n'', F_n'' only on the base point component. The extension to the whole $K(\mathbb{A})$ can be done by artificially on $K_0(\mathbb{Z}[\pi_0\mathbb{A}])$.

4. IDENTIFYING VERSCHIEBUNG AND FROBENIUS

In the previous section we defined the Frobenius and Verschiebung operations using various definitions of the nil-terms. In this section we prove that these definitions in fact coincide. Recall from Theorem 2.7 that for a topological monoid M there are natural homotopy equivalences $\delta_{\pm}: \text{Nil}^{\text{fd}}(*, M) \rightarrow \Omega \text{NA}_{\pm}^{\text{fd}}(*, M)$.

Theorem 4.1. *The following diagrams commute up to a preferred homotopy*

$$\begin{array}{ccc} \text{Nil}^{\text{fd}}(*, M) & \xrightarrow{V_n'} & \text{Nil}^{\text{fd}}(*, M) & \text{Nil}^{\text{fd}}(*, M) & \xrightarrow{F_n'} & \text{Nil}^{\text{fd}}(*, M) \\ \delta_{\pm} \downarrow & & \downarrow \delta_{\pm} & \delta_{\pm} \downarrow & & \downarrow \delta_{\pm} \\ \Omega \text{NA}_{\pm}^{\text{fd}}(*, M) & \xrightarrow{V_n} & \Omega \text{NA}_{\pm}^{\text{fd}}(*, M) & \Omega \text{NA}_{\pm}^{\text{fd}}(*, M) & \xrightarrow{F_n} & \Omega \text{NA}_{\pm}^{\text{fd}}(*, M), \end{array}$$

for all $n \in \mathbb{N}_{\times}$.

For the proof we need the notion of the projective line category [KW95, page 4].

Definition 4.2 (Projective line category). The *projective line category* $\mathbb{P}_{\text{fd}}(M)$ is given by the full subcategory of $\mathbb{D}_{\text{fd}}(M \times \mathbb{Z})$ whose objects (Y_-, Y, Y_+) satisfy the additional property that the induced maps $Y_-(t) \rightarrow Y$ and $Y_+(t) \rightarrow Y$ are weak homotopy equivalences. A *cofibration* in $\mathbb{P}_{\text{fd}}(M)$ is a morphism which is a cofibration in $\mathbb{D}_{\text{fd}}(M \times \mathbb{Z})$. A *weak homotopy equivalence* from (Y_-, Y, Y_+) to (X_-, X, X_+) is a morphism in which $Y_- \rightarrow X_-$, $Y \rightarrow X$ and $Y_+ \rightarrow X_+$ are weak homotopy equivalences of spaces. Let $\mathbb{P}_{\text{fd}}^{h_{\mathbb{N}^+}}(M)$ be the full Waldhausen subcategory of $\mathbb{P}_{\text{fd}}(M)$ with objects (Y_-, Y, Y_+) such that Y_+ is acyclic.

Note that the operations of section 3.1 restrict to $\mathbb{P}_{\text{fd}}^{h_{\mathbb{N}^+}}(M)$. Further we need a pair of exact functors from [KW95] which relate the projective line category to the nil-category. There is an exact functor

$$\begin{aligned} \Phi: \text{NIL}_{\text{fd}}(*, M) &\rightarrow \mathbb{P}_{\text{fd}}^{h_{\mathbb{N}^+}}(M) \\ (Y, f) &\mapsto (Y_f, Y_f(t), *) \end{aligned}$$

where Y_f is the homotopy coequalizer of the pair of maps

$$Y \otimes \mathbb{N}_- \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[t^{-1}]{} \end{array} Y \otimes \mathbb{N}_- .$$

where t^{-1} is the map $(y, r) \mapsto (y, r - 1)$. An explicit model for the homotopy coequalizer of two maps $\alpha, \beta: U \rightarrow V$ is the quotient of $V \coprod (U \times [0, 1])$ by $(u, 0) \sim \alpha(u)$, $(u, 1) \sim \beta(u)$ and $* \times [0, 1] \sim$ the base point of V . Note that if f is a cofibration then Y_f is homotopy equivalent to Y with the \mathbb{N}_- -action given by $(t^k, y) \mapsto f^{\circ k}(y)$.

There is also an exact functor

$$\begin{aligned} \Psi: \mathbb{P}_{\text{fd}}^{h_{\mathbb{N}^+}}(M) &\rightarrow \text{NIL}_{\text{fd}}(*, M) \\ (Y_-, Y, Y_+) &\mapsto (Y_-, t^{-1}). \end{aligned}$$

Lemma 4.3. *The following diagrams commute up to a preferred homotopy*

$$\begin{array}{ccc}
\Omega|h\mathcal{S}_\bullet \text{NIL}_{\text{fd}}(*, M)| & \xrightarrow{V'_n} & \Omega|h\mathcal{S}_\bullet \text{NIL}_{\text{fd}}(*, M)| \\
\Phi \downarrow & & \downarrow \Phi \\
\Omega|h\mathcal{S}_\bullet \mathbb{P}_{\text{fd}}^{h\mathbb{N}_+}(M)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet \mathbb{P}_{\text{fd}}^{h\mathbb{N}_+}(M)| \\
\\
\Omega|h\mathcal{S}_\bullet \text{NIL}_{\text{fd}}(*, M)| & \xrightarrow{F'_n} & \Omega|h\mathcal{S}_\bullet \text{NIL}_{\text{fd}}(*, M)| \\
\Phi \downarrow & & \downarrow \Phi \\
\Omega|h\mathcal{S}_\bullet \mathbb{P}_{\text{fd}}^{h\mathbb{N}_+}(M)| & \xrightarrow{F_n} & \Omega|h\mathcal{S}_\bullet \mathbb{P}_{\text{fd}}^{h\mathbb{N}_+}(M)|.
\end{array}$$

for all $n \in \mathbb{N}_\times$.

Proof. We will first reduce to the case where f is a cofibration. Consider the functor $\Psi \circ \Phi: \text{NIL}_{\text{fd}}(*, M) \rightarrow \text{NIL}_{\text{fd}}(*, M)$ which maps (Y, f) to (Y_f, t) . By construction we have that t is a cofibration. Since $\Psi \circ \Phi$ induces a homotopy equivalence [KW95, Lemma 3.5] we can precompose our diagrams with $\Psi \circ \Phi$ and therefore assume that all objects in $\text{NIL}_{\text{fd}}(*, M)$ are of the form (Y, f) where f is a cofibration. Thus we can use for Y_f the space Y with the \mathbb{N}_- -action given via f .

To prove the lemma we need to define a preferred natural homotopy between $V_n \circ \Phi$ and $\Phi \circ V_n$ and between $F_n \circ \Phi$ and $\Phi \circ F_n$.

