

## ON INVERTING THE KOSZUL COMPLEX

KAMAL KHURI-MAKDISI

ABSTRACT. Let  $V$  be an  $n$ -dimensional vector space. We give a direct construction of an exact sequence that gives a  $GL(V)$ -equivariant “resolution” of each symmetric power  $S^t V$  in terms of direct sums of tensor products of the form  $\wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_p} V$ . This exact sequence corresponds to inverting the relation in the representation ring of  $GL(V)$  that is described by the Koszul complex, and has appeared before in work by B. Totaro, analogously to a construction of K. Akin involving the normalized bar resolution. Our approach yields a concrete description of the differentials, and provides an alternate direct proof that  $\text{Ext}_{\wedge(V^*)}^t(k, k) = S^t(V)$ .

Let  $k$  be a field, and let  $V$  be an  $n$ -dimensional vector space over  $k$  with  $n \geq 2$ . We view the alternating and symmetric powers  $\wedge^i V$  and  $S^t V$  of  $V$  as representations of  $G = GL(V) \cong GL(n, k)$ ; we allow any  $i, t \in \mathbf{Z}$ , with the understanding that  $\wedge^i V$  and  $S^t V$  are zero unless  $0 \leq i \leq n$  or  $t \geq 0$ . Working in the representation ring of  $G$  (i.e., the Grothendieck group of algebraic representations of  $G$ ), we can write  $S^t V$  as a polynomial in the fundamental representations  $\wedge^1 V = V, \wedge^2 V, \dots, \wedge^n V$ ; as we shall see below, the terms of this polynomial can be ordered in a natural way with alternating signs. This suggests that there should exist an exact sequence of representations of  $G$  that concretely realizes this polynomial expression for  $S^t V$ . This sequence, given in equation (3) below, has appeared in Sections 2 and 4 of [Tot97], as a modification of a construction of [Aki89], via the normalized bar resolution of  $k$  as a module for the exterior algebra  $\wedge(V^*)$ . It is known that there is a natural isomorphism

$$(1) \quad \text{Ext}_{\wedge(V^*)}^t(k, k) = S^t V,$$

and computing the Ext group using the normalized bar resolution as in [Tot97] produces the exact sequence (3). It is however not immediate to write down the differentials explicitly, as this involves chasing through various dualizations and natural isomorphisms. Moreover, one needs to have prior knowledge of (1) to use this approach.

In this note, we give a direct, self-contained construction of the exact sequence (3), with explicit differentials, and without using (1). We give a straightforward proof that the sequence is exact; this involves a total induction on  $t$  that uses the Koszul complex to analyze the sequence (3) in terms of the previous cases. Our proof generalizes the argument of [KM03] from  $SL(2)$  to  $GL(n)$ . We hope that the structure of our argument, as well as our identification of (3) as an inversion of the Koszul complex, will be of interest in other contexts. Combining our proof of exactness with a separate calculation to show that our sequence is the same as the one coming from the normalized bar resolution, we obtain an independent proof of (1).

---

2000 *Mathematics Subject Classification.* 20G05, 15A72, 16E05.  
April 26, 2007.

The first few cases of the exact sequence, for  $1 \leq t \leq 4$ , are

$$\begin{aligned}
& 0 \rightarrow V \rightarrow S^1V \rightarrow 0, & 0 \rightarrow \wedge^2V \rightarrow V \otimes V \rightarrow S^2V \rightarrow 0, \\
& 0 \rightarrow \wedge^3V \rightarrow (V \otimes \wedge^2V) \oplus (\wedge^2V \otimes V) \rightarrow V \otimes V \otimes V \rightarrow S^3V \rightarrow 0, \\
(2) \quad & 0 \rightarrow \wedge^4V \rightarrow (V \otimes \wedge^3V) \oplus (\wedge^2V \otimes \wedge^2V) \oplus (\wedge^3V \otimes V) \rightarrow \\
& \rightarrow (V \otimes V \otimes \wedge^2V) \oplus (V \otimes \wedge^2V \otimes V) \oplus (\wedge^2V \otimes V \otimes V) \rightarrow \\
& \rightarrow V \otimes V \otimes V \otimes V \rightarrow S^4V \rightarrow 0.
\end{aligned}$$

