Ratio of Price to Expectation and Complete Bernstein Functions

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Abstract

For a game with positive expectation and some negative profit, a unique price exists, at which the optimal proportion of investment reaches its maximum. For a game with parallel translated profit, the ratio of this price to its expectation tends to converge toward less than or equal to 1/2 if its expectation converges to 0^+ . In this paper, we will investigate such properties by using the integral representations of a complete Bernstein function and establish several Abelian and Tauberian theorems.

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1. Introduction

Consider a coin-flipping game such that profit is 9 dollars or -2 dollars if a tossed coin yields heads or tails, respectively. For simplicity, we will omit the currency notation. Let $t \in [0, 1]$ be the proportion of investment. Then, the investor repeatedly invests t of his/her current capital (see [12, 13]). For example, let c > 0 be the current capital; when the investor plays the game once, his/her capital will be 9ct/u + c(1-t) or -2ct/u + c(1-t) if a tossed coin yields heads or tails, respectively, where u > 0 is the price of the game such that u/(u + 2) > t. Let the initial capital be 1. After N attempts, if the investor has capital c_N , then the growth rate (geometric mean) is given by $c_N^{1/N}$. As the value $G_u(t) := \lim_{N\to\infty} \left(\exp(t) + c(t) + c(t$

In general, a game (a(x), F(x)) would mean that if the investor invests 1 unit (which price is *u* dollars), then he/she receives a(x) dollars (including the invested money) in accordance with a distribution function F(x), defined on an interval $I \subseteq (-\infty, \infty)$ such that $\int_I d(F(x)) = 1$. It is assumed that the profit function a(x)is measurable and non-constant (a.e.) with respect to F(x). When no confusion arises, we write dF for d(F(x)) and use the following notation:

(1.1)
$$E := \int_{I} a(x) dF, \ \xi := \text{ess inf}_{x \in I} a(x), \ H_{\xi} := \int_{I} \frac{1}{a(x) - \xi} dF.$$

In this paper, we always assume that E > 0 and $\xi > -\infty$. If $\int_{a(x)=\xi} dF > 0$, we define $H_{\xi} = \infty$ and $1/H_{\xi} = 0$. Since a(x) is non-constant, we have $\xi < E$, $H_{\xi} > 0$, $1/H_{\xi} < \infty$, and $\xi + 1/H_{\xi} < E$.

In order to explain the background of this paper, we will define notations such as $w_{\beta}(z)$ and $G_u(t)$ in this paragraph. However, this paper utilizes neither such notations nor their related properties, except in the first paragraph of Section 2. We denote the integral $\int_I (a(x) - \beta)/(a(x)z - z\beta + \beta)dF$ by $w_{\beta}(z)$, which is holomorphic with respect to two complex variables (z, β) $(z := t + si, \beta := u + hi, i := \sqrt{-1}, \{t, s, u, h\} \subset \mathbb{R})$ near each point (t_0, u_0) such that $0 < t_0 < u_0/(u_0 - \xi)$ and

 $u_0 > \max(0, \xi)$. We denote $\exp(\int_I \log(a(x)t/u - t + 1) dF)$ by $G_u(t)$ and term it as the limit expectation of growth rate for each u > 0 and $0 \le t \le 1$ with $\xi t/u - t + 1 > 0$. We say that t_u is the optimal proportion of investment with respect to u > 0, if

(1.2)
$$\overline{\lim}_{\substack{\rho \to t_u \\ 0 \le \rho \le 1 \\ \xi \rho/u - \rho + 1 > 0}} \int_I \log \frac{a(x)t/u - t + 1}{a(x)\rho/u - \rho + 1} dF \le 0$$

for each $0 \leq t \leq 1$ with $\xi t/u - t + 1 > 0$. A game (a(x), F(x)) is said to be *effective* if $\int_{a(x)>1} a(x)^{\nu} dF < \infty$ for some $\nu > 0$. If a game is effective, $G_u(t)$ is continuous (see [9, Theorem 4.1]) and the inequality (1.2) implies that $G_u(t_u) = \sup_{0 \leq t \leq 1, \ \xi t/u - t + 1 > 0} G_u(t)$, which suggests that t_u is optimal for maximizing the limit expectation of growth rate.

For a game with *parallel translated profit* (a(x) - m, F(x)) (m < E), we use underlined notations such as $\underline{a}(x) := a(x) - m$, $\underline{E} := E - m$, $\underline{\xi} := \xi - m$, and $\underline{H}_{\underline{\xi}} := H_{\xi}$.

From [9, Lemma 3.16], if $m \in (\xi, E)$, then a unique price $\underline{u}_{\max} \in (0, E - m)$ exists such that \underline{t}_u is strictly increasing in the interval $0 < u < \underline{u}_{\max}$ and strictly decreasing in the interval $\underline{u}_{\max} < u < E - m$. It should be noted that \underline{u}_{\max} is a function with respect to $m \in (\xi, E)$, and it satisfies $\underline{t}_{\underline{u}_{\max}} = \max_{0 < u < E - m} \underline{t}_u$. In a sense, \underline{u}_{\max} is considered to be the price in which the broker's commission income is maximized.

Under mild restrictions, we will show that $\lim_{m\to E^-} \underline{u}_{\max}/\underline{E} = 1/2$ (see Theorem 3.19). In such a case, it suggests that the so-called half price sale makes a profit. For example, in the case of the abovementioned coin-flipping game, we obtain

$$\lim_{n \to E^-} \frac{\underline{u}_{\max}}{\underline{E}} = \lim_{m \to (7/2)^-} \frac{11\sqrt{(m+2)(9-m)/2 - (m+2)(9-m)}}{(7/2-m)^2} = \frac{1}{2},$$

where -2 < m < 7/2 (see Corollary 3.20).

Defining $\Psi(c) := 1 / \int_I (a(x) + c)^{-1} dF - c \ (c \in (-\xi, \infty))$, we obtain the following: LEMMA 1.1. $\lim_{c \to \infty} \Psi(c) = E$.

PROOF. Assume $c > \max(1, -2\xi)$. Then, we have a(x) + c/2 > 0 and $0 < \int_I c/(a(x) + c)dF \le \int_I 2dF = 2$. If $E < \infty$, then, by applying Lebesgue's monotone convergence and dominated convergence theorems to the equation

$$\frac{1}{\int_{I} \frac{1}{a(x)+c} dF} - c = E + \frac{E^2}{c-E} - \frac{\int_{I} \frac{a(x)^2}{a(x)+c} dF}{\left(1 - \frac{E}{c}\right) \int_{I} \frac{c}{a(x)+c} dF} \quad (c \neq E),$$

we obtain the conclusion (even if $E \leq 0$). Assume $E = \infty$. Since a(x) is non-constant with respect to F(x), we observe that

$$\Psi'(c) = \frac{\int_{I} \frac{1}{(a(x)+c)^{2}} dF - \left(\int_{I} \frac{1}{a(x)+c} dF\right)^{2}}{\left(\int_{I} \frac{1}{a(x)+c} dF\right)^{2}} > 0,$$

which implies that $\Psi(c)$ is increasing with respect to c. Putting $\lim_{c\to\infty} \Psi(c) = M$ (including ∞),

$$a_N(x) := \begin{cases} N, & a(x) > N, \\ a(x), & a(x) \le N, \end{cases} \text{ and } b_{c,N} := \frac{1}{\int_I \frac{1}{a_N(x) + c} dF} - c \quad (N > \max(1, \xi)). \end{cases}$$

Then, the following properties hold:

(1) $a_N(x)$ is nondecreasing with respect to N. (2) $b_{c,N}$ is nondecreasing with respect to N. (3) From the above arguments, we obtain $\lim_{c\to\infty} b_{c,N} = \int_I a_N(x) dF$ $< \infty$, which is nondecreasing with respect to N. (4) By applying Lebesgue's (monotone convergence) theorem, we obtain $\lim_{N\to\infty} b_{c,N} = \Psi(c)$, which is increasing with respect to c. (5) Further, from Lebesgue's (monotone convergence) theorem, we obtain $\lim_{N\to\infty} (\lim_{c\to\infty} b_{c,N}) = \lim_{N\to\infty} \int_I a_N(x) dF = \int_I a(x) dF = \infty$.