We start with the Verschiebung operations. We have

$$V_n \circ \Phi((Y, f)) = (Y \otimes_{\varphi_n} \mathbb{N}_-, Y(t) \otimes_{\varphi_n} \mathbb{Z}, *)$$

where t^{-1} acts on Y via f and

$$\Phi \circ V_n((Y, f)) = ((\vee^n Y), (\vee^n Y)(t), *)$$

where t^{-1} acts on $(\vee^n Y)$ via

$$\begin{pmatrix} 0 & & & f \\ \text{id} & \ddots & & \\ & \ddots & 0 & \\ & & \text{id} & 0 \end{pmatrix}.$$

Points in $Y(t) \otimes_{\varphi_n} \mathbb{Z}$ are triples (y, z, i) where $y \in Y$, $0 \leq i \leq n-1$ and $z \in \mathbb{Z}$ modulo the equivalence relation

$$\begin{aligned}
(y, z, i) &\sim (f(y), z+1, i) \\
(*, z, i) &\sim (*, z', i').
\end{aligned}$$

The \mathbb{Z} -action is given by $t \cdot (y, z, i) \mapsto (y, z, i+1)$ for $i \leq n-2$ and $t \cdot (y, z, n-1) \mapsto (y, z-1, 1)$.

Points in $(\vee^n Y)(t)$ are triples (y, i, z) where $y \in Y$, $0 \leq i \leq n-1$ and $z \in \mathbb{Z}$ modulo the equivalence relation

$$\begin{aligned}
(y, i, z) &\sim (y, i+1, z+1) \text{ for } i \neq n-1 \\
(y, n-1, z) &\sim (f(y), 0, z+1) \\
(*, i, z) &\sim (*, i', z').
\end{aligned}$$

The \mathbb{Z} -action is given by $t \cdot (y, i, z) \mapsto (y, i, z-1)$.

The map

$$\begin{aligned} Y(t) \otimes_{\varphi_n} \mathbb{Z} &\rightarrow (\vee^n Y)(t) \\ (y, z, i) &\mapsto (y, 0, z \cdot n + i) \end{aligned}$$

and a similar map $Y \otimes_{\varphi_n} \mathbb{N}_- \rightarrow (\vee^n Y)$ induce a natural equivalence of the functors and hence a preferred homotopy between $V_n \circ \Phi$ and $\Phi \circ V_n$.

Now the Frobenius operations. We have

$$F_n \circ \Phi((Y, f)) = (Y_{\varphi_n}, Y(t)_{\varphi_n}, *)$$

where t^{-1} acts on Y via f and

$$\Phi \circ F_n((Y, f)) = (Y, Y(t), *)$$

where now t^{-1} acts on Y via $f^{\circ n}$. But this is just the space Y_{φ_n} , so we can choose the identity on the first coordinate. It remains to find a preferred natural isomorphism $Y_{\varphi_n}(t) \rightarrow Y(t)_{\varphi_n}$.

Points in $Y(t)_{\varphi_n}$ are pairs (y, z) where $y \in Y$ and $z \in \mathbb{Z}$ modulo the equivalence relation

$$(y, z) \sim (f(y), z - 1).$$

The \mathbb{Z} -action is given by $t \cdot (y, z) \mapsto (y, z + n)$.

Points in $Y_{\varphi_n}(t)$ are pairs (y, z) where $y \in Y$ and $z \in \mathbb{Z}$ modulo the equivalence relation

$$(y, z) \sim (f^{\circ n}(y), z - 1).$$

The \mathbb{Z} -action is given by $t \cdot (y, z) \mapsto (y, z + 1)$.

The map

$$\begin{aligned} Y_{\varphi_n}(t) &\rightarrow Y(t)_{\varphi_n} \\ (y, z) &\mapsto (y, n \cdot z) \end{aligned}$$

induces a preferred natural equivalence of functors and hence a preferred natural homotopy between $F_n \circ \Phi$ and $\Phi \circ F_n$. \square

Proof of Theorem 4.1. Let us briefly recall how are the terms NA_+^{fd} and Nil^{fd} identified in [KW95] (analogous proof works for NA_-^{fd}). First it is proven that the inclusion functors induce a homotopy fibration sequence [KW95, Proposition 2.2]:

$$\Omega|h\mathcal{S}_{\bullet}\mathbb{P}_{\text{fd}}^{h\mathbb{N}_+}(M)| \rightarrow \Omega|h\mathcal{S}_{\bullet}\mathbb{P}_{\text{fd}}(M)| \rightarrow \Omega|h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)|.$$

Recall that NA_+^{fd} is defined to be the homotopy fiber of the augmentation map

$$\epsilon: \Omega|h\mathcal{S}_{\bullet}\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \rightarrow \Omega|h\mathcal{S}_{\bullet}\mathbb{C}_{\text{fd}}(M)|.$$

We define $\Omega|h\mathcal{S}_{\bullet}\mathbb{P}_{\text{fd}}(M)|^{\epsilon}$ to be the homotopy fiber of the augmentation map

$$\epsilon: \Omega|h\mathcal{S}_{\bullet}\mathbb{P}_{\text{fd}}(M)| \rightarrow \Omega|h\mathcal{S}_{\bullet}\mathbb{C}_{\text{fd}}(M)|$$

which is induced by the functor which maps (Y_-, Y, Y_+) to Y/\mathbb{Z} . Since

$$\Omega|h\mathcal{S}_{\bullet}\text{NIL}_{\text{fd}}(*, M)| \simeq \Omega|h\mathcal{S}_{\bullet}\mathbb{P}_{\text{fd}}^{h\mathbb{N}_+}(M)|$$

we obtain the following diagram where the vertical and horizontal sequences of maps are homotopy fibration sequences

$$\begin{array}{ccccc}
& & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)|^\epsilon & \dashrightarrow & \text{NA}_+^{\text{fd}}(*, M) \\
& \nearrow \text{dashed} & \downarrow & & \downarrow \\
\Omega|h\mathcal{S}_\bullet\text{NIL}_{\text{fd}}(M, *)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \\
& & \downarrow & & \downarrow \\
& & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)|.
\end{array}$$

This implies that we obtain the dashed homotopy fibration sequence. The map

$$\Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)|^\epsilon \rightarrow \text{NA}_+^{\text{fd}}(*, M)$$

is null homotopic and there is a homotopy equivalence

$$\Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)|^\epsilon \simeq \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)|.$$

All these constructions are natural in M and hence we obtain a natural homotopy equivalence $\delta_+ : \text{Nil}^{\text{fd}}(*, M) \xrightarrow{\simeq} \Omega \text{NA}_+^{\text{fd}}(*, M)$.