In general, the sequence has the form

$$(3) \quad 0 \rightarrow T_t^1 \xrightarrow{\delta} T_t^2 \xrightarrow{\delta} \cdots \xrightarrow{\delta} T_t^t \xrightarrow{\pi} S^tV \rightarrow 0,$$

for suitable differentials  $\delta$  and  $\pi$ , where, for  $1 \leq p \leq t$ , the  $p$ th term  $T_t^p$  is given by

$$\begin{aligned}
(4) \quad T_t^p &= \bigoplus_{\substack{1 \leq i_1, \dots, i_p \leq n \\ i_1 + \dots + i_p = t}} (\wedge^{i_1}V \otimes \cdots \otimes \wedge^{i_p}V) \\
&\cong \bigoplus_{\substack{\ell_1, \dots, \ell_n \geq 0 \\ \ell_1 + 2\ell_2 + \dots + n\ell_n = t \\ \ell_1 + \ell_2 + \dots + \ell_n = p}} \frac{(\ell_1 + \dots + \ell_n)!}{\ell_1! \cdots \ell_n!} [V^{\otimes \ell_1} \otimes (\wedge^2V)^{\otimes \ell_2} \cdots \otimes (\wedge^nV)^{\otimes \ell_n}].
\end{aligned}$$

We shall see in the proof of Theorem 2 that it is reasonable to allow  $p = 0$ , provided we define  $T_0^0 = k$  and  $T_t^0 = 0$  for  $t \neq 0$ . Note also that  $T_t^t = V^{\otimes t}$  and that  $T_t^1 = \wedge^tV$ , which may be zero (our conventions imply that  $T_t^p = 0$  unless  $p \leq t \leq np$ ; note also that the condition  $i_1, \dots, i_p \leq n$  in (4) is redundant since otherwise  $\wedge^iV = 0$ .) Our choice of letters  $t$  and  $p$  refers to the ‘‘total degree’’ (i.e., the effect of a scalar matrix in  $G$ ) and ‘‘partial degree’’ (i.e., the number of parts) of a decomposable tensor  $\alpha_1 \otimes \cdots \otimes \alpha_p \in \wedge^{i_1}V \otimes \cdots \otimes \wedge^{i_p}V \subset T_t^p$ . The multinomial coefficient  $\frac{(\ell_1 + \dots + \ell_n)!}{\ell_1! \cdots \ell_n!}$  refers to a direct sum of several copies of the representation  $V^{\otimes \ell_1} \otimes \cdots \otimes (\wedge^nV)^{\otimes \ell_n}$  of  $G$ . In the special case  $n = 2$ , we see that  $T_t^{t-i} \cong \binom{t-i}{i} [V^{\otimes(t-2i)} \otimes (\wedge^2V)^{\otimes i}]$ , and we recover the result of [KM03].

Before describing the differentials in (3), we pause to explain why

$$(5) \quad T_t^1 - T_t^2 + \cdots + (-1)^{t-1}T_t^t + (-1)^tS^tV = 0$$

in the representation ring of  $G$ . This gives the polynomial expression of  $S^tV$  in terms of the  $\wedge^iV$ , and arises from inverting the relation between symmetric and alternating powers that is expressed by the exactness of the Koszul complex

$$(6) \quad 0 \rightarrow \wedge^nV \otimes S^*V \rightarrow \cdots \rightarrow \wedge^2V \otimes S^*V \rightarrow V \otimes S^*V \rightarrow S^*V \rightarrow k \rightarrow 0.$$

Here  $S^*V = \bigoplus_{t \geq 0} S^tV$  is the symmetric algebra on  $V$ ; it is naturally isomorphic to the polynomial algebra  $k[e_1, \dots, e_n]$ , with  $\{e_1, \dots, e_n\}$  a basis for  $V$ . The last term  $k$  is the trivial representation of  $G$ , i.e., the unit element of the representation ring of  $G$ ; this term should be viewed as  $k[e_1, \dots, e_n]/\langle e_1, \dots, e_n \rangle$ .