Therefore, if $M < \infty$, then $\Psi(c) \leq M$ and $b_{c,N} \leq M$, which contradicts the fact that $\lim_{N\to\infty} (\lim_{c\to\infty} b_{c,N}) = \infty$. Hence, $M = E = \infty$, that is, $\lim_{c\to\infty} \Psi(c) = E$.

2. Parallel translated profit

We consider a game with parallel translated profit (a(x) - m, F(x)) to have sufficiently small positive expectation, if $\xi + 1/H_{\xi} < m < E$. In this case, it is easy to observe that $\underline{E} = E - m > 0$, $\underline{\xi} = \xi - m < 0$ and $\underline{\xi} + 1/\underline{H}_{\underline{\xi}} = \xi - m + 1/H_{\xi} < 0$. Therefore, from [9, Lemma 4.27], $\eta_m := \lim_{u \to 0^+} \underline{t}_u/u$ exists such that $0 < \eta_m < -1/\underline{\xi} = 1/(m - \xi)$. For each $u \in (0, E - m)$ and $t \in (0, u/(u - \underline{\xi}))$, we have $\underline{w}_u(\underline{t}_u) = 0$ where $\underline{w}_u(t) = \int_I (a(x) - m - u) / ((a(x) - m)t - tu + u) dF$ and $\underline{t}_u \in (0, u/(u - \underline{\xi}))$. It should be noted that $(a(x) - m)\underline{t}_u/u - \underline{t}_u + 1 > 0$ and $(a(x) - m)\eta_m + 1 \ge 1 - (m - \xi)\eta_m > 0$ for each $x \in I$. The equation $\underline{w}_u(\underline{t}_u) = 0$ can be written as $\int_I ((a(x) - m)\underline{t}_u/u - \underline{t}_u + 1)^{-1} dF = 1$. Hence, we have

(2.1)
$$\int_{I} \frac{1}{(a(x) - m)\eta_m + 1} dF = 1,$$

because on the set $\{x \mid a(x) \ge m\}$, $1/((a(x) - m)\underline{t}_u/u - \underline{t}_u + 1)$ is strictly increasing with respect to sufficiently small u > 0 (see [9, Lemmas 3.12, 3.15 and 3.16]), and on the set $\{x \mid \xi \le a(x) < m\}$, $1/((a(x) - m)\underline{t}_u/u - \underline{t}_u + 1)$ converges uniformly to $1/((a(x) - m)\eta_m + 1) \ (u \to 0^+)$.

Since $\Psi(c)$ $(-\xi < c < \infty)$ is strictly increasing from $\xi + 1/H_{\xi}$ to E (see the proof of Lemma 1.1), the equation $m = \Psi(c)$ has a unique solution $c = c_m$ for each $\xi + 1/H_{\xi} < m < E$. Since the equation $m = 1/\int_{I} (a(x) + c_m)^{-1} dF - c_m$ is equivalent to

$$\int_{I} \frac{1}{(a(x) - m)\frac{1}{m + c_m} + 1} dF = 1,$$

from (2.1), we obtain that $\eta_m = 1/(m+c_m)$.

LEMMA 2.1. c_m is strictly increasing from $-\xi$ to ∞ with respect to $m \in (\xi + 1/H_{\xi}, E)$.

PROOF. As $\Psi(c)$ is strictly increasing from $\xi + 1/H_{\xi}$ to E, the relation $m = \Psi(c_m)$ leads to the conclusion.

LEMMA 2.2. η_m is strictly decreasing from H_{ξ} to 0 with respect to $m \in (\xi + 1/H_{\xi}, E)$.

PROOF. Since $\eta_m = 1/(m + c_m)$, Lemma 2.1 leads to the conclusion. LEMMA 2.3. $\lim_{m \to E^-} m\eta_m = 0$.

PROOF. From $\lim_{m\to E^-} c_m = \infty$ and Lebesgue's theorem, we obtain the equality $m\eta_m = m/(m+c_m) = \int_I a(x)/(a(x)+c_m)dF$, which implies the conclusion.

LEMMA 2.4.
$$\underline{t}_u = u\eta_{m+u} \ (0 < u < E - m)$$

PROOF. From the property of η_{m+u} , we observe that $\int_I ((a(x) - (m+u))\eta_{m+u} + 1)^{-1} dF = 1$, which can be written as $\int_I ((a(x) - m)(u\eta_{m+u})/u - u\eta_{m+u} + 1)^{-1} dF = 1$. This suggests that $\underline{w}_u(u\eta_{m+u}) = 0$. Therefore, by the uniqueness of \underline{t}_u , we arrive at the conclusion.

LEMMA 2.5. $\underline{u}_{\text{max}}$ can be uniquely determined by the system

(2.2)
$$\begin{cases} m+v = \Psi(c), \\ v = (m+c)\Psi'(c), \end{cases}$$

with two unknown variables $v (= \underline{u}_{\max})$ and $c (= c_{m+\underline{u}_{\max}})$, for each $m \in (\xi+1/H_{\xi}, E)$.

PROOF. From $m = \Psi(c_m)$, we obtain $m + \underline{u}_{\max} = \Psi(c_{m+\underline{u}_{\max}})$. Since $\underline{t}_{\underline{u}_{\max}}$ = $\max_{0 \le u \le E-m} \underline{t}_u$, we find that $\underline{t}'_{\underline{u}_{\max}} = 0$. From $\underline{t}_u = u\eta_{m+u} = u/(m+u+u)$ c_{m+u}), we obtain $\underline{t}'_u = (m + c_{m+u} - uc'_{m+u})/(m+u+c_{m+u})^2$. Thus, $m + c_{m+\underline{u}_{\max}} - \underline{u}_{\max}c'_{m+\underline{u}_{\max}} = 0$. Using $\Psi'(c_m)c'_m = 1$, we have $m + c_{m+\underline{u}_{\max}} - \underline{u}_{\max}/\Psi'(c_m+\underline{u}_{\max})$ = 0, which implies (2.2). On the other hand, from

$$\Psi''(c) = \frac{2\left(\left(\int_{I} \frac{1}{(a(x)+c)^{2}} dF\right)^{2} - \int_{I} \frac{1}{a(x)+c} dF \int_{I} \frac{1}{(a(x)+c)^{3}} dF\right)}{\left(\int_{I} \frac{1}{a(x)+c} dF\right)^{3}} < 0 \quad (c > -\xi)$$

 $\Psi(c)$ is a strictly concave function due to Schwarz's inequality. Therefore, a line $y - m = \Psi'(c)(x + m)$ that is tangent to $\Psi(c)$ and passes through the point (-m, m) is uniquely determined. This implies the uniqueness of the solution of (2.2). \Box

EXAMPLE 2.6. The game $(x, \int_0^x 1/(\pi(t+1)\sqrt{t})dt)$ $(x \in (0,\infty))$ has the following properties: $\xi = 0, \xi + 1/H_{\xi} = 0 < m < E = \infty, \Psi(c) = \sqrt{c}, \eta_m = 1/(m(m+1)), c_m = m^2, \underline{t}_u = u/((m+u)(m+u+1)), \text{ and } \underline{u}_{\max} = \sqrt{m(m+1)}.$

EXAMPLE 2.7. The game $(x, \int_0^x 8r\sqrt{rt}/(\pi(t+r)^3)dt)$ $(x \in (0,\infty), r > 0)$ has the following properties: $\xi = 0, \xi + 1/H_{\xi} = r/3 < m < E = 3r, \Psi(c) = (c-r)^3/(c^2 - 6cr - 3r^2 + 8r\sqrt{cr}) - c, \eta_m = (3r - m)^2/(m+r)^3, c_m = r(r-3m)^2/(3r-m)^2, \underline{t}_u = u(3r - m - u)^2/(r + m + u)^3$, and $\underline{u}_{\max} = (3r - m)(m+r)/(m+9r)$.

REMARK. Since $\underline{E} = E - m > 0$, the assumption E > 0 can be dismissed as long as we consider a game (a(x) - m, F(x)) with $\xi + 1/H_{\xi} < m < E$. It is clear that even if $E \leq 0, \xi + 1/H_{\xi} < E$ holds, provided $\xi = \text{ess inf}_{x \in I} a(x) > -\infty$.