Define the Verschiebung operations on $\Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)|$ to be the identity and define the Frobenius operation on $\Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)|$ to be the map which is induced by the functor \vee^n defined in the proof of Proposition 3.6.

We prove the result only for the Verschiebung operations. The proof for the Frobenius operation is analogous with ϵ replaced by ι . We have the following diagram

$$\begin{array}{ccccccc}
& & & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)|^\epsilon & & \text{NA}_+^{\text{fd}}(*, M) & \\
& & & \nearrow V_n & & \nearrow V_n & \\
& & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)|^\epsilon & \downarrow & \text{NA}_+^{\text{fd}}(*, M) & \downarrow & \text{NA}_+^{\text{fd}}(*, M) \\
& & \downarrow & & \downarrow & & \downarrow \\
\Omega|h\mathcal{S}_\bullet\text{NIL}_{\text{fd}}(*; M)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \\
& \nearrow V_n & \downarrow & & \downarrow & & \downarrow \\
\Omega|h\mathcal{S}_\bullet\text{NIL}_{\text{fd}}(*; M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| & \xrightarrow{V_n} & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| \\
& & \downarrow & & \downarrow & & \downarrow \\
\Omega|h\mathcal{S}_\bullet\text{NIL}_{\text{fd}}(*; M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{P}_{\text{fd}}(M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{D}_{\text{fd}}(M \times \mathbb{N}_+)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)| & \longrightarrow & \Omega|h\mathcal{S}_\bullet\mathbb{C}_{\text{fd}}(M)|
\end{array}$$

The vertices in the top layer of the diagram are defined as homotopy fibers of the corresponding vertical maps. The horizontal sequences are homotopy fibration sequences. After using the trick of precomposing with $\Psi \circ \Phi$ (as in Lemma 4.3) we obtain by the proof of Proposition 3.6 that the diagram is strictly commutative except at the left horizontal square where it is commutative only up to a preferred homotopy by Lemma 4.3. This diagram can be replaced by a (preferred) homotopy equivalent diagram which is strictly commutative. Adding the dashed arrows explained above we obtain a self-map of the dashed homotopy fibration sequence (consisting of three maps so that the corresponding squares strictly commute). The statement of theorem then follows since the natural homotopy equivalence δ_+ is obtained from the “connecting” map of the dashed homotopy fibration sequence. \square

If M is a topological monoid, there exists a natural homotopy equivalence $\varepsilon : A^{\text{fd}}(*, M) \xrightarrow{\simeq} K(\mathbb{S}[M])$. This was mentioned in Remark 2.5 when $M = \Omega X$ and for general M the construction of ε is sketched in the proof of the following theorem.

Theorem 4.4. *Let $L = \mathbb{N}_+, \mathbb{N}_-$ or \mathbb{Z} . Then the following diagrams commute up to a preferred homotopy*

$$\begin{array}{ccc}
A^{\text{fd}}(*, M \times L) & \xrightarrow{V_n} & A^{\text{fd}}(*, M \times L) & A^{\text{fd}}(*, M \times L) & \xrightarrow{F_n} & A^{\text{fd}}(*, M \times L) \\
\downarrow \varepsilon & & \downarrow \varepsilon & \downarrow \varepsilon & & \downarrow \varepsilon \\
K(\mathbb{S}[M \times L]) & \xrightarrow{V_n''} & K(\mathbb{S}[M \times L]) & K(\mathbb{S}[M \times L]) & \xrightarrow{F_n''} & K(\mathbb{S}[M \times L])
\end{array}$$

for all $n \in \mathbb{N}_\times$.

Proof. Recall briefly the definition of $\varepsilon: A^{\text{fd}}(*, M \times L) \rightarrow K(\mathbb{S}[M \times L])$ using the notation of [Wal85] and [DGM04]. There $A^{\text{fd}}(*, M \times L)$ is defined by the \mathcal{S}_\bullet -construction on the category $R_{\text{fd}}(*, M \times L)$, of all finitely-dominated objects which has the structure of a Waldhausen category in a similar way as $\mathbb{C}_{\text{fd}}(*, M)$. The required identification goes via a sequence of the following weak equivalences natural in $M \times L$ [Wal85, page 385-389], [DGM04, page 113]:

$$\begin{aligned}
\Omega|h\mathcal{S}_\bullet R_{\text{fd}}(*, M \times L)| &\simeq \Omega|\text{colim}_n h\mathcal{N}_\bullet R_{\text{fd}}^n(*, M \times L)| \\
&\simeq K_0(\mathbb{Z}[\pi_0\mathbb{A}]) \times \text{colim}_{n,k} |hR_k^n(*, M \times L)|^+ \\
&\simeq K_0(\mathbb{Z}[\pi_0\mathbb{A}]) \times \text{colim}_{n,k} B\mathcal{H}_k^n(M \times L)^+ \\
&\simeq K_0(\mathbb{Z}[\pi_0\mathbb{A}]) \times \text{colim}_k B\widehat{\text{GL}}_k(\mathbb{S}[M \times L])^+
\end{aligned}$$

The first line is Proposition 2.2.2 in [Wal85] (slightly modified, see also Remark on page 389), the second line is Segal's group completion theorem [Seg74], the third line is Proposition 2.2.5 in [Wal85] and the last line is explained in [DGM04, page 113].

Now the identification of the Verschiebung operations follows from naturality of the above weak equivalences and the fact that in both cases the Verschiebung is induced by the monoid morphism $\varphi_n: M \times L \rightarrow M \times L$.

The identification of the Frobenius operations is a tedious verification. Informally one defines the Frobenius operations in all the lines of the above identification, on the first two lines they are defined similarly as on $\mathbb{C}_{\text{fd}}(*, M \times L)$ and on the third line they are defined similarly as on $\widehat{\text{GL}}_k(\mathbb{S}[M \times L])^+$. The details are left for the reader. \square

5. IDENTITIES

In this section we prove certain useful identities satisfied by the Frobenius and Verschiebung operations. As a consequence we obtain Corollary 1.1 about the non-finiteness generation of the nil-terms. Further recall that the nil-terms are infinite loop spaces, so we can think of them as of spectra and hence as of \mathbb{S} -modules. Using this and some of the identities proven here we obtain Corollaries 5.2, 5.3 about the structure of an $\mathbb{S}[\mathbb{N}_\times]$ -module and an $\mathbb{S}[\mathbb{N}_\times * \mathbb{N}_\times]$ -module on the nil-terms and hence a structure of a $\mathbb{Z}[\mathbb{N}_\times]$ -module and a $\mathbb{Z}[\mathbb{N}_\times * \mathbb{N}_\times]$ -module on the homotopy groups of the nil-terms. Here $\mathbb{N}_\times = \{1, 2, \dots\}$ is considered as a multiplicative monoid and $\mathbb{N}_\times * \mathbb{N}_\times$ denotes the free product of monoids.