The morphisms in (6) (see for instance Section XXI.4 of [Lan02] or Section VII.2 of [MacL63]) increase the degree in each  $S^*V$  component by one, so by taking  $G$ -equivariant Hilbert series with a formal parameter  $x$ , we obtain the identity of formal power series in the representation ring of  $G$ :

$$(7) \quad \left( \sum_{t=0}^{\infty} S^tV \cdot x^t \right) (1 - V \cdot x + \wedge^2V \cdot x^2 - \cdots + (-1)^n \wedge^n V \cdot x^n) = 1.$$

Now replace  $x$  by  $-x$  and invert the sum over the  $\wedge^i V$  to yield

$$(8) \quad \begin{aligned} \sum_{t=0}^{\infty} (-1)^t S^t V \cdot x^t &= \frac{1}{1 + V \cdot x + \wedge^2 V \cdot x^2 + \cdots + \wedge^n V \cdot x^n} \\ &= \sum_{p=0}^{\infty} (-1)^p (V \cdot x + \wedge^2 V \cdot x^2 + \cdots + \wedge^n V \cdot x^n)^p. \end{aligned}$$

For  $t \geq 1$ , the coefficient of  $x^t$  in  $(V \cdot x + \wedge^2 V \cdot x^2 + \cdots + \wedge^n V \cdot x^n)^p$  is zero unless  $1 \leq p \leq t$ , in which case this coefficient is the class of  $T_t^p$ . This proves (5).

We now define the differential  $\delta$  of (3) on each direct summand  $\wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_p} V$  of  $T_t^p$ , where  $t = i_1 + \cdots + i_p$ . We start with the case  $p = 1$ . Given  $v_1, \dots, v_i \in V$ , we introduce the notations  $v_A$ , for nonempty  $A \subset \{1, \dots, i\}$ , and  $s(A, B)$ , for disjoint nonempty  $A, B \subset \{1, \dots, i\}$ , by

$$(9) \quad \begin{aligned} v_A &= v_{a_1} \wedge \cdots \wedge v_{a_j} \in \wedge^j V, \text{ where } A = \{a_1, \dots, a_j\} \text{ with } a_1 < \cdots < a_j, \\ s(A, B) &\in \{\pm 1\} \text{ such that } v_{A \cup B} = s(A, B) v_A \wedge v_B. \end{aligned}$$

Note that  $s(A, B)$  depends only on the sets  $A$  and  $B$ , and not on the choice of  $v_1, \dots, v_i$ . We then define, for a decomposable tensor  $v_1 \wedge \cdots \wedge v_i \in \wedge^i V = T_i^1$ ,

$$(10) \quad \delta(v_1 \wedge \cdots \wedge v_i) = \sum_{\substack{\emptyset \neq A, B \subset \{1, \dots, i\} \\ A \cup B = \{1, \dots, i\} \\ A \cap B = \emptyset}} s(A, B) v_A \otimes v_B \in T_i^2.$$

For example,  $\delta(v_1) = 0$ ,  $\delta(v_1 \wedge v_2) = v_1 \otimes v_2 - v_2 \otimes v_1$ , and  $\delta(v_1 \wedge v_2 \wedge v_3 \wedge v_4)$  is

$$(11) \quad \begin{aligned} &(v_1 \wedge v_2 \wedge v_3) \otimes v_4 - (v_1 \wedge v_2 \wedge v_4) \otimes v_3 + (v_1 \wedge v_3 \wedge v_4) \otimes v_2 - (v_2 \wedge v_3 \wedge v_4) \otimes v_1 \\ &+ (v_1 \wedge v_2) \otimes (v_3 \wedge v_4) - (v_1 \wedge v_3) \otimes (v_2 \wedge v_4) + [3 \text{ more terms}] + (v_3 \wedge v_4) \otimes (v_1 \wedge v_2) \\ &+ v_1 \otimes (v_2 \wedge v_3 \wedge v_4) - v_2 \otimes (v_1 \wedge v_3 \wedge v_4) + v_3 \otimes (v_1 \wedge v_2 \wedge v_4) - v_4 \otimes (v_1 \wedge v_2 \wedge v_3). \end{aligned}$$

We claim that  $\delta$  is well defined, i.e., that the right hand side of (10) is an alternating form in the vectors  $v_1, \dots, v_i$ . The easiest way to verify this is to check that if two adjacent vectors  $v_\ell, v_{\ell+1}$  are equal, then the right hand side of (10) vanishes.