3. Complete Bernstein functions

A C^{∞} function $f: (0, \infty) \to \mathbb{R}$ with a continuous extension to $[0, \infty)$ is called a *Bernstein function* if $f \ge 0$ and $(-1)^k f^{(k)}(x) \le 0$ for each $k \in \{1, 2, 3, ...\}$ (see [5, Definition 1.2.1]).

A function $f:(0,\infty) \to \mathbb{R}$ is called a *complete Bernstein function* if there exists a Bernstein function ϕ such that $f(x) = x^2 \int_0^\infty e^{-sx} \phi(s) ds$ (see [14, Definition 1.4]).

THEOREM 3.1 [14, Theorem 1.5]. Each of the following five properties of f: $(0, \infty) \to \mathbb{R}$ implies the other four:

- (1) f is a complete Bernstein function.
- (2) f can be represented as $f(x) = \tau x + b + \int_0^\infty x/(x+t)\sigma(dt)$ with $\tau, b \ge 0$ and a measure σ on $(0, \infty)$.
- (3) f extends analytically on $\mathbb{C} \setminus (-\infty, 0]$ such that $f(\overline{z}) = \overline{f(z)}$ and $\operatorname{Im} z \operatorname{Im} f(z) \ge 0$. (In other words, f preserves the upper and lower half-planes in \mathbb{C}).
- (4) f is a Bernstein function with representation $f(x) = \tau x + b$ $+ \int_0^\infty (1 - e^{-sx})\beta(s)ds$, where $\tau, b \ge 0$, $\beta(s) = \int_0^\infty e^{-st}\rho(dt)$, and $\int_0^\infty 1/(t(t+1))\rho(dt) < \infty$. (In fact, $\rho(dt) = t\sigma(dt)$ of (2).)
- (5) x/f(x) is a complete Bernstein function or $f \equiv 0$.

Note that the triple (τ, b, ρ) given above is uniquely determined by f (see [5, Theorem 1.2.3]).

LEMMA 3.2. The function $\Psi(c) = 1/\int_I (a(x) + c)^{-1} dF - c$: $(-\xi, \infty) \to (\xi + 1/H_{\xi}, E)$ extends analytically on $\mathbb{C} \setminus (-\infty, -\xi]$ and preserves the upper and lower half-planes.

PROOF. From Lemma 1.1, we obtain $\Psi((-\xi, \infty)) \subset (\xi + 1/H_{\xi}, E)$. Putting $c = u + yi \in \mathbb{C} \setminus (-\infty, -\xi], \{u, y\} \subset \mathbb{R}$ and s = a(x) + u, then

$$\Psi(c) = \frac{1}{\int_I \frac{1}{s+yi} dF} - u - yi = \frac{1}{\int_I \frac{s}{s^2 + y^2} dF - yi \int_I \frac{1}{s^2 + y^2} dF} - u - yi.$$

If $y \ge 0$, then due to Schwarz's inequality, we observe that

$$\operatorname{Im}\Psi(c) = \frac{y\left(\int_{I} \frac{1}{s^{2}+y^{2}} dF \int_{I} \frac{s^{2}}{s^{2}+y^{2}} dF - \left(\int_{I} \frac{1}{\sqrt{s^{2}+y^{2}}} \frac{s}{\sqrt{s^{2}+y^{2}}} dF\right)^{2}\right)}{\left(\int_{I} \frac{s}{s^{2}+y^{2}} dF\right)^{2} + y^{2} \left(\int_{I} \frac{1}{s^{2}+y^{2}} dF\right)^{2}} \gtrless 0.$$

Set $\alpha(v) := \int_{a(x) \le v+\xi} dF$. Then, $\alpha(v)$ is a right continuous nondecreasing function such that $\alpha(v) \ge 0$, $\alpha(\xi^-) = 0$ and $\alpha(\infty) = 1$. Thus, the Stieltjes transform $\int_I (a(x) + c)^{-1} dF = \int_{0^-}^{\infty} (v + c + \xi)^{-1} d(\alpha(v))$ is analytic with respect to $t = c + \xi \in \mathbb{C} \setminus (-\infty, 0]$ (see [15, Corollary VIII.2b.1]). It is easy to verify that $\Psi(c)$ has no singular point in $\mathbb{C} \setminus (-\infty, -\xi]$.

THEOREM 3.3. $\Psi(c) - \xi - 1/H_{\xi}$ is a complete Bernstein function with respect to $t = c + \xi > 0$.

PROOF. From Theorem 3.1 (3) and Lemma 3.2, we arrive at the conclusion. $\hfill \Box$

LEMMA 3.4. $\lim_{c\to\infty} \Psi(c)/c = 0$. PROOF. From $\Psi(c)/c = 1/(1 - \int_I a(x)/(a(x) + c)dF) - 1$ and

$$\lim_{c \to \infty} \int_I \frac{a(x)}{a(x) + c} dF = \lim_{c \to \infty} \int_{a(x) > 0} \frac{a(x)}{a(x) + c} dF + \lim_{c \to \infty} \int_{\xi \le a(x) < 0} \frac{a(x)}{a(x) + c} dF = 0,$$

we arrive at the conclusion by applying Lebesgue's monotone convergence and dominated convergence theorems.

LEMMA 3.5. $\Psi(c)$ can be written as

(3.1)
$$\Psi(c) = \xi + \frac{1}{H_{\xi}} + \int_0^\infty \frac{c+\xi}{t(t+c+\xi)} \rho(dt) \quad (c > -\xi)$$

with $\int_0^\infty t^{-1}\rho(dt) = E - \xi - 1/H_{\xi}$ and $\int_0^\infty 1/(t(t+1))\rho(dt) < \infty$. PROOF. From Theorems 3.1 and 3.3, $\Psi(c)$ can be written as

$$\Psi(c) - \xi - 1/H_{\xi} = \tau(c+\xi) + b + \int_0^\infty \frac{c+\xi}{t(t+c+\xi)} \rho(dt) \quad (c > -\xi),$$

where $\tau \ge 0$, $b \ge 0$, and $\int_0^\infty 1/(t(t+1))\rho(dt) < \infty$. Since

$$\frac{\Psi(c)}{c} = \tau + \frac{\tau\xi + b + \xi + 1/H_{\xi}}{c} + \frac{c+\xi}{c} \int_0^\infty \frac{1}{t(t+c+\xi)} \rho(dt) dt$$

we have $\tau = 0$ by applying Lemma 3.4 and Lebesgue's theorem. Since $\partial \left(\frac{(c+\xi)}{t(t+c+\xi)} \right)$ $\partial c = (t+c+\xi)^{-2} > 0, (c+\xi)/(t(t+c+\xi))$ is increasing with respect to $c > -\xi$. From $\Psi(-\xi) = \xi + 1/H_{\xi}$ and Lebesgue's theorem, we obtain b = 0. From $\lim_{c \to \infty} \Psi(c) = E$ and Lebesgue's theorem, we obtain that $E - \xi - 1/H_{\xi} = \int_{0}^{\infty} t^{-1}\rho(dt)$.

LEMMA 3.6. The condition $E < \infty$ is equivalent to $\int_0^{\infty} t^{-1}\rho(dt) < \infty$. PROOF. Lemma 3.5 shows that $\int_0^{\infty} t^{-1}\rho(dt) = E - \xi - 1/H_{\xi}$, which implies the conclusion because ξ and $1/H_{\xi}$ are finite.