Proposition 5.1. *For $i \geq 0$ and $m, n \in \mathbb{N}_\times$ the following identities hold on $\text{NA}_+^{\text{fd}}(*, M)$:*

$$\begin{aligned} F_1 &= V_1 = 1 \\ V_n V_m &= V_{n \cdot m} \\ F_n F_m &= F_{n \cdot m} \\ F_n V_n(x) &\simeq n \cdot x \\ F_n V_m &\simeq V_m F_n \text{ if } m, n \text{ are coprime.} \end{aligned}$$

Proof. The first identity is obvious, the proof of the next two follows from the formula $\varphi_{mn} = \varphi_n \circ \varphi_m$. For the fourth identity let (Y, f) be an object in $\text{NIL}_{\text{fd}}(*; M)$. Then matrix manipulation yields

$$\begin{aligned} F_n V_n((Y, f)) &= F_n \left((\vee^n Y, \begin{pmatrix} 0 & & f \\ \text{id} & \ddots & \\ & \ddots & 0 \\ & & \text{id} & 0 \end{pmatrix}) \right) \\ &\cong \left(\vee^n Y, \begin{pmatrix} f & & & \\ & f & & \\ & & \ddots & \\ & & & f \end{pmatrix} \right) \cong \vee^n(Y, f), \end{aligned}$$

where the \cong -signs mean that the corresponding functors are isomorphic. The resulting map induces multiplication by n on $\text{NIL}_{\text{fd}}(*; M)$ and therefore also on $\text{NA}_+^{\text{fd}}(*, M)$.

For the last identity we will show that there is a cofibration sequence of functors

$$(5.1) \quad V_m F_n \Psi \circ \Phi \xrightarrow{\Xi} F_n V_m \Psi \circ \Phi \longrightarrow G,$$

where Φ, Ψ are the functors from section 4 and G is functor which induces the trivial map on $\text{Nil}^{\text{fd}}(*, M)$. The fact that $\Psi \circ \Phi \simeq \text{id}$ and Waldhausen additivity theorem [Wal85, Proposition 1.3.2(4)] now imply the identity. Recall that the effect of $\Psi \circ \Phi$ on an object (Y, f) in $\text{NIL}_{\text{fd}}(*, M)$ is that it turns the map f to a cofibration, so we may assume that f is a cofibration. To obtain the cofibration sequence (5.1) note that both values $V_m F_n(Y, f)$ and $F_n V_m(Y, f)$ are of the shape $(\vee_m Y, ?) = (Y_1 \vee \dots \vee Y_m, ?)$, where the maps $?$ permute all Y_i . More precisely,

since m and n are coprime the maps ? are of the following shape:

$$\begin{array}{ccc}
 & V_m F_n & F_n V_m \\
 & \downarrow & \downarrow \\
 & Y_1 & Y_{\sigma(1)} \\
 & \downarrow \text{id} & \downarrow f^{l_{\sigma(1)}} \\
 & Y_2 & Y_{\sigma(2)} \\
 & \downarrow \text{id} & \downarrow f^{l_{\sigma(2)}} \\
 & Y_3 & Y_{\sigma(3)} \\
 & \downarrow \text{id} & \downarrow f^{l_{\sigma(3)}} \\
 & \vdots & \vdots \\
 & \downarrow \text{id} & \downarrow f^{l_{\sigma(m-1)}} \\
 & Y_m & Y_{\sigma(m)}
 \end{array}$$

f^n on the left, $f^{l_{\sigma(m)}}$ on the right, and f^n on the far left.

where σ is a permutation on m letters and $\sum_i l_{\sigma(i)} = n$. Define the natural transformation:

$$V_m F_n \xrightarrow{\Xi} F_n V_m$$

$$\begin{array}{ccc}
 Y_1 & \xrightarrow{\text{id}} & Y_{\sigma(1)} \\
 \downarrow \text{id} & & \downarrow f^{l_{\sigma(1)}} \\
 Y_2 & \xrightarrow{f^{l_{\sigma(1)}}} & Y_{\sigma(2)} \\
 \downarrow \text{id} & & \downarrow f^{l_{\sigma(2)}} \\
 Y_3 & \xrightarrow{f^{l_{\sigma(1)} + l_{\sigma(2)}}} & Y_{\sigma(3)} \\
 \downarrow \text{id} & & \downarrow f^{l_{\sigma(3)}} \\
 \vdots & & \vdots \\
 \downarrow \text{id} & & \downarrow f^{l_{\sigma(m-1)}} \\
 Y_m & \xrightarrow{f^{n - l_{\sigma(m)}}} & Y_{\sigma(m)}
 \end{array}$$

f^n on the left, $f^{l_{\sigma(m)}}$ on the right, and $f^{n - l_{\sigma(m)}}$ on the far left.

and the functor G as a cofiber of Ξ . An inspection shows that the morphism of $G(Y, f)$ is a strictly lower triangular matrix and therefore is trivial in $\text{Nil}^{\text{fd}}(*, M)$. \square

Corollary 5.2. *The Verschiebung operations V_n for $n \in \mathbb{N}_{\times}$ define a structure of an $\mathbb{S}[\mathbb{N}_{\times}]$ -module on $\text{NA}_{\pm}^{\text{fd}}(*, M)$ and hence a structure of a $\mathbb{Z}[\mathbb{N}_{\times}]$ -module on $\pi_i \text{NA}_{\pm}^{\text{fd}}(*, M)$.*

The statement follows from the identity $V_n V_m = V_{n \cdot m}$ of Proposition 5.1.

Corollary 5.3. *The Frobenius and Verschiebung operations F_n, V_n for $n \in \mathbb{N}_{\times}$ define a structure of an $\mathbb{S}[\mathbb{N}_{\times} * \mathbb{N}_{\times}]$ -module on $\text{NA}_{\pm}^{\text{fd}}(*, M)$ and hence a structure of a $\mathbb{Z}[\mathbb{N}_{\times} * \mathbb{N}_{\times}]$ -module on $\pi_i \text{NA}_{\pm}^{\text{fd}}(*, M)$.*

The statement follows from the identities $F_n F_m = F_{n \cdot m}$, $V_n V_m = V_{n \cdot m}$ of Proposition 5.1.

Now we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Since F_m and V_m satisfy the relation stated in Theorem 5.1 and $F_m(x) = 0$ for m bigger than a certain number M , we can apply a trick which is due to Farrell [Far77].