We now define the general action of  $\delta$  on  $\alpha_1 \otimes \cdots \otimes \alpha_p \in \wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_p} V \subset T_t^p$ , for  $1 \leq p \leq t - 1$ , by

$$(12) \quad \begin{aligned} \delta(\alpha_1 \otimes \cdots \otimes \alpha_p) &= \delta(\alpha_1) \otimes \alpha_2 \otimes \cdots \otimes \alpha_p - \alpha_1 \otimes \delta(\alpha_2) \otimes \cdots \otimes \alpha_p \\ &+ \cdots + (-1)^{p-1} \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \delta(\alpha_p) \in T_t^{p+1}. \end{aligned}$$

Finally, the last differential  $\pi : T_t^t \rightarrow S^t V$  is the natural projection from  $T_t^t = V^{\otimes t}$ .

**Lemma 1.** *The differentials  $\delta : T_t^p \rightarrow T_t^{p+1}$  and  $\pi$  as defined above are  $G$ -homomorphisms that satisfy  $\delta\delta = 0$  and  $\pi\delta = 0$ . In other words, (3) is a complex of  $G$ -representations.*

*Proof.* From the definition of  $\delta(v_1 \wedge \cdots \wedge v_i)$ , we see that it respects the  $G$ -action. (This would have been less transparent if we had defined  $\delta$  from the beginning in terms of basis elements  $e_{a_1} \wedge \cdots \wedge e_{a_i}$  of  $\wedge^i V$ , for a fixed basis  $\{e_1, \dots, e_n\}$  of  $V$ .) This implies the  $G$ -linearity in general. As for  $\delta\delta = 0$ , one first checks directly that  $\delta\delta(v_1 \wedge \cdots \wedge v_i) = 0$ . The crucial ingredient is that if  $A, B$ , and  $C$  are disjoint nonempty subsets of  $\{1, \dots, i\}$ , then  $s(A \cup B, C)s(A, B) = s(A, B \cup C)s(B, C)$ . This follows from comparing  $v_A \wedge v_B \wedge v_C$  with  $v_{A \cup B \cup C}$ . This settles the case  $p = 1$ , and

we then proceed inductively for larger  $p$ . In particular, writing  $\beta = \alpha_2 \otimes \cdots \otimes \alpha_p$ , we have  $\delta\delta(\alpha \otimes \beta) = \delta[\delta(\alpha) \otimes \beta - \alpha \otimes \delta(\beta)]$ , and the reader should be careful to note the  $+$  sign in expanding the first term:  $\delta[\delta(\alpha) \otimes \beta] = \delta(\delta(\alpha)) \otimes \beta + \delta(\alpha) \otimes \delta(\beta)$ , because  $\delta(\alpha)$  has partial degree 2. This proves our result except at the last step, involving  $\pi : T_t^t \rightarrow S^t V$ , where one can show directly that  $\pi\delta = 0$ .  $\square$

**Theorem 2.** *The complex (3) is exact if  $t \geq 1$ .*

*Proof.* It is more convenient to show instead that the truncated complex

$$(13) \quad T_t^\bullet : 0 \rightarrow T_t^1 \xrightarrow{\delta} T_t^2 \xrightarrow{\delta} \cdots \xrightarrow{\delta} T_t^t \rightarrow 0$$

has zero cohomology everywhere except at the last term, where the cohomology is  $S^t V$ . The proof is by induction on  $t$ , the case  $t = 1$  being trivial. For the inductive step, we introduce a filtration on  $T_t^\bullet$ , and compute  $H^*(T_t^\bullet)$  by the spectral sequence of the filtered complex (see for instance Proposition XX.9.1 of [Lan02], but note that the indices on the filtration there run opposite to ours; Section XI.3 of [MacL63] is another reference). We define our filtration on  $T_t^\bullet$  by taking  $\text{Fil}^f(T_t^p) \subset T_t^p$  to be

$$(14) \quad \text{Fil}^f(T_t^p) = \bigoplus_{\substack{i_1 \leq f \\ 1 \leq i_1, \dots, i_p \leq n \\ i_1 + \cdots + i_p = t}} (\wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_p} V).$$