LEMMA 3.7. A function $f(x) \ge 0$ (x > 0) is a complete Bernstein function if and only if 1/(x + f(x)) is a Stieltjes transform. In this case, a right continuous nondecreasing function $0 \leq G(t) \leq 1$ exists such that $1/(x + f(x)) = \int_{0}^{\infty} (x + f(x)) dx$ $+t)^{-1}d(G(t)).$

PROOF. Let f(x) be a complete Bernstein function. Therefore, in accordance with Theorem 3.1 (2), x + f(x) is a complete Bernstein function. Further, in accordance with Theorem 3.1 (5), x/(x + f(x)) is a complete Bernstein function. Thus, from Theorem 3.1 (2), we have $x/(x+f(x)) = \tilde{\tau}x + \tilde{b} + \int_0^\infty x/(x+t)\tilde{\sigma}(dt)$ (x > 0) with $\tilde{\tau}, \tilde{b} \ge 0$ and a measure $\tilde{\sigma}$ on $(0, \infty)$. We can obtain $\tilde{\tau} = 0$ as follows. From Theorem 3.1 (2), we have $f(x) = \tau x + b + \int_0^\infty x/(x+t)\sigma(dt) \ (\tau, b \ge 0)$. Thus,

$$\frac{1}{\tilde{\tau}x+\tilde{b}+\int_0^\infty \frac{x}{x+t}\tilde{\sigma}(dt)} = 1+\tau+\frac{b}{x}+\int_0^\infty \frac{1}{x+t}\sigma(dt).$$

If $\tilde{\tau} > 0$, then the process $x \to \infty$ leads to $0 = 1 + \tau$, which contradicts the fact that $\tau \geq 0$. Therefore, the right continuous nondecreasing function $G(t) := \tilde{b} + \int_0^t \tilde{\sigma}(dt)$ (if $x \ge 0$) or 0 (if x < 0) yields the Stieltjes transform

$$\frac{1}{x+f(x)} = \frac{\widetilde{b}}{x} + \int_0^\infty \frac{1}{x+t} \widetilde{\sigma}(dt) = \int_{0^-}^\infty \frac{1}{x+t} d(G(t)).$$

If $\int_{0^-}^{\infty} d(G(t)) > 1$, then $\liminf_{x\to\infty} f(x) = \liminf_{x\to\infty} x(1/\int_{0^-}^{\infty} x(x+t)^{-1} d(G(t)) -1) < 0$, which contradicts the assumption that $f(x) \ge 0$. Thus, we find that $0 \le G(t) \le 1.$

On the other hand, assume that 1/(x + f(x)) is a Stieltjes transform such that $1/(x + f(x)) = \hat{\tau} + \frac{\hat{b}}{x} + \int_0^\infty (x + t)^{-1} \hat{\sigma}(dt)$, where $\hat{\tau}$ and \hat{b} are constants, and $\hat{\sigma}$ is a measure on $(0, \infty)$. Since $f(x) \ge 0$, by applying Lebesgue's theorem, we obtain $\lim_{x\to\infty} 1/(x + f(x)) = 0 = \hat{\tau}$. Put $G(t) = \hat{b} + \int_0^t \hat{\sigma}(dt)$ (if $x \ge 0$) or 0 (if x < 0). Then, we obtain $f(x) = 1/\int_{0^-}^{\infty} (x+t)^{-1} d(G(t)) - x$. As mentioned above, $\int_{0^-}^{\infty} d(G(t)) > 1$ causes a contradiction. Thus, we have $\int_{0^-}^{\infty} (dG(t)) \le 1$. Putting x = u + yi and s = t + u, as in the proof of Lemma 3.2, we observe that if $y \ge 0$, then

(3.2)
$$\operatorname{Im} f(u+yi) = \frac{y\left(\begin{array}{c} \left(1-\int_{0^{-}}^{\infty} dG\right)\int_{0^{-}}^{\infty}\frac{1}{s^{2}+y^{2}}dG + \int_{0^{-}}^{\infty}\frac{1}{s^{2}+y^{2}}dG\int_{0^{-}}^{\infty}\frac{s^{2}}{s^{2}+y^{2}}dG}{-\left(\int_{0^{-}}^{\infty}\frac{1}{\sqrt{s^{2}+y^{2}}}\frac{s}{\sqrt{s^{2}+y^{2}}}dG\right)^{2}}\right)} \\ = \frac{y\left(\begin{array}{c} \left(\int_{0^{-}}^{\infty}\frac{1}{s^{2}+y^{2}}dG\right)^{2} + y^{2}\left(\int_{0^{-}}^{\infty}\frac{1}{s^{2}+y^{2}}dG\right)^{2}\right)}{\left(\int_{0^{-}}^{\infty}\frac{s}{s^{2}+y^{2}}dG\right)^{2} + y^{2}\left(\int_{0^{-}}^{\infty}\frac{1}{s^{2}+y^{2}}dG\right)^{2}} \\ \ge 0. \end{cases}$$

Thus, the analytic function f(u + yi) on $\mathbb{C} \setminus (-\infty, 0]$ (see [15, Corollary VIII.2b.1]) preserves the upper and lower half-planes. This implies, in accordance with Theorem 3.1 (3), that f(x) is a complete Bernstein function.

We characterize the relation between the subset of complete Bernstein functions such that $\tau = 0$ and all the probability measures on $[0, \infty)$.

THEOREM 3.8. A complete Bernstein function $f(x) = \tau x + b + \int_0^\infty x/(x + t)\sigma(dt)$ can be written as $f(x) = 1/\left(\int_{0^-}^\infty (x + t)^{-1}d(G(t))\right) - x$ with a distribution function $0 \le G(t) \le 1$ with $G(\infty) = 1$ and $G(0^-) = 0$ if and only if $\tau = 0$.

PROOF. From Lemma 3.7, for a complete Bernstein function $\tau x + b + \int_0^\infty x/(x+t)\sigma(dt)$, a right continuous nondecreasing function $0 \leq G(t) \leq 1$ exists such that $1/(x + \tau x + b + \int_0^\infty x/(x+t)\sigma(dt)) = \int_{0^-}^\infty (x+t)^{-1}d(G(t))$. Therefore, we observe that $\int_{0^-}^\infty x/(x+t)d(G(t)) = 1/(1+\tau+b/x+\int_0^\infty (x+t)^{-1}\sigma(dt))$, which implies $\int_{0^-}^\infty d(G(t)) = 1/(1+\tau)$ as $x \to \infty$. Thus, if $\tau = 0$, then $\int_{0^-}^\infty d(G(t)) = 1$. In this case, we have $f(x) = 1/(\int_{0^-}^\infty (x+t)^{-1}d(G(t))) -x$. The converse is obtained by applying Lebesgue's theorem to the equation $1/\int_{0^-}^\infty x/(x+t)d(G(t)) - 1 = \tau + b/x + \int_0^\infty (x+t)^{-1}\sigma(dt)$.

LEMMA 3.9. $\lim_{c\to\infty} \Psi^{(n)}(c) = 0 \ (n = 1, 2, 3, ...).$

PROOF. Using (3.1), we have (see [15, Corollary VIII.2b.2])

(3.3)
$$\Psi'(c) = \int_0^\infty \frac{1}{(t+c+\xi)^2} \rho(dt), \Psi''(c) = -\int_0^\infty \frac{2}{(t+c+\xi)^3} \rho(dt), \dots$$
$$\Psi^{(n)}(c) = (-1)^{n-1} n! \int_0^\infty \frac{1}{(t+c+\xi)^{n+1}} \rho(dt).$$

By applying Lebesgue's (monotone convergence) theorem, we conclude that $\lim_{c\to\infty} \Psi^{(n)}(c) = 0$ (n = 1, 2, 3, ...).

LEMMA 3.10. If $E < \infty$, then $\lim_{c\to\infty} c^n \Psi^{(n)}(c) = 0$ (n = 1, 2, 3, ...). PROOF. From Lemma 3.6, $\int_0^\infty t^{-1} \rho(dt) < \infty$. From properties such as

$$c^{n}\Psi^{(n)}(c) = (-1)^{n-1}n! \frac{c^{n}}{(c+\xi)^{n}} \int_{0}^{\infty} \frac{t(c+\xi)^{n}}{(t+c+\xi)^{n+1}} \frac{\rho(dt)}{t}$$
$$\lim_{c \to \infty} \frac{c^{n}}{(c+\xi)^{n}} = 1, \quad \lim_{c \to \infty} \frac{t(c+\xi)^{n}}{(t+c+\xi)^{n+1}} = 0,$$
$$\left| \frac{t(c+\xi)^{n}}{(t+c+\xi)^{n+1}} \right| < 1 \qquad (t, \ c+\xi > 0),$$

we can apply Lebesgue's (dominated convergence) theorem and obtain $\lim_{c\to\infty} c^n \Psi^{(n)}(c) = 0$ (n = 1, 2, 3, ...).