For $n \in \mathbb{N}$ let $h\mathcal{S}_\bullet^n \text{NIL}_{\text{fd}}(*; M)$ be the full subcategory of $h\mathcal{S}_\bullet \text{NIL}_{\text{fd}}(*; M)$ consisting of objects of nilpotency degree smaller or equal to n . We have

$$\begin{aligned} K_i(\text{NIL}_{\text{fd}}(*; M)) &= \pi_i \Omega |h\mathcal{S}_\bullet \text{NIL}_{\text{fd}}(*; M)| \\ &= \text{colim}_n \pi_i \Omega |h\mathcal{S}_\bullet^n \text{NIL}_{\text{fd}}(*; M)| \end{aligned}$$

and

$$K_i(\mathbb{C}_{\text{fd}}(M)) = \pi_i \Omega |h\mathcal{S}_\bullet^0 \text{NIL}_{\text{fd}}(*; M)|.$$

Assume now that $\pi_i \text{NA}_+^{\text{fd}}(*, M)$ is a finitely generated abelian group. Thus we can find an N such that the generators of $K_i(\text{NIL}_{\text{fd}}(*; M))$ are contained in $\pi_i \Omega |h\mathcal{S}_\bullet^N \text{NIL}_{\text{fd}}(*; M)|$. This implies that there is an ℓ such that F_m is the trivial map for $m \geq \ell$. Let T be the torsion subgroup of $\pi_i \text{NA}_+^{\text{fd}}(*, M)$. Since $\pi_i \text{NA}_+^{\text{fd}}(*, M)$ is finitely generated we have that $|T|$ is finite. Choose $t \in \mathbb{N}$ such that $t \cdot |T| + 1 \geq \ell$. By Theorem 5.1 we get that $F_{t \cdot |T| + 1} V_{t \cdot |T| + 1}$ is a monomorphism. On the other hand, since $t \cdot |T| + 1 \geq \ell$, the group $\pi_i \text{NA}_+^{\text{fd}}(*, M)$ is in the kernel of $F_{t \cdot |T| + 1}$. Thus $\pi_i \text{NA}_+^{\text{fd}}(*, M)$ is the trivial group.

The proof for p -primary subgroups is identical. \square

6. A CALCULATION VIA TRACE INVARIANTS

This section contains the proof of Theorem 1.2. It is a calculation of the homotopy type of the p -completion of the spaces $\text{NA}_\pm^{\text{fd}}(*)$ and their homotopy groups as $\mathbb{Z}_p[\mathbb{N}_\times]$ -modules, where p is an odd prime and \mathbb{Z}_p denotes p -adic integers, in a certain range. We follow a general scheme of Madsen [Mad94, section 4.5] to study the linearization map $A^{\text{fd}}(S^1) \rightarrow K(\mathbb{Z}[\mathbb{Z}])$. The idea of the calculation is based on a calculation of the linearization map $A(*) \rightarrow K(\mathbb{Z})$ by Bökstedt and Madsen [BM94], who in turn claim the original idea to come from Goodwillie.

Throughout we fix an odd prime p and we denote the p -completion of a space or of a spectrum by $\hat{}$. For details on p -completion of spectra see [DGM04, Appendix A.1.11]. We note that connective spectra are p -good (the p -completion map induces an isomorphism on $\tilde{H}_*(-; \mathbb{F}_p)$) and the p -completion commutes with fibration and cofibration sequences of spectra. The homotopy groups of a p -completion of a spectrum become \mathbb{Z}_p -modules and the Verschiebung operations induce a structure of a $\mathbb{Z}_p[\mathbb{N}]$ -module on $\pi_* \text{NA}_\pm^{\text{fd}}(X)^\wedge$.

The scheme for calculating the homotopy type of $\text{NA}_+^{\text{fd}}(*)$ as suggested by Madsen at the end of [Mad94, section 4.5] is as follows. The product $\text{NA}_+^{\text{fd}}(*) \times \text{NA}_-^{\text{fd}}(*)$ is the homotopy cofiber of the assembly map $S_+^1 \wedge A(*) \rightarrow A(S^1)$. The product $\text{NK}_+(\mathbb{Z}) \times \text{NK}_-(\mathbb{Z})$ is the homotopy cofiber of the assembly map $S_+^1 \wedge K(\mathbb{Z}) \rightarrow K(\mathbb{Z}[\mathbb{Z}])$ and is known to be contractible. For a space X , denote by $F_K(X)$ the homotopy fiber of the linearization map $l: A(X) \rightarrow K(\mathbb{Z}[\pi_1 X])$ and recall that the two assembly maps are compatible with l and so we obtain a homotopy fibration sequence

$$(6.1) \quad S_+^1 \wedge F_K(*) \rightarrow F_K(S^1) \rightarrow \text{NA}_+^{\text{fd}}(*) \times \text{NA}_-^{\text{fd}}(*) .$$

So it is enough to study $F_K(S^1)$, factor out the image of the assembly map and identify the two summands. This is the approach we take, we describe the space

$F_K(S^1)_p^\wedge$ in a certain range in Proposition 6.3. The proof of Theorem 1.2 assuming 6.3 is given at the end of this section as well as the proofs of its corollaries. In fact our calculation applies more generally, so we determine the homotopy type of the space $F_K(B\pi)_p^\wedge$ for a discrete group π in a certain range.

Proposition 6.1. *Let p be an odd prime and let π be a discrete group. Then there is a $(4p - 7)$ -connected map*

$$\left(\bigvee_{[g] \in [\pi]} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (B\pi_+) \right)_p^\wedge \longrightarrow F_K(BZ_\pi(g))_p^\wedge,$$

where $[\pi]$ denotes the set of conjugacy classes of elements of π and $Z_\pi(g)$ is the centralizer of an element $g \in \pi$.

Corollary 6.2. *Let p be an odd prime and let π be a discrete group such that the assembly map $B\pi_+ \wedge K(\mathbb{Z}) \rightarrow K(\mathbb{Z}\pi)$ is a homotopy equivalence. Then there is a $(4p - 7)$ -connected map*

$$\left(\bigvee_{[g] \in [\pi] \setminus [1]} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (BZ_\pi(g)_+) \right)_p^\wedge \longrightarrow \text{Wh}(B\pi)_p^\wedge$$

where $\text{Wh}(B\pi)$ denotes the homotopy cofiber of the assembly map $B\pi_+ \wedge A(*) \rightarrow A(B\pi)$.