Thus  $f$  is a bound on the degree  $i_1$  of the “first term”  $\alpha_1$  in any tensor  $\alpha_1 \otimes \cdots \otimes \alpha_p \in \text{Fil}^f(T_t^p)$ . We have  $T_t^p = \text{Fil}^t(T_t^p) \supset \cdots \supset \text{Fil}^0(T_t^p) = 0$ . The reader is encouraged at this point to look ahead to the first diagram in Example 3, which illustrates the complex and its filtration in the case  $t = 4$ .

The  $E_0$  term in our spectral sequence is given by the associated graded complex  $\text{Gr}(T_t^\bullet) = \bigoplus_{f=1}^t \text{Gr}^f(T_t^\bullet)$ , and the spectral sequence abuts to the cohomology  $H^*(T_t^\bullet)$  that we wish to compute. Since  $\text{Gr}^f(T_t^p) = \text{Fil}^f(T_t^p)/\text{Fil}^{f-1}(T_t^p)$ , we have

$$(15) \quad \text{Gr}^f(T_t^p) \cong \bigoplus_{\substack{i_1 = f \\ i_1 + \cdots + i_p = t}} (\wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_p} V) \cong \wedge^f V \otimes T_{t-f}^{p-1}, \quad \text{for } 1 \leq f \leq t.$$

Here we slightly abuse notation, since  $\text{Gr}^t(T_t^1) = \wedge^t V$ , corresponding to taking  $T_0^0 = k$  in the rightmost term of (15) when  $p = 1$  and  $f = t$ ; if  $p = 1$  and  $f \neq t$ , we take  $T_{t-f}^0$  and  $\text{Gr}^f(T_t^1)$  to be zero, consistently with (4). Now  $\delta$  respects the filtration, and descends to a differential  $\bar{\delta} : \text{Gr}^f(T_t^p) \rightarrow \text{Gr}^f(T_t^{p+1})$  that can be identified with  $1 \otimes (-\delta) : \wedge^f V \otimes T_{t-f}^{p-1} \rightarrow \wedge^f V \otimes T_{t-f}^p$ . This is illustrated in the second diagram in Example 3. Applying our inductive hypothesis, and noting that the presence of  $-\delta$  instead of  $\delta$  makes no difference, we see that the  $\bar{\delta}$ -cohomology of  $\text{Gr}^f(T_t^\bullet)$  is concentrated in degree  $\bullet = t - f + 1$ , where it is naturally isomorphic to  $\wedge^f V \otimes S^{t-f} V$  (this holds even if  $f = t$ ). Hence the  $E_1$  term of our spectral sequence is

$$(16) \quad 0 \rightarrow \wedge^t V \otimes S^0 V \rightarrow \wedge^{t-1} V \otimes S^1 V \rightarrow \cdots \rightarrow \wedge^1 V \otimes S^{t-1} V \rightarrow 0,$$

with differentials induced from the “portion” of the original differentials  $\delta$  on

$$(17) \quad 0 \rightarrow \wedge^t V \rightarrow \wedge^{t-1} V \otimes V \rightarrow \wedge^{t-2} V \otimes V \otimes V \rightarrow \cdots \rightarrow \wedge^1 V \otimes V^{\otimes(t-1)} \rightarrow 0.$$

This “portion” corresponds to considering in (10) only the terms where  $B$  has cardinality  $|B| = 1$ . Note that (17) is not a complex, as the composition of the “partial”  $\delta$ s is not zero, but the induced maps in (16) do give a complex, due

to the presence of the symmetric powers  $S^i V$  instead of the tensor powers  $V^{\otimes i}$ . Comparing the differentials with those in, say, Section XXI.4 of [Lan02], we see that (16) is the part of the Koszul complex (6) in total degree  $t$ , after we drop the final terms  $S^* V \rightarrow k \rightarrow 0$ . (The fact that the Koszul complex, as written, starts with  $\wedge^n V$ , whereas we have started with  $\wedge^t V$ , is immaterial, since the “missing” vector spaces are all zero, either from  $\wedge^i V$  with  $i > n$ , or from  $S^j V$  with  $j < 0$ .) We have  $t \geq 2$ , so we can ignore the term  $k$  in the Koszul complex, since that term appears only in total degree 0. Hence we obtain that the  $E_2$  term of our spectral sequence is degenerate, consisting of a single instance of  $S^t V$  in one corner. Thus  $E_2 = E_\infty$ , and we obtain that the complex (13) has the desired cohomology.  $\square$