LEMMA 3.11. $\lim_{m\to E^-} \underline{u}_{\max} = \lim_{c\to\infty} c\Psi'(c)$ if one of them exists. In particular, if $E < \infty$, $\lim_{m\to E^-} \underline{u}_{\max} = 0$.

PROOF. From (2.2), we obtain

(3.4)
$$\underline{u}_{\max} = \frac{\left(1 + \frac{\Psi(c_{m+\underline{u}_{\max}})}{c_{m+\underline{u}_{\max}}}\right)}{1 + \Psi'(c_{m+\underline{u}_{\max}})} c_{m+\underline{u}_{\max}} \Psi'(c_{m+\underline{u}_{\max}}).$$

From $\liminf_{m\to E^-} c_{m+\underline{u}_{\max}} = \liminf_{m\to E^-} \Psi^{-1}(m+\underline{u}_{\max}) \ge \liminf_{m\to E^-} \Psi^{-1}(m)$ = ∞ , we obtain $\lim_{m\to E^-} c_{m+\underline{u}_{\max}} = \infty$. Therefore, using Lemmas 3.4 and 3.9, we have $\lim_{m\to E^-} \underline{u}_{\max} = \lim_{c\to\infty} c\Psi'(c)$, provided one of them exists. The rest of this lemma is deduced from Lemma 3.10.

LEMMA 3.12 [10, Lemma 1.1.2]. If $\lim_{c \to (-\xi)^+} \Psi'(c) = \infty$, $\lim_{c \to (-\xi)^+} (-1)^{n-1} \Psi^{(n)}(c) = \infty$ (n = 1, 2, 3, ...).

PROOF. From (3.3), we have $(-1)^{n-1}\Psi^{(n)}(c) = n! \int_0^\infty (t+c+\xi)^{-n-1}\rho(dt) \ge 0$ $(c > -\xi, n = 1, 2, 3, ...)$. Since $\Psi^{(3)}(c) \ge 0$, we observe that $\Psi'(a) - \Psi'(c) = \int_c^a \Psi^{(2)}(x)dx \ge \Psi^{(2)}(c)(a-c)$ for each $-\xi < c < a$. This implies that $\Psi^{(2)}(c) \le (\Psi'(a) - \Psi'(c))/(a-c)$. Thus, using $\lim_{c \to (-\xi)^+} \Psi'(c) = \infty$, we have $\lim_{c \to (-\xi)^+} \Psi^{(2)}(c) = -\infty$. For each $n \in \{3, 4, 5, ...\}$, we find that $(-1)^n(\Psi^{(n-1)}(a) - \Psi^{(n-1)}(c)) = \int_c^a (-1)^n \Psi^{(n)}(x)dx \ge (-1)^n \Psi^{(n)}(c)(a-c)$, which implies that $(-1)^{n-1}\Psi^{(n)}(c) \ge (-1)^{n-2}\Psi^{(n-1)}(c)/(a-c) + (-1)^{n-1}\Psi^{(n-1)}(a)/(a-c)$. Therefore, by induction on n, we arrive at the conclusion.

LEMMA 3.13. $\lim_{c \to (-\xi)^+} (c+\xi)^{n+1} \Psi^{(n)}(c) = 0 \ (n = 1, 2, 3, ...).$

PROOF. From (3.3), we have $(c + \xi)^{n+1}\Psi^{(n)}(c) = (-1)^{n-1}n! \int_0^\infty (c + \xi)^{n+1}/(t + c + \xi)^{n+1}\rho(dt)$. Thus, applying Lebesgue's (monotone convergence) theorem, we arrive at the conclusion.

THEOREM 3.14. If $H_{\xi} = \infty$, then $\lim_{m \to (\xi+1/H_{\xi})^+} \underline{u}_{\max} = 0$. If $H_{\xi} < \infty$, then $\lim_{m \to (\xi+1/H_{\xi})^+} \underline{u}_{\max} > 0$, which can assume any positive value, exists.

PROOF. From (3.3), we have $\Psi'(c) > 0$ and $\Psi''(c) < 0$ for $c > -\xi$. As shown in the proof of Lemma 2.5, $c'_{m+\underline{u}_{\max}} = 1/\Psi'(m + \underline{u}_{\max}) > 0$. Therefore, $\lambda := \lim_{m \to (\xi+1/H_{\xi})^+} c_{m+\underline{u}_{\max}}$ exists such that $-\xi \leq \lambda < \infty$. Since $\underline{u}_{\max} = \Psi(c_{m+\underline{u}_{\max}})$ -m, $\lim_{m \to (\xi+1/H_{\xi})^+} \underline{u}_{\max} = \Psi(\lambda) -\xi -1/H_{\xi}$. From $\Psi''(c) < 0$, $\Psi'(\lambda^+) > 0$ exists (including $+\infty$). From the proof of Lemma 3.11, we observe that \underline{u}_{\max} $= (c_{m+\underline{u}_{\max}} + \Psi(c_{m+\underline{u}_{\max}})) / (1 + 1/\Psi'(c_{m+\underline{u}_{\max}}))$, which induces $\Psi(\lambda) - \xi - 1/H_{\xi}$ $= (\lambda + \Psi(\lambda)) / (1 + 1/\Psi'(\lambda^+))$.

If $H_{\xi} = \infty$, then $\Psi(-\xi) = \xi$. For each $c > -\xi$, put $m := (\Psi(c) - c\Psi'(c)) / (1 + \Psi'(c))$ and $v := \Psi(c) - m$. Then

$$\begin{cases} m+v = \Psi(c), \\ v = (m+c)\Psi'(c). \end{cases}$$

Moreover, from $\partial m/\partial c = -(c + \Psi(c))\Psi''(c)/(1 + \Psi'(c))^2 > 0$, we obtain $\xi < m < E$. Thus, in accordance with (2.2), we can consider $v = \underline{u}_{\max}$ and $c = c_{m+\underline{u}_{\max}}$. Therefore, $-\xi \leq \lambda \leq c_{m+\underline{u}_{\max}} = c$ for each $c > -\xi$, which implies that $\lambda = -\xi$. Thus, $\lim_{m \to (\xi+1/H_{\xi})^+} \underline{u}_{\max} = \Psi(-\xi) - \xi = 0$.

If $H_{\xi} < \infty$, then from $(\lambda + \Psi(\lambda)) / (1 + 1/\Psi'(\lambda^+)) \ge 1/(H_{\xi}(1 + 1/\Psi'(\lambda^+)))$ > 0, $\lim_{m \to (\xi+1/H_{\xi})^+} \underline{u}_{\max} > 0$. In Example 2.7, we observe that $\xi + 1/H_{\xi} = r/3$ and $\underline{u}_{\max} = (3r - m)(m + r)/(m + 9r)$. Thus, we obtain $\lim_{m \to (r/3)^+} \underline{u}_{\max} = 8r/21$ (r > 0), which implies the conclusion.

The following Lemma is similar to [11, Lemma 2.10].

LEMMA 3.15. $|\Psi^{(n+1)}(c)/\Psi^{(n)}(c)| < (n+1)/(c+\xi)$ and $\Psi'(c)/(\Psi(c) - \xi - 1/H_{\xi}) < 1/(c+\xi)$ ($c > -\xi$, n = 1, 2, 3, ...).

PROOF. From (3.3), we have

$$\begin{split} \Psi^{(n+1)}(c) \Big| &= (n+1)! \int_0^\infty \frac{1}{(t+c+\xi)^{n+2}} \rho(dt) \\ &= n! \int_0^\infty \frac{n+1}{t+c+\xi} \times \frac{1}{(t+c+\xi)^{n+1}} \rho(dt) \\ &< \frac{n+1}{c+\xi} \times n! \int_0^\infty \frac{1}{(t+c+\xi)^{n+1}} \rho(dt) \\ &= \frac{n+1}{c+\xi} \left| \Psi^{(n)}(c) \right| \ (n=1,2,3,\ldots). \end{split}$$

Moreover, from (3.1), we observe that

$$(c+\xi)\Psi'(c) = \int_0^\infty \frac{c+\xi}{t(t+c+\xi)}\rho(dt) - \int_0^\infty \frac{(c+\xi)^2}{t(t+c+\xi)^2}\rho(dt) < \Psi(c) - \xi - 1/H_{\xi}.$$

CORORALLY 3.16. $\liminf_{m \to E^-} \underline{u}'_{\max} \ge -1/2$.