If $\pi = \mathbb{Z}$ we obtain the result for $F_K(S^1)$ and $\text{Wh}(S^1) = \text{NA}_+^{\text{fd}}(*) \times \text{NA}^{\text{fd}}(*)$, since $S^1 = B\mathbb{Z}$. In this case we are also interested in the Frobenius and Verschiebung operations. Recall that these were defined also on $A^{\text{fd}}(S^1)$. They restrict to $F_K(S^1)$ and so we have a structure of a $\mathbb{Z}_p[\mathbb{N}_\times]$ -module and a $\mathbb{Z}_p[\mathbb{N}_\times * \mathbb{N}_\times]$ -module on the p -completion of $F_K(S^1)$. Below $S^1(n)$ denotes just a copy of S^1 indexed by $n \in \mathbb{Z}$.

Proposition 6.3. *If p is an odd prime the following holds:*

- (1) *There is a $(4p - 7)$ -connected map*

$$\bigvee_{n \in \mathbb{Z}} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(n)_+) \longrightarrow F_K(S^1)_p^\wedge,$$

hence

$$\begin{aligned} \pi_{2p-2} F_K(S^1)_p^\wedge &\cong \bigoplus_{n \in \mathbb{Z}} \mathbb{F}_p \{\beta_n\} \\ \pi_{2p-1} F_K(S^1)_p^\wedge &\cong \bigoplus_{n \in \mathbb{Z}} \mathbb{F}_p \{\gamma_n\} \\ \pi_i F_K(S^1)_p^\wedge &\cong 0 \text{ for } i < 2p - 2, 2p - 1 < i \leq 4p - 7 \end{aligned}$$

where

$$\begin{aligned} \beta_n &\in \pi_{2p-2} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(n)_+) \cong \mathbb{F}_p \\ \gamma_n &\in \pi_{2p-1} \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(n)_+) \cong \mathbb{F}_p \end{aligned}$$

represent a certain choice of generators of these \mathbb{Z}_p -modules.

- (2) *The $\mathbb{Z}_p[\mathbb{N}_\times]$ -module structure on $\pi_* F_K(S^1)_p^\wedge$ is given by*

$$\begin{aligned} (n, \beta_m) &\mapsto \beta_{nm} \\ (n, \gamma_m) &\mapsto n \cdot \gamma_{nm}. \end{aligned}$$

Corollary 6.4. *The $\mathbb{Z}_p[\mathbb{N}_\times]$ -module $\pi_{2p-2} F_K(S^1)_p^\wedge$ is generated by the elements $\beta_{-1}, \beta_0, \beta_1$, and the $\mathbb{Z}_p[\mathbb{N}_\times]$ -module $\pi_{2p-1} F_K(S^1)_p^\wedge$ is not finitely generated.*

Corollary 6.5. *The $\mathbb{Z}_p[\mathbb{N}_\times * \mathbb{N}_\times]$ -module $\pi_{2p-1} \text{NA}_\pm^{\text{fd}}(*, M)_p^\wedge$ is not finitely generated.*

A standard strategy when studying $K(\mathbb{A})$ for an \mathbb{S} -algebra \mathbb{A} is to use invariants of K -theory such as topological Hochschild homology $\mathrm{THH}(\mathbb{A})$ or topological cyclic homology $\mathrm{TC}(\mathbb{A}, p)$ for a prime p , which we think of as functors from \mathbb{S} -algebras to spectra. This approach is particularly convenient in the relative situation, and general results can be found for example in [DGM04], [Mad94]. However, calculations for an arbitrary \mathbb{S} -algebra \mathbb{A} are still very hard. In the special case, when $\mathbb{A} = \mathbb{S}[\Omega X]$, there is a convenient relationship between K -theory, the above invariants and the free loop space of X (see Theorem 6.7 below). Hence, a possible approach to our problem is to replace the linearization map by a map between K -theory of the \mathbb{S} -algebras of the form $\mathbb{S}[\Omega X]$, which in turn can be understood via the free loop space. It turns out that this works if we restrict ourselves to a certain dimension range. This is the already mentioned trick used by Bökstedt and Madsen [BM94].

We prove the following proposition which replaces the space $F_K(B\pi)_p^\wedge$ for a discrete group by another (more approachable) space. For a space X we denote $Q(X) := \Omega^\infty \Sigma^\infty X$ and we let $SG = Q(S^0)_1$ be the identity component which is a topological monoid with respect to the composition product.

Proposition 6.6. *If π is a discrete group, p an odd prime, then there is a $(4p-7)$ -cartesian square*

$$\begin{array}{ccc} A(BSG \times B\pi)_p^\wedge & \longrightarrow & A(B\pi)_p^\wedge \\ \downarrow & & \downarrow \\ A(B\pi)_p^\wedge & \longrightarrow & K(\mathbb{Z}[\pi])_p^\wedge, \end{array}$$

where the left vertical arrow is induced by the projection map and the right vertical arrow is the linearization map.

Hence in the range $* \leq 4p-7$ it is enough to study the homotopy fiber of the left vertical map. The proof of Proposition 6.6 is given later. We first prove Proposition 6.1 assuming 6.6.

We denote $\Lambda X = \mathrm{Map}(S^1, X)$, $\mathrm{ev}: \Lambda X \rightarrow X$ is the evaluation map $\mathrm{ev}(\lambda) = \lambda(1)$. For a map $f: Y \rightarrow X$ the symbol $\Lambda(f: Y \rightarrow X)$ denotes the pullback of the diagram

$$Y \xrightarrow{f} X \xleftarrow{\mathrm{ev}} \Lambda X.$$

Goodwillie constructs in [Goo90] a natural map $A(X) \rightarrow \Sigma_+^\infty \Lambda X$. Given a map of spaces $f: Y \rightarrow X$ the composition $A(Y) \rightarrow A(X) \rightarrow \Sigma_+^\infty \Lambda X$ factors through $\Sigma_+^\infty \Lambda(f: Y \rightarrow X)$. This factorization has the following property, see [Goo90, Corollary 3.3].

Theorem 6.7 (Goodwillie). *If $f: Y \rightarrow X$ is a k -connected map, then the square*

$$\begin{array}{ccc} A(Y) & \longrightarrow & \Sigma_+^\infty \Lambda(f: Y \rightarrow X) \\ \downarrow & & \downarrow \\ A(X) & \longrightarrow & \Sigma_+^\infty \Lambda X \end{array}$$

is $(2k-1)$ -cartesian.

