

# Least-Squares Prices of Games

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## Abstract

What are the prices of random variables? In this paper, we define the least-squares prices of coin-flipping games, which are proved to be minimal, positive linear, and arbitrage-free. These prices depend both on a set of games that are available for investing simultaneously and on a risk-free interest rate. In addition, we show a case in which the mean-variance portfolio theory is inappropriate in our incomplete market.

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## 1. Introduction

Consider the following two coin-flipping games:

**Game A** : Profit is 19 or 1 if a tossed coin yields heads or tails, respectively.

**Game B** : Profit is 10 if a tossed coin yields heads or tails.

In general, game  $B$  is preferable to game  $A$  (see [7, Example 9.2]). Despite the fact that the expectations concerning the two games are equal, the price of  $B$  should be higher than that of  $A$ . However, if game  $C$  is available for investing simultaneously, the three prices of these games should be the same; this is because the mixed game  $(A + C)/2$  is equal to  $B$ .

**Game C** : Profit is 1 or 19 if a tossed coin yields heads or tails, respectively.

Therefore, the price of a game should change in accordance with the set of games that are available for investing simultaneously. As F. Black and M. Scholes demonstrate, the price of an option depends on the risk-free continuously compound interest rate  $r > 0$  (see [1, page 643]). It is noteworthy that if  $r = 0$ , no investor will invest his/her money, because no gain is expected. In this paper (except in Remarks 3.5 and 3.6), we assume that  $r$  is 0.05. The term “arbitrage-free” implies that no investor has an opportunity to earn a profit exceeding the risk-free interest rate.

Here, we introduce the pricing method of a coin-flipping game.

**Theorem 1.1.** *Suppose that a game  $A := (a, b)$  involves a profit  $a$  or  $b$  ( $a, b > 0$ ) if a tossed coin yields heads or tails, respectively. Put  $E^A := (a + b)/2$ . If  $E^A/\sqrt{ab} \leq e^r$ , then the price of game  $A$  is given by  $u_r^A = \sqrt{ab}/e^r$ , and the optimal proportion of investment is 1. Otherwise,  $u_r^A = \kappa a + (1 - \kappa)b$  (if  $a \geq b$ ) or  $\kappa b + (1 - \kappa)a$  (if  $a < b$ ), where  $\kappa := (1 - \sqrt{1 - 1/e^{2r}})/2$ , and the optimal proportion of investment is  $u_r^A(E^A - u_r^A) / ((a - u_r^A)(u_r^A - b))$ .*

**Proof.** Using Remark 3.1 under the conditions of this theorem and solving the simultaneous quadratic equations, we obtain the conclusion.  $\square$

**Game A** : As  $E^A/\sqrt{ab} = 10/\sqrt{19} \doteq 2.294 > e^{0.05} \doteq 1.051$  and  $\kappa \doteq 0.3458$ , we obtain the price  $u_r^A \doteq 7.224$  and the optimal proportion of investment  $t_{u_r^A} \doteq 0.274$ .

Now, we explain the term “optimal proportion of investment.” Let  $t \in [0, 1]$  be a proportion of investment; then, the investor repeatedly invests  $t$  of his/her current capital. For example, let  $c$  be the current capital; when the investor plays game  $A = (19, 1)$  once, his/her capital will be  $19ct/u + c(1 - t)$  or  $ct/u + c(1 - t)$  if a tossed coin

yields heads or tails, respectively, where  $u$  is the price of the game. Let the initial capital be 1. After  $N$  attempts, if the investor has capital  $c_N$ , then the growth rate (geometric mean) is given by  $c_N^{1/N}$ . As the value  $\lim_{N \rightarrow \infty}$  (expectation of  $c_N^{1/N}$ ) is a function with respect to  $t \in [0, 1]$ , it reaches its maximum of  $G_u$  at  $t = t_u$ , where  $G_u$  is called the limit expectation of the growth rate. The price  $u_r^A$  is determined by the equation  $G_u = e^r$ . Thus, the optimal proportion of investment  $t_{u_r^A}$  is determined (see Remark 3.1). It is noteworthy that the value  $\lim_{N \rightarrow \infty}$  (variance of  $c_N^{1/N}$ ) is 0.

**Game B**: As  $E^B/\sqrt{ab} = 10/\sqrt{100} = 1 < e^{0.05} \doteq 1.051$ , we obtain the price  $u_r^B \doteq 9.512$ . In this case, the optimal proportion of investment is 1. This implies that the investor should invest his/her entire current capital in each attempt.

In Section 2, we will introduce the least-squares price  $u_r^{A, \Omega}$  of game  $A$  in a set  $\Omega$  of games and prove some properties of  $u_r^{A, \Omega}$ .

**Example 1.1.**  $\Omega = \{(19, 1), (4, 16)\}$ . Using Theorem 1.1, we have  $u_r^{(19, 1)} \doteq 7.224$  and  $u_r^{(4, 16)} \doteq 8.149$ . As  $0.4(19, 1) + 0.6(4, 16) = (10, 10)$ , using Lemma 2.2, we obtain  $u_r^{(19, 1), \Omega} = u_r^{(4, 16), \Omega} = 10/e^r \doteq 9.512$ , where each price reaches its maximum.