**Example 3.** We include a diagram of the filtered complex  $T_4^\bullet$  below, drawn in such a way that we obtain a second quadrant spectral sequence. Thus the diagram represents the fourth exact sequence of (2), omitting the final term  $S^4 V$ . The terms of our complex in a given partial degree  $p$  lie on a single SW-NE diagonal. Note that this is not a double complex, due to the diagonal arrows. For example, the differential  $\delta : \wedge^4 V \rightarrow (V \otimes \wedge^3 V) \oplus (\wedge^2 V \otimes \wedge^2 V) \oplus (\wedge^3 V \otimes V)$  is represented by the three arrows emanating from  $\wedge^4 V$  below, and each arrow corresponds to the terms in one of the three lines of formula (11).

$$\begin{array}{ccccccc}
 & & & & & & V \otimes \wedge^3 V \\
 & & & & & & \downarrow \\
 & & & & & & \wedge^2 V \otimes \wedge^2 V \longrightarrow V \otimes [(\wedge^2 V \otimes V) \oplus (V \otimes \wedge^2 V)] \\
 & & & & & & \downarrow \\
 & & & & & & \wedge^3 V \otimes V \\
 & & & & & & \downarrow \\
 \wedge^4 V & \longrightarrow & \wedge^3 V \otimes V & \longrightarrow & \wedge^2 V \otimes V \otimes V & \longrightarrow & V \otimes V \otimes V \otimes V
 \end{array}$$

The filtration corresponds to taking all columns to the right of some vertical line in the above diagram. Hence each part  $\text{Gr}^f(T_4^\bullet)$  of the graded complex  $E_0$  sees *only* the vertical arrows in a single column. Note the “common factors” of  $\wedge^2 V$  and of  $\wedge^1 V = V$  in the rightmost two columns above. We also see trivial “common factors” of  $\wedge^4 V$  and  $\wedge^3 V$  in the leftmost two columns. Identifying the remaining factors in terms of the previous complexes  $T_t^\bullet$  for  $t \leq 3$ , we can represent  $E_0$  as the diagram

$$\begin{array}{cccc}
 & & & V \otimes T_3^1 \\
 & & & \downarrow \delta \\
 & & & V \otimes T_3^2 \\
 & & \wedge^2 V \otimes T_2^1 & \downarrow \delta \\
 & & \downarrow \delta & \downarrow \delta \\
 \wedge^4 V & \wedge^3 V \otimes T_1^1 & \wedge^2 V \otimes T_2^2 & V \otimes T_3^3
 \end{array}$$

We now compute  $E_1$ . From the inductive hypothesis, only the bottom row survives, with differentials induced from the bottom row of our first diagram for  $T_4^\bullet$ :

$$\wedge^4 V \longrightarrow \wedge^3 V \otimes S^1 V \longrightarrow \wedge^2 V \otimes S^2 V \longrightarrow V \otimes S^3 V$$

This is the truncated Koszul complex of (16), and hence computing  $E_2$  leaves us with just  $S^4 V$  in the SE corner.

**Remark 4.** We conclude this note by showing the equivalence between (3) and the construction in Sections 2 and 4 of [Tot97] via the normalized bar resolution. The normalized bar resolution is a resolution of  $k$  as a left module over an associative  $k$ -algebra  $\Lambda$  with unit, equipped with an augmentation map  $\varepsilon : \Lambda \rightarrow k$ . We write  $\bar{\Lambda} = \Lambda/k$  for the quotient of vector spaces, which can if needed be identified with the ideal  $\ker \varepsilon$ , although it is preferable not to view  $\bar{\Lambda}$  as a module over  $\Lambda$ . Then the normalized bar resolution of  $k$  is

$$(18) \quad \cdots \rightarrow \Lambda \otimes \bar{\Lambda} \otimes \bar{\Lambda} \otimes \bar{\Lambda} \xrightarrow{\partial_3} \Lambda \otimes \bar{\Lambda} \otimes \bar{\Lambda} \xrightarrow{\partial_2} \Lambda \otimes \bar{\Lambda} \xrightarrow{\partial_1} \Lambda \xrightarrow{\varepsilon} k \rightarrow 0.$$