Proof of Proposition 6.1. By Theorem 6.6 we have a $(4p-7)$ -connected map

$$\mathrm{hofiber}(A(BSG \times B\pi) \rightarrow A(B\pi))_p^\wedge \rightarrow F_K(B\pi)_p^\wedge$$

whose source can be understood using Proposition 6.7. If $f: Z \times X \rightarrow X$ is the projection, then $\Lambda(f: Z \times X \rightarrow X) = Z \times \Lambda X$. If Z has a base point the vertical maps in the diagram of Theorem 6.7 have sections and the homotopy fibers of these

vertical maps are homotopy equivalent to the homotopy cofibers of the corresponding sections. Thus, if Z is a k -connected space, we obtain a $(2k - 1)$ -connected map

$$(6.2) \quad \text{hofiber}(A(Z \times X) \rightarrow A(X)) \rightarrow \Sigma^\infty Z \wedge (\Lambda X_+).$$

(We have a homotopy cofibration sequence $* \times \Lambda X \rightarrow Z \times \Lambda X \rightarrow Z \wedge (\Lambda X_+)$.)

For a discrete group π we have a decomposition

$$\Lambda B\pi \simeq \coprod_{[g] \in [\pi]} BZ_\pi(g),$$

where $[\pi]$ denotes the set of conjugacy classes of elements in π and $Z_\pi(g)$ is the centralizer of an element $g \in \pi$. Further, by [Tod58] we have a $(4p - 6)$ -connected map

$$BSG_p^\wedge \rightarrow \Sigma^{2p-2} \mathbb{H}\mathbb{F}_p.$$

Plugging $Z = BSG$, $X = B\pi$ in (6.2) and p -completing yields the statement of Proposition 6.1. \square

Now we prove Proposition 6.6. As already mentioned Bökstedt and Madsen in [BM94, Proposition 9.11] show such a result for the functor $\text{TC}(-, p)$ for an odd prime p instead of K -theory, and with π the trivial group. This is our starting point. We will not recall the definition of $\text{TC}(-, p)$ here, since it is considerably complicated. We need to know that for a prime p there is a functor from \mathbb{S} -algebras to spectra $\mathbb{A} \mapsto \text{TC}(\mathbb{A}, p)$ which comes with a natural transformation $\text{trc}: K(\mathbb{A}) \rightarrow \text{TC}(\mathbb{A}, p)$, called the *cyclotomic trace map* which has certain properties. A theorem of Dundas says that under certain assumptions relative K -theory equals relative TC (Theorem 6.8 below). Further we need another property of TC which we also only cite.

Theorem 6.8 (Dundas). *If $\mathbb{A} \rightarrow \mathbb{B}$ is a map of \mathbb{S} -algebras inducing an isomorphism $\pi_0(\mathbb{A}) \rightarrow \pi_0(\mathbb{B})$, then the following square*

$$\begin{array}{ccc} K(\mathbb{A})_p^\wedge & \longrightarrow & K(\mathbb{B})_p^\wedge \\ \downarrow & & \downarrow \\ \text{TC}(\mathbb{A}, p)_p^\wedge & \longrightarrow & \text{TC}(\mathbb{B}, p)_p^\wedge. \end{array}$$

is homotopy cartesian.

For the proof see [Dun97], [Mad94].

Proposition 6.9. *The functor $\mathbb{A} \mapsto \text{TC}(\mathbb{A}, p)$ has the following property. If*

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{B} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{D} \end{array}$$

is a k -cartesian square of connective \mathbb{S} -algebras, then

$$\begin{array}{ccc} \text{TC}(\mathbb{A}, p) & \longrightarrow & \text{TC}(\mathbb{B}, p) \\ \downarrow & & \downarrow \\ \text{TC}(\mathbb{C}, p) & \longrightarrow & \text{TC}(\mathbb{D}, p) \end{array}$$

is a $(k - 1)$ -cartesian square.

For the proof combine the statements [DGM04, Proposition 1.4.2] and [DGM04, Proposition 2.2.7]. We also need the following result from homotopy theory.

Lemma 6.10. *There is a $(4p - 6)$ -cartesian square of \mathbb{S} -algebras*

$$(6.3) \quad \begin{array}{ccc} \mathbb{S}[SG \times \pi]_p^\wedge & \xrightarrow{\theta} & \mathbb{S}[\pi]_p^\wedge \\ \varepsilon \downarrow & & \downarrow l \\ \mathbb{S}[\pi]_p^\wedge & \xrightarrow{l} & \mathbb{HZ}[\pi]_p^\wedge. \end{array}$$

Proof. The maps in the diagram are defined as follows. The map l is the linearization map. The map ε is induced by the projection $SG \times \pi \rightarrow \pi$. To obtain the map θ when $\pi = 1$ take the adjoint of the inclusion $SG \hookrightarrow Q(S^0)$, which is a map of the form $\mathbb{S}[SG] \rightarrow \mathbb{S}$. For $\pi \neq 1$ adjoin π to this map.

If $\pi = 1$ the diagram (6.3) is just the diagram from [BM94, Proposition 9.11]. But adjoining π is the same as taking the wedge of π -many copies of this diagram for $\pi = 1$. So if the diagram for $\pi = 1$ is $(4p - 6)$ -cartesian after p -completion then also the diagram with adjoined π is $(4p - 6)$ -cartesian after p -completion. \square

Proof of Proposition 6.6. By Proposition 6.9 applying $\mathrm{TC}(-, p)_p^\wedge$ to the diagram of Lemma 6.10 yields a $(4p - 7)$ -cartesian square. Recalling $SG \simeq \Omega BSG$ and denoting $\mathrm{TC}(X, p) = \mathrm{TC}(\mathbb{S}[\Omega X], p)$ we obtain a $(4p - 7)$ -cartesian square

$$(6.4) \quad \begin{array}{ccc} \mathrm{TC}(BSG \times B\pi, p)_p^\wedge & \longrightarrow & \mathrm{TC}(B\pi, p)_p^\wedge \\ \downarrow & & \downarrow \\ \mathrm{TC}(B\pi, p)_p^\wedge & \longrightarrow & \mathrm{TC}(\mathbb{Z}[\pi], p)_p^\wedge. \end{array}$$

We have a similar square with K -theory instead of $\mathrm{TC}(-, p)$ which maps via the cyclotomic trace map trc into the square (6.4). By Theorem 6.8 the sides of the cube containing both K -theory and $\mathrm{TC}(-, p)$ are homotopy cartesian and hence also the whole cube is homotopy cartesian. But then the side consisting of K -theories is as cartesian as the opposite side which is the diagram (6.4). \square