PROOF. From Lemma 2.5, we have $\underline{u}_{\max} = (m + c_{m+\underline{u}_{\max}})\Psi'(c_{m+\underline{u}_{\max}})$, $\Psi'(c_{m+\underline{u}_{\max}})$, $c'_{m+\underline{u}_{\max}} = 1$ and $m = \left(\Psi(c_{m+\underline{u}_{\max}}) - c_{m+\underline{u}_{\max}}\Psi'(c_{m+\underline{u}_{\max}})\right)/(1 + \Psi'(c_{m+\underline{u}_{\max}}))$. It follows that

$$\underline{u}_{\max}' = -1 - \frac{1 + \Psi'(c_{m+\underline{u}_{\max}})}{(m + c_{m+\underline{u}_{\max}})c'_{m+\underline{u}_{\max}}\Psi''(c_{m+\underline{u}_{\max}})}$$
$$= -1 - \frac{\Psi'(c_{m+\underline{u}_{\max}})}{(c_{m+\underline{u}_{\max}} + \xi)\Psi''(c_{m+\underline{u}_{\max}})} \times \frac{(c_{m+\underline{u}_{\max}} + \xi)(1 + \Psi'(c_{m+\underline{u}_{\max}}))^2}{c_{m+\underline{u}_{\max}}(1 + \frac{\Psi(c_{m+\underline{u}_{\max}})}{c_{m+\underline{u}_{\max}}})}$$

On the other hand, from to Lemmas 2.1, 3.4, and 3.9, we have

$$\lim_{m \to E^-} \frac{c_{m+\underline{u}_{\max}} + \xi}{c_{m+\underline{u}_{\max}}} \cdot \frac{(1 + \Psi'(c_{m+\underline{u}_{\max}}))^2}{1 + \frac{\Psi(c_{m+\underline{u}_{\max}})}{c_{m+\underline{u}_{\max}}}} = 1.$$

From Lemma 3.15, we observe that $-\Psi''(c)/\Psi'(c) < 2/(c+\xi)$, which implies that $-1 - \Psi'(c)/((c+\xi)\Psi''(c)) > -1/2$. Therefore, we conclude that

$$\lim \inf_{m \to E^-} \underline{u}'_{\max} = \lim \inf_{c \to \infty} \left(-1 - \frac{\Psi'(c)}{(c+\xi)\Psi''(c)} \right) \ge -\frac{1}{2}.$$

LEMMA 3.17. Assume $E < \infty$. Then,

(3.5)
$$\lim_{m \to E^{-}} \frac{\underline{u}_{\max}}{\underline{E}} = \frac{1}{\frac{1}{\lim_{c \to \infty} \frac{c\Psi'(c)}{\overline{E} - \Psi(c)}} + 1} = \frac{1}{\frac{1}{\lim_{c \to \infty} \frac{\int_0^\infty (c/(c+t))^2 \rho(dt)}{\int_0^\infty c/(c+t)\rho(dt)}} + 1}$$

if one of three limits exists.

PROOF. From (2.2) and (3.4), we have

$$\frac{\underline{u}_{\max}}{\underline{E}} = \frac{1}{\frac{E - \Psi(c_{m+\underline{u}_{\max}})}{c_{m+\underline{u}_{\max}}\Psi'(c_{m+\underline{u}_{\max}})\left(1 + \Psi(c_{m+\underline{u}_{\max}})/c_{m+\underline{u}_{\max}}\right)} + \frac{c_{m+\underline{u}_{\max}} + E}{c_{m+\underline{u}_{\max}}\Psi(c_{m+\underline{u}_{\max}})}}$$

Thus, from Lemmas 1.1, 2.1, 3.4, and 3.10, we obtain $\lim_{m\to E^-} \underline{u}_{\max}/\underline{E} = 1/(1/\lim_{c\to\infty} (c\Psi'(c)))$ $/(E - \Psi(c))) + 1$. Using (3.1) and (3.3) we observe that

(3.6)
$$\frac{c\Psi'(c)}{E-\Psi(c)} = \frac{\int_0^\infty \left((c+\xi)/(t+c+\xi)\right)^2 \rho(dt)}{(1+\frac{\xi}{c})\int_0^\infty (c+\xi)/(t+c+\xi)\rho(dt)},$$

which yields the desired equation.

LEMMA 3.18.

(3.7)
$$-\lim_{m \to E^{-}} \underline{u}'_{\max} = 1 + \lim_{c \to \infty} \frac{\Psi'(c)}{(c+\xi) \Psi''(c)} = 1 - \lim_{c \to \infty} \frac{\int_{0}^{\infty} (c/(c+t))^{2} \rho(dt)}{2 \int_{0}^{\infty} (c/(c+t))^{3} \rho(dt)}$$

if one of three limits exists. In this case, if $E < \infty$, its value is equal to $\lim_{m \to E^-} \underline{u}_{\max}$ $/\underline{E}$.

PROOF. From (3.3), we have $\Psi'(c)/((c+\xi) \Psi''(c)) = -\int_0^\infty ((c+\xi)/(t+c+c)) dt dt$ $\xi(t)^2 \rho(dt) / \left(2 \int_0^\infty ((c+\xi)/(t+c+\xi))^3 \rho(dt)\right).$ From the proof of Corollary 3.16, we have $\lim_{m \to E^-} \underline{u}'_{\max} = -1 - \lim_{c \to \infty} \Psi'(c)/((c+\xi) \Psi''(c))$ if one of them exists. In this case, if $E < \infty$, then from $\lim_{m \to E^-} \underline{u}_{\max} = 0$ (Lemma 3.11) and by using the mean value theorem, we obtain $\lim_{m\to E^-} \underline{u}_{\max}/\underline{E} = -\lim_{m\to E^-} \int_m^E \underline{u}'_{\max}(t) dt/\underline{E}$

THEOREM 3.19. If $\int_0^{\infty} \rho(dt) < \infty$ and $\int_0^{\infty} 1/(t(t+1))\rho(dt) < \infty$, we have $E < \infty$ (Lemma 3.6). By applying Lebesgue's theorem, we observe that $\lim_{c\to\infty} (\int_0^{\infty} c/(c+1))\rho(dt) < \infty$. $t)\rho(dt) / \int_0^\infty (c/(c+t))^2 \rho(dt) = 1$. Thus, from Lemma 3.17, we obtain $\lim_{m \to E^-} \underline{u}_{\max}$ $/\underline{E} = 1/2.$

CORORALLY 3.20. $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} = 1/2$ for any finite game $\{(a_j, p_j)\}$ such that $\sum_{j=1}^n a_j p_j > 0$, $\sum_{j=1}^n p_j = 1$, $0 \le p_j < 1$, and $1 \le j \le n$.

PROOF. The complete Bernstein function

$$\Psi(c) - \xi - \frac{1}{H_{\xi}} = \frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j + c}} - c - \xi - 1/H_{\xi} \quad (c > -\xi)$$

is a rational function with respect to c, which is analytic on $\mathbb{C}\setminus(-\infty, -\xi]$ and preserves the upper and lower half-planes (Theorems 3.1 and 3.3). Therefore, using [8, Theorem 2.2 (vi)], we obtain the representation $\Psi(c) - \xi - 1/H_{\xi} = -\sum_{j=1}^{m} e_j/(c)$ $(+d_j) + k$, where $e_j > 0$, $d_j > \xi$ and k is a constant. Since $\Psi(-\xi) = \xi + 1/H_{\xi}$, $k = \sum_{j=1}^m e_j/(d_j - \xi)$ holds. Defining $\sigma(dt)$ as the sum of Dirac measures $\sum_{j=1}^m e_j/(d_j - \xi)$. $-\xi$) $\delta_{d_i-\xi}$, we observe that

$$\int_0^\infty \frac{c+\xi}{t+c+\xi} \sigma(dt) = -\sum_{j=1}^m \frac{e_j}{c+d_j} + \sum_{j=1}^m \frac{e_j}{d_j-\xi} = \Psi(c) - \xi - \frac{1}{H_\xi}$$

Using Theorem 3.1, we obtain $\int_0^\infty \rho(dt) = \int_0^\infty t\sigma(dt) = \sum_{j=1}^m e_j < \infty$, which, in accordance with Theorem 3.19, implies the conclusion.