Here  $\otimes = \otimes_k$ , as before, and the  $\Lambda$ -module structure is via multiplication on the leftmost factor  $\Lambda$  in each tensor product. The differential  $\partial_p$  for  $p \geq 1$  is given by the following formula, where we write  $\bar{\lambda} \in \bar{\Lambda}$  for the image of  $\lambda \in \Lambda$  (see Section X.2 of [MacL63], with  $C = k$ , for the proof that this is well defined and that (18) is exact):

$$(19) \quad \begin{aligned} \partial_p(\lambda \otimes \bar{\lambda}_1 \otimes \cdots \otimes \bar{\lambda}_p) &= \lambda \lambda_1 \otimes \bar{\lambda}_2 \otimes \cdots \otimes \bar{\lambda}_p \\ &- \lambda \otimes \overline{\lambda_1 \lambda_2} \otimes \bar{\lambda}_3 \otimes \cdots \otimes \bar{\lambda}_p + \cdots + (-1)^{p-1} \lambda \otimes \bar{\lambda}_1 \otimes \cdots \otimes \overline{\lambda_{p-1} \lambda_p} \\ &+ (-1)^p \varepsilon(\lambda_p) \lambda \otimes \bar{\lambda}_1 \otimes \cdots \otimes \bar{\lambda}_{p-1}. \end{aligned}$$

We apply the above resolution (18) in the case when the algebra  $\Lambda$  is the exterior algebra  $\Lambda = \wedge(V^*)$  on the dual vector space  $V^*$ . Thus we can view  $\bar{\Lambda}$  as  $\bigoplus_{i \geq 1} \wedge^i(V^*) = \bigoplus_{1 \leq i \leq n} \wedge^i(V^*)$ . Now apply the functor  $\text{Hom}_\Lambda(-, k)$  and note that  $\text{Hom}_\Lambda(\Lambda \otimes M, k)$  is naturally isomorphic to the dual  $k$ -vector space  $M^*$ . We obtain that  $\text{Ext}_\Lambda^\bullet(k, k)$  can be computed from the complex

$$(20) \quad 0 \rightarrow k \xrightarrow{\partial_1^*=0} \bar{\Lambda}^* \xrightarrow{\partial_2^*} \bar{\Lambda}^* \otimes \bar{\Lambda}^* \xrightarrow{\partial_3^*} \cdots.$$

We can identify  $\bar{\Lambda}^*$  with  $\bigoplus_{i \geq 1} \wedge^i V$ . The complex (20) is graded by total degree  $t$ , which is compatible with the differentials  $\partial^*$ . The components in degree  $t \geq 1$  of (20) are exactly our  $T_t^p$ ; recall that, by our conventions, in order for  $T_t^p$  to be nonzero it is necessary to have  $1 \leq p \leq t$ . It remains to compare the resulting differentials  $\partial^*$  with the differentials  $\delta$ , so as to show that the part of (20) in total degree  $t$  is essentially the same as the complex (13).

**Proposition 5.** *In the above context, the restriction to total degree  $t$  of  $\partial^*$  in (20) is the negative of  $\delta$ , as defined in (10) and (12).*

*Proof.* We shall show only the key step, namely that  $\partial_2^* = -\delta$  for  $\delta$  as in (10). We leave it to the reader to subsequently verify that the recurrence relating  $\partial_{p+1}^*$  to  $\partial_2^*$  is the same as (12) for  $\delta$ . It is enough to compare  $\partial_2^*(\alpha)$  with  $\delta(\alpha)$  for  $\alpha = v_1 \wedge \cdots \wedge v_i \in \wedge^i V = (\wedge^i(V^*))^* \subset \bar{\Lambda}^* \subset \text{Hom}_k(\Lambda, k)$ . More precisely,  $\alpha$  annihilates  $\wedge^{i'}(V^*)$  for  $i' \neq i$ , and acts on  $v_1^* \wedge \cdots \wedge v_i^* \in \wedge^i(V^*)$  by

$$(21) \quad \langle \alpha, v_1^* \wedge \cdots \wedge v_i^* \rangle = \det(\langle v_j, v_k^* \rangle)_{1 \leq j, k \leq i}.$$