Proof of Proposition 6.3. 1. This follows immediately from Proposition 6.1 since $S^1 = B\mathbb{Z}$ and from the observation that the spectrum $\Sigma^{2p-2} \mathbb{H}\mathbb{F}_p \wedge (S^1(n)_+)$ already is p -complete. For part 2. we need some notation. Recall that we have a homotopy equivalence

$$(6.5) \quad \Lambda S^1 \simeq \prod_{n \in \mathbb{Z}} S^1(n).$$

By $d\alpha_1 \in \pi_{2p-2} BSG = H_{2p-2}(BSG)$, $1_n \in H_0(S^1(n))$, $s_n \in H_1(S^1(n))$ are denoted the generators of the respective groups. Further we denote

$$\begin{aligned} \beta_n &= 1_n \cdot d\alpha_1 \in \pi_{2p-2}(S^1(n)_+ \wedge BSG) \cong H_{2p-2}(S^1(n)_+ \wedge BSG) \\ \gamma_n &= s_n \cdot d\alpha_1 \in \pi_{2p-1}(S^1(n)_+ \wedge BSG) \cong H_{2p-1}(S^1(n)_+ \wedge BSG). \end{aligned}$$

2. The Verschiebung operations give $\pi_* F_K(S^1)_p^\wedge$ a structure of a $\mathbb{Z}_p[\mathbb{N}_\times]$ -module. Recall that they are induced by maps of \mathbb{S} -algebras $\varphi_n: \mathbb{S}[M \times \mathbb{Z}] \rightarrow \mathbb{S}[M \times \mathbb{Z}]$. These induce the Verschiebung operations also on $\Sigma_+^\infty \Lambda B(M \times \mathbb{Z}) \simeq \Sigma_+^\infty \Lambda X \times \Lambda S^1$, where $M = \Omega X$. Therefore the natural map

$$A^{\mathrm{fd}}(X \times S^1) \simeq K(\mathbb{S}[M \times \mathbb{Z}]) \rightarrow \Sigma_+^\infty \Lambda B(M \times \mathbb{Z}) \simeq \Sigma_+^\infty \Lambda X \times \Lambda S^1$$

is compatible with the Verschiebung operation. In more detail, the operation V_n is induced by $\varphi_n: \mathbb{Z} \rightarrow \mathbb{Z}$ given by multiplication by n . This induces an n -fold cover map on S^1 (which is the target S^1 in ΛS^1). The effect on $\tilde{H}_*(\Lambda S^1_+; \mathbb{F}_p)$ is $1_m \mapsto 1_{nm}$ and $s_m \mapsto ns_{nm}$. \square

Proof of Corollary 6.4. This follows immediately from Proposition 6.3 (2). \square

Proof of Proposition 1.2 and Corollary 6.2. This follows from the homotopy fibration sequence (6.1) and the calculation of [BM94, Proposition 9.11] that $F_K(*)_p^\wedge$ is $(2p-3)$ -connected and $\pi_{2p-2}F_K(*)_p^\wedge = \mathbb{F}_p\{d\alpha_1\}$. Further we have $\pi_*S_+^1 \wedge F_K(*)_p^\wedge = \mathbb{F}_p\{\beta_0, \gamma_0\}$ where $\beta_0 = 1_0 \cdot d\alpha_1$, $\gamma_0 = s_0 \cdot d\alpha_1$. The same argument works for general discrete π . \square

Proof of Corollary 1.3. This follows from Proposition 1.2 and Corollary 6.4. \square

Proof of Corollary 6.5. Suppose that $\pi_{2p-1} \text{NA}_{\pm}^{\text{fd}}(*, M)_p^\wedge$ is finitely generated as a $\mathbb{Z}_p[\mathbb{N}_\times * \mathbb{N}_\times]$ -module with the generating set $A = \{a_1, \dots, a_k\}$ and assume $a_1 = \gamma_1$. We claim that the following three statements hold

$$(6.6) \quad \bigcup_{n \in \mathbb{N}_\times \setminus \{1\}} \text{Im}(V_n(\bigoplus_{i \in \mathbb{N}_\times} \mathbb{F}_p\{\gamma_i\})) \subset \bigoplus_{\substack{i \in \mathbb{N}_\times \\ \text{for } i \neq p^l}} \mathbb{F}_p\{\gamma_i\}$$

$$(6.7) \quad \left| \bigcup_{\substack{n \in \mathbb{N}_\times \\ a_i \in A}} F_n(a_i) \cap \bigoplus_{l \in \mathbb{N}_\times} \mathbb{F}_p\{\gamma_{p^l}\} \right| < \infty$$

$$(6.8) \quad \bigcup_{n \in \mathbb{N}_\times \setminus \{1\}} \text{Im}(F_n(\bigoplus_{\substack{i \in \mathbb{N}_\times \\ \text{for } i \neq p^l}} \mathbb{F}_p\{\gamma_i\})) \subset \bigoplus_{\substack{i \in \mathbb{N}_\times \\ \text{for } i \neq p^l}} \mathbb{F}_p\{\gamma_i\} \cup \bigcup_{n \in \mathbb{N}_\times} F_n(\mathbb{F}_p\{\gamma_1\}).$$

The statement (6.6) follows immediately from Proposition 6.3 (2). The statement (6.7) follows since for any x we have $F_n(x) = 0$ when $n \geq N(x)$ for some $N(x)$. To prove (6.8) let

$$\gamma_m \in \bigoplus_{\substack{i \in \mathbb{N} \\ \text{for } i \neq p^l}} \mathbb{F}_p\{\gamma_i\}.$$

Again by Proposition 6.3 (2) we have $\gamma_m = V_m(z\gamma_1)$ for some $z \in \mathbb{F}_p$. Then

$$\begin{aligned} F_n(\gamma_m) &= F_n V_m(z\gamma_1) \\ &= \text{gcd}(m, n) F_{n'} V_{m'}(z\gamma_1) \\ &= \text{gcd}(m, n) V_{m'} F_{n'}(z\gamma_1) \end{aligned}$$

where $\text{gcd}(m, n)$ denotes the greatest common divisor, $m = \text{gcd}(m, n) \cdot m'$ and $n = \text{gcd}(m, n) \cdot n'$. If $m' = 1$ then $F_n(\gamma_m) \in \bigcup_{n \in \mathbb{N}_\times} F_n(\mathbb{F}_p\{\gamma_1\})$ and if $m' > 1$ then by (6.6) we have $F_n(\gamma_m) \in \bigoplus_{\substack{i \in \mathbb{N}_\times \\ \text{for } i \neq p^l}} \mathbb{F}_p\{\gamma_i\}$.

By (6.6) and (6.8) the subgroup

$$\bigoplus_{\substack{i \in \mathbb{N}_\times \\ \text{for } i \neq p^l}} \mathbb{F}_p\{\gamma_i\} \oplus \bigcup_{\substack{n \in \mathbb{N}_\times \\ a_i \in A}} F_n(a_i)$$

is invariant under the Frobenius and Verschiebung operations. Furthermore all the generators are contained in this subgroup. But this gives a contradiction since by (6.7) there are elements in the complement of this subgroup in $\bigoplus_{n \in \mathbb{N}_\times} \mathbb{F}_p\{\gamma_n\}$. \square

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