4. Abelian theorems

In the following paragraphs, we assume that a nonzero measure $\rho(dt)$ originates from (3.1). For a function f(x) > 0, $\omega_f := \limsup_{x \to \infty} \log f(x) / \log x$ is termed the upper order (see [1, Section 2.2.2]). We will show that $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E}$ can be calculated by the upper order of the function $\int_0^x \rho(dt)$.

A measurable function f(x) > 0 is said to be regularly varying of index r, written as $f \in R_r$, if $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^r$ for each $\lambda > 0$ (see [1, Section 1.4.2]). It is easy to verify that $\omega_f = r$ if $f \in R_r$. The notation l(x) is used only for a slowly varying function such that $l(x) \in R_0$. We write $f(x) \sim cg(x)$ when $\lim_{x\to\infty} f(x)/g(x) = c$. If c = 0, the relation $f(x) \sim cg(x)$ suggests that f(x) = o(g(x)) (see [1, Preface]).

LEMMA 4.1. If $\int_0^x \rho(dt) \in R_r$, $0 \le r \le 2$. In this case, if $E < \infty$, $0 \le r \le 1$.

PROOF. We can write $\int_0^x \rho(dt) = x^r l(x)$ with $l(x) \in R_0$. Assuming r < 0, then $\lim_{x\to\infty} \int_0^x \rho(dt) = \lim_{x\to\infty} x^r l(x) = 0$ (see [1, Proposition 1.3.6]), which contradicts the fact that $\int_0^\infty \rho(dt) > 0$. Assuming r > 2, we have $\lim_{x\to\infty} \int_0^x \rho(dt) / (x(x+1)) = \lim_{x\to\infty} x^{r-2} l(x) \cdot \lim_{x\to\infty} 1/(1+1/x) = \infty$. On the other hand, for each x > 0, we have $\int_0^x \rho(dt) / (x(x+1)) \le \int_0^\infty (t(t+1))^{-1} \rho(dt) < \infty$ (Lemma 3.5), which is a contradiction.

If $E < \infty$, then $\int_0^\infty t^{-1} \rho(dt) < \infty$ (Lemma 3.6). Thus, for each x > 0, we have $\int_0^x \rho(dt)/x \le \int_0^\infty t^{-1} \rho(dt) < \infty$, which implies that $r \le 1$.

LEMMA 4.2. Suppose $0 \le r < n$. Then, $\int_0^x \rho(dt) \sim x^r l(x) \ (x \to \infty)$ if and only if $\int_0^\infty (x/(x+t))^n \rho(dt) \sim \Gamma(n-r)\Gamma(r+1)x^r l(x)/\Gamma(n) \ (x \to \infty)$.

PROOF. The nondecreasing function $U(x) := \int_0^x \rho(dt)$ satisfies $U(0^-) = 0$. Since $0 < n - r \le n$, using [1, Theorem 1.7.4], we obtain that $U(x) \sim x^r l(x)$ $(x \to \infty)$ is equivalent to

$$\int_0^\infty \frac{d(U(t))}{(x+t)^n} \sim \frac{\Gamma(n-r)\Gamma(r+1)}{\Gamma(n)} x^{r-n} l(x) \quad (x \to \infty),$$

which implies the conclusion.

LEMMA 4.3. If $\int_0^x \rho(dt) \in R_r$ and $r \neq 2$, then $\lim_{c\to\infty} \Psi'(c)/(c\Psi''(c)) = 1/(r-2)$.

PROOF. From Lemma 4.1, we have $0 \le r < 2$. From Lemma 4.2, we observe that

$$\lim_{x \to \infty} \frac{\int_0^\infty (x/(x+t))^2 \rho(dt)}{\int_0^\infty (x/(x+t))^3 \rho(dt)} = \frac{\Gamma(3)\Gamma(2-r)}{\Gamma(2)\Gamma(3-r)} = \frac{2}{2-r}.$$

Therefore, the relation (3.7) implies the conclusion.

THEOREM 4.4. If $E < \infty$ and $\int_0^x \rho(dt) \in R_r$, then $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} = (1-r) / (2-r)$ with $0 \le r \le 1$.

PROOF. From Lemmas 3.18 and 4.3, we obtain $\lim_{m\to E^-} \underline{u}_{\max} / \underline{E} = 1 + \lim_{c\to\infty} \Psi'(c) / ((c+\xi)\Psi''(c)) = (1-r)/(2-r).$

Whenever we use the notation q(t), it is understood that $\rho(dt) = q(t)dt$ with $q(t) \ge 0$.

LEMMA 4.5. If $q(t) \in R_{\alpha}$, then $\alpha \leq 1$. In addition, if $E < \infty$, then $\alpha \leq 0$.

PROOF. We can write $q(t) = t^{\alpha}l(t)$ with $l(t) \in R_0$. From [1, Corollary 1.4.2], X > 0 exists such that l(x) is locally bounded in $[X, \infty)$. Assuming $\alpha > 1$, then using [1, Propositions 1.3.6 and 1.5.8], we obtain $\lim_{x\to\infty} \int_X^x \rho(dt)/(x(x+1))$ $= \lim_{x\to\infty} x^{\alpha-1}l(x)/(\alpha+1) \cdot \lim_{x\to\infty} 1/(1+1/x) = \infty$. On the other hand, for

each x > X, we observe that $\int_X^x \rho(dt)/(x(x+1)) \le \int_X^x (t(t+1))^{-1} \rho(dt) \le \int_0^\infty (t(t+1))^{-1} \rho(dt) < \infty$ (Lemma 3.5), which is a contradiction. When $E < \infty$, we have $\int_X^x \rho(dt)/x \le \int_0^\infty t^{-1} \rho(dt) < \infty$ (Lemma 3.6). Thus, arguments similar to the one above yield the conclusion.

 $(\alpha \leq -1).$

PROOF. As the proof of Lemma 4.5, if $\alpha > -1$, using [1, Proposition 1.5.8], we obtain $\int_X^x t^{\alpha} l(t) dt \sim x^{\alpha+1} l(x)/(\alpha+1) \in R_{\alpha+1}$. If $\alpha = -1$, then from [1, Proposition 1.5.9a], we have $\int_X^x t^{-1}l(t)dt \in R_0$. If $\alpha < -1$, then the nondecreasing function $\int_0^x q(t)dt = \int_0^x t^{\alpha}l(t)dt$ is bounded as will be shown below, which suggests that $\int_0^x q(t)dt \in R_0$. Put $\varepsilon := -(\alpha + 1)/2 > 0$. Then, from $\lim_{t\to\infty} t^{-\varepsilon}l(t) = 0$, Y > 0 exists such that $0 \le t^{-\varepsilon}l(t) \le 1$ for each $t \ge Y$. Therefore, for each $x \ge Y$, we find that $\int_0^x q(t)dt \le \int_0^Y q(t)dt + \int_Y^x t^{-\varepsilon}l(t)t^{-1-\varepsilon}dt \le \int_0^Y q(t)dt + 1/(\varepsilon Y^{\varepsilon}) < \infty$.

CORORALLY 4.7. If $E < \infty$ and $q(t) \in R_{\alpha}$, then $\alpha \leq 0$ and

(4.1)
$$\lim_{m \to E^-} \frac{\underline{u}_{\max}}{\underline{E}} = \begin{cases} \frac{1}{2}, & \text{if } \alpha \leq -1, \\ \frac{\alpha}{\alpha - 1}, & \text{if } -1 < \alpha \leq 0. \end{cases}$$

PROOF. It is the direct consequence of Theorem 4.4 and Lemmas 4.5 and 4.6.