Unraveling the definitions, we obtain that  $\partial_2^*(\alpha) \in (\bar{\Lambda} \otimes \bar{\Lambda})^*$  acts on  $\bar{\lambda}_1 \otimes \bar{\lambda}_2$  by

$$(22) \quad \langle \partial_2^*(\alpha), \bar{\lambda}_1 \otimes \bar{\lambda}_2 \rangle = \varepsilon(\lambda_1) \langle \alpha, \bar{\lambda}_2 \rangle - \langle \alpha, \overline{\lambda_1 \wedge \lambda_2} \rangle + \varepsilon(\lambda_2) \langle \alpha, \bar{\lambda}_1 \rangle.$$

The value depends only on the classes  $\bar{\lambda}_1, \bar{\lambda}_2 \in \bar{\Lambda}$ , and it is enough to test the action of  $\partial_2^*(\alpha)$  in the situation when  $\lambda_1 = v_1^* \wedge \cdots \wedge v_{i_1}^* \in \wedge^{i_1}(V^*)$  and  $\lambda_2 =$

$v_{i_1+1}^* \wedge \cdots \wedge v_{i_1+i_2}^* \in \wedge^{i_2}(V^*)$ , with  $i_1, i_2 \geq 1$ . Hence we have

$$(23) \quad \langle \partial_2^*(\alpha), \bar{\lambda}_1 \otimes \bar{\lambda}_2 \rangle = -\langle \alpha, v_1^* \wedge \cdots \wedge v_{i_1+i_2}^* \rangle.$$

The above quantity vanishes unless  $i_1 + i_2 = i$ , in which case it is the negative of the determinant of (21). We now wish to compare this value to

$$(24) \quad \langle \delta(\alpha), \bar{\lambda}_1 \otimes \bar{\lambda}_2 \rangle = \sum_{A,B} s(A,B) \langle v_A, \bar{\lambda}_1 \rangle \cdot \langle v_B, \bar{\lambda}_2 \rangle$$

with  $A$  and  $B$  as in (10). This is nonzero only if there exist choices with  $|A| = i_1$  and  $|B| = i_2$ , which forces  $i = i_1 + i_2$ ; in that case, the resulting sum in (24) reduces to the general Laplace expansion of  $\det((v_j, v_k^*))_{1 \leq j, k \leq i}$  in terms of minors of sizes  $i_1$  and  $i_2$ , thereby completing our proof.  $\square$

Our results thus give a direct calculation of the cohomology of (20), and hence of  $\text{Ext}_\Lambda^i(k, k)$ , in terms of our calculation of the cohomology of (13). Thus we obtain as a consequence a direct proof of (1), using only the exactness of the Koszul complex.

#### REFERENCES

- [Aki89] Kaan Akin, *Extensions of symmetric tensors by alternating tensors*, J. Algebra **121** (1989), no. 2, 358–363. MR 992770 (90d:20021)
- [KM03] Kamal Khuri-Makdisi, *An exact sequence in the representation theory of  $\text{SL}(2)$* , Comm. Algebra **31** (2003), no. 9, 4153–4160. MR 1995526 (2004f:20087)
- [Lan02] Serge Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556 (2003e:00003)
- [MacL63] Saunders Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York, 1963. MR 0156879 (28 #122)
- [Tot97] Burt Totaro, *Projective resolutions of representations of  $\text{GL}(n)$* , J. Reine Angew. Math. **482** (1997), 1–13. MR 1427655 (98h:20080)

MATHEMATICS DEPARTMENT AND CENTER FOR ADVANCED MATHEMATICAL SCIENCES, AMERICAN UNIVERSITY OF BEIRUT, BLISS STREET, BEIRUT, LEBANON

*E-mail address:* `kmakdisi@aub.edu.lb`