5. Tauberian theorems

Given a measurable function $f: (0,\infty) \to \mathbb{R}$, let $\check{f}(z) := \int_0^\infty t^{-z-1} f(t) dt$ be its Mellin transform for $z \in \mathbb{C}$ such that the integral converges absolutely (see [1, 2, 3]). For example, putting $k(x) := 2x^2$ (0 < x < 1) or 0 ($x \ge 1$), we obtain k(z) = 2/(2-z) ($-\infty < \operatorname{Re} z < 2$). In addition, putting h(x) := x (0 < x < 1) or 0 $(x \ge 1)$, we obtain $\check{h}(z) = 1/(1-z)$ $(-\infty < \operatorname{Re} z < 1)$. Given measurable functions $(x \ge 1)$, we obtain h(x) = 1/(1-x) ($-\infty$) ($-\infty$ of these functions for x > 0 such that the integral converges absolutely.

THEOREM 5.1. If $E < \infty$ and $\lim_{m \to E^-} \underline{u}'_{\max}$ exists, then $\lim_{m \to E^-} \underline{u}_{\max}/\underline{E}$ = (1-r)/(2-r) ($0 \le r \le 1$) and $\int_0^x \rho(dt) \in R_r$, where r is the upper order of $\int_0^\infty (x/(x+t))^n \rho(dt)$ (n > 1).

PROOF. Putting $K(x) := \int_0^\infty (x/(x+t))^3 \rho(dt)$, we observe that $\check{k}(2^-) = \infty$ and

$$\frac{\left(k*K\right)\left(x\right)}{K(x)} = \frac{\int_{0}^{\infty} \left(x/(x+t)\right)^{2} \rho(dt)}{\int_{0}^{\infty} \left(x/(x+t)\right)^{3} \rho(dt)} \ge 1$$

From Lemma 3.18, $\lim_{x\to\infty} (k * K)(x)/K(x) = c \ge 1$ exists. As K(x) is an increasing function, from [1, Theorem 5.2.3 and Section 2.1.2], we obtain c = $k(\omega_K) = 2/(2-\omega_K), \omega_K < 2 \text{ and } K(x) \text{ is regularly varying. Thus, } \lim_{m \to E^-} \underline{u}_{\max}/\underline{E}$ $= -\lim_{m \to E^-} \underline{u}'_{\max} = 1 - 1/(2 - \omega_K) = (1 - \omega_K)/(2 - \omega_K).$ From Lemma 4.2, it follows that $\int_0^x \rho(dt) \in R_{\omega_K}$. Moreover, from Lemma 4.1, we have $0 \le \omega_K \le 1$. This implies that the upper order of $\int_0^\infty (x/(x+t))^n \rho(dt)$ is always ω_K for each n > 1.

THEOREM 5.2. If $E < \infty$ and $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} \neq 0$ exists, then $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} = -\lim_{m \to E^-} \underline{u}'_{\max} = (1-r)/(2-r) \ (0 \le r < 1)$ and $\int_0^x \rho(dt) \in R_r$, where r is the upper order of $\int_0^\infty (x/(x+t))^n \rho(dt) \ (n > 1)$. PROOF. This proof is formally the same as that in Theorem 5.1. Putting

 $S(x) := \int_0^\infty (x/(x+t))^2 \rho(dt)$, we observe that $\check{h}(1^-) = \infty$ and

$$\frac{(h*S)(x)}{S(x)} = \frac{\int_0^\infty x/(x+t)\rho(dt)}{\int_0^\infty (x/(x+t))^2 \rho(dt)} \ge 1.$$

From Lemma 3.17, $\lim_{x\to\infty} (h * S)(x)/S(x) = c \ge 1$ exists. As S(x) is an increasing function, from [1, Theorem 5.2.3 and Section 2.1.2], we obtain $c = \check{h}(\omega_S) = 1/(1$ $-\omega_S$), $\omega_S < 1$, and that S(x) is regularly varying. Thus, $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} = 1/(c+1) = (1-\omega_S)/(2-\omega_S)$. From Lemma 4.2, it follows that $\int_0^x \rho(dt) \in R_{\omega_S}$. Moreover, from Lemma 4.1, we have $0 \leq \omega_S < 1$. This implies that the upper order of $\int_0^\infty (x/(x+t))^n \rho(dt)$ is always ω_K for each n > 1. In this case, based on Lemmas 3.18 and 4.3, $\lim_{m\to E^-} \underline{u}'_{\text{max}}$ exists.

CORORALLY 5.3. If $E < \infty$, the following are equivalent.

- (a) $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} \neq 0$ exists.
- (b) $\int_0^x \rho(dt) \in R_r \ (r \neq 1).$ (c) $\lim_{m \to E^-} \underline{u}'_{\max} \neq 0$ exists.

PROOF. Using Theorems 4.4, 5.1, and 5.2, and Lemmas 3.18 and 4.3, we arrive at the conclusion.

It is noteworthy that $\int_0^x \rho(dt) \in R_1$ includes $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} = 0$. However, the converse is not necessarily true because a nonregularly varying function $\int_{e}^{x} \rho(dt)$

 $= (2 + \sin(\log x))x/(1 + \log x)^{3/2} - (2 + \sin 1)e/(2\sqrt{2})$ ($x \ge e$) provides an example with $\lim_{m\to E^-} \underline{u}_{\max}/\underline{E} = 0$. The details are left to the reader. In this direction, we observe that $\lim_{m\to E^-} \underline{u}_{\max}/\underline{E} = 0$ if and only if $\int_0^\infty (x+t)^{-1}\rho(dt)$ is normalized slowly varying. Because, since $\lim_{c\to\infty} c\Psi'(c)/(E - \Psi(c)) = 0$ (Lemma 3.17), $E - \Psi(c) = \int_0^\infty (t+c+\xi)^{-1}\rho(dt)$ (Lemma 3.5) is normalized slowly varying (see [1, (1.3.4)]).

LEMMA 5.4 [1, Theorem 1.7.2]. If $\int_0^x f(t)dt \sim cx^r l(x) \ (x \to \infty)$, where f(x) is nondecreasing or nonincreasing in an interval (T, ∞) (T > 0), then $f(x) \sim$ $crx^{r-1}l(x) \ (x \to \infty)$

CORORALLY 5.5. Assuming $E < \infty$, $\rho(dt) = q(t)dt$ and q(t) is nonincreasing in an interval (T, ∞) (T > 0). When $\lim_{m \to E^-} \underline{u}_{\max} / \underline{E} \neq 0$ exists, the following properties hold:

(1) If $\omega_q \leq -1$, then $\lim_{m \to E^-} \underline{u}_{\max}/\underline{E} = 1/2$ and $\int_0^x q(t)dt$ is slowly varying. (2) If $\omega_q > -1$, then $\omega_q < 0$, $\lim_{m \to E^-} \underline{u}_{\max}/\underline{E} = -\omega_q/(1-\omega_q)$, and q(t) is regularly varying.

PROOF. Put $S(x) := \int_0^\infty (x/(x+t))^2 q(t) dt$. Then, by applying Theorem 5.2, we obtain $\lim_{m\to E^-} \underline{u}_{\max}/\underline{E} = (1-\omega_S)/(2-\omega_S)$, $(0 \le \omega_S < 1)$ and $\int_0^x q(t)dt \in R_{\omega_S}$. Thus, by Lemma 5.4 we find that $\int_0^x q(t)dt \sim x^{\omega_S} l(x) \ (x \to \infty)$ and q(t) $\sim \omega_S t^{\omega_S - 1} l(t) \ (t \to \infty).$

(1) Assume $\omega_q < -1$. From $\limsup_{t\to\infty} \log q(t) / \log t < -1$, we obtain $\int_0^\infty q(t) dt$ $<\infty, \int_0^x q(t)dt \in R_0$, and $\omega_S = 0$. Next, assume $\omega_q = -1$. If $\omega_S \neq 0$, we have $\omega_q = \omega_S - 1 = -1$, which is a contradiction.

(2) Assume $\omega_q > -1$ and $\omega_S \neq 0$. Then, we find that $\omega_q = \omega_S - 1 < 0$ and $\lim_{m\to E^-} \underline{u}_{\max} / \underline{E} = -\omega_q / (1 - \omega_q).$ Next, assume $\omega_q > -1$ and $\omega_S = 0$. Then, $q(t) = o(t^{-1}l(t))$ $(t \to \infty)$ and $\omega_q \leq -1$, thus contradicting the assumption. \Box

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