**Example 1.2.**  $\Omega = \{(19, 1), (16, 4)\}$ . As Example 1.1, we have  $u_r^{(19, 1)} \doteq 7.224$  and  $u_r^{(16, 4)} \doteq 8.149$ . Observe that  $p(19, 1) + (1-p)(16, 4) = (3p+16, 4-3p)$  and  $10/\sqrt{(3p+16)(4-3p)} \geq 5/4 > e^{0.05} \doteq 1.051$  for each  $p \in [0, 1]$ . In this case, using Lemma 2.3 with the linearity  $u_r^{(3p+16, 4-3p)} = 6p\kappa + 12\kappa - 3p + 4$ , we obtain  $u_r^{(19, 1), \Omega} = u_r^{(19, 1)}$  and  $u_r^{(16, 4), \Omega} = u_r^{(16, 4)}$ , where each price is unchanged.

**Example 1.3.**  $\Omega = \{(12, 8), (11, 9)\}$ . Using Theorem 1.1, we have  $u_r^{(12, 8)} \doteq 9.320$  and  $u_r^{(11, 9)} \doteq 9.465$ . Observe that  $p(12, 8) + (1-p)(11, 9) = (p+11, 9-p)$  and  $10/\sqrt{(p+11)(9-p)} \leq 5/(2\sqrt{6}) \doteq 1.021 < e^{0.05} \doteq 1.051$  for each  $p \in [0, 1]$ . In this case, by the fact that  $u_r^{(p+11, 9-p)} = \sqrt{(p+11)(9-p)}/e^r$  and using numerical calculations according to Definition 2.1, we obtain  $u_r^{(12, 8), \Omega} \doteq 9.345$  and  $u_r^{(11, 9), \Omega} \doteq 9.469$ , where  $u_r^A < u_r^{A, \Omega} < E^A/e^r$  for each  $A \in \Omega$ .

For a better understanding of the background, we present our incomplete market assumptions as follows (compare with [8, Sections 2.1 and 8.2]).

1. Frictionless Market: There are no transactions costs or taxes, and all securities are perfectly divisible.

2. Price-Taker: The investor's actions cannot affect the probability distribution of returns on the securities. Every security has a positive expectation.

3. No Arbitrage Opportunities: There exists a unique riskless standard asset, that is not necessarily tradable. Further, there exists a security, wherein the limit expectation of the growth rate is equal to that of the riskless standard asset. The limit expectation of the growth rate of any security never exceeds that of the standard asset. The standard asset is usually provided by the riskless rate of interest.

4. No Short Sales: Combined with suitable transactions, all necessary short sales must be included in the securities (probability distribution of returns), that have positive expectations. For example,  $-(19, 1) + 2(16, 4) = (13, 7)$ .

## 2. Least-Squares Prices

Let  $\Psi := \{G_j := (c_j, d_j) : c_j, d_j > 0, j = 1, 2, \dots, m\}$  be a finite set of coin-flipping games, which are completely correlated. Denote the convex cone  $\{\sum_{j=1}^m k_j G_j : k_j \geq 0, j = 1, 2, \dots, m\}$  by  $\widehat{\Psi}$ . Then, a basis  $\Omega := \{A_i : i = 1 \text{ or } i = 1, 2\}$  exists such that  $\widehat{\Psi} = \widehat{\Omega}$  (see Remark 3.3). This is because, if  $\min_{j=1,2,\dots,m} c_j/d_j = \max_{j=1,2,\dots,m} c_j/d_j$ , we can choose  $\Omega := \{A_1 := G_1\}$ . If not, we can choose  $\Omega := \{A_1 := G_{j_0}, A_2 := G_{j_1}\}$  such that  $c_{j_0}/d_{j_0} = \min_{j=1,2,\dots,m} c_j/d_j$  and  $c_{j_1}/d_{j_1} = \max_{j=1,2,\dots,m} c_j/d_j$ . Since the set  $\Omega = \{A_i, 1 \leq i \leq n\}$  ( $n = 1$  or  $2$ ) is a basis of the convex cone  $\widehat{\Omega}$ , if  $\sum_{i=1}^n k_i A_i = \sum_{i=1}^n k'_i A_i$ , then  $k_i = k'_i$  for each  $1 \leq i \leq n$ .

Set  $S := \{(t_i) \in R^n : 0 \leq t_i \leq 1, 1 \leq i \leq n\}$  and  $Q := \{(p_i) \in R^n : \sum_{i=1}^n p_i = 1, p_i \geq 0, 1 \leq i \leq n\}$ .

From Theorem 1.1, we can verify that  $0 < u_r^A \leq E^A/e^r$  and  $u_r^{kA} := u_r^{(ka, kb)} = k u_r^A$  for each  $A = (a, b)$  and  $k > 0$ .

**Definition 2.1.** By defining the function

$$(2.1) \quad L((t_i)) := \sup_{(p_i) \in Q} \frac{u_r^{\sum_{i=1}^n p_i A_i}}{\sum_{i=1}^n p_i (u_r^{A_i} + t_i (E^{A_i}/e^r - u_r^{A_i}))} \quad ((t_i) \in S),$$

we have  $L((0)) = \sup_{(p_i) \in Q} (u_r^{\sum_{i=1}^n p_i A_i} / \sum_{i=1}^n p_i u_r^{A_i}) \geq u_r^{A_1}/u_r^{A_1} = 1$  and

$$L((1)) = \sup_{(p_i) \in Q} \frac{u_r^{\sum_{i=1}^n p_i A_i}}{\sum_{i=1}^n p_i E^{A_i}/e^r} = \sup_{(p_i) \in Q} \frac{u_r^{\sum_{i=1}^n p_i A_i}}{E^{\sum_{i=1}^n p_i A_i}/e^r} \leq 1.$$

Since the set  $T := \{(t_i) \in S : L((t_i)) \leq 1\}$  is not null, convex, closed, and thus compact, there is a unique point  $(x_i) \in T$  such that  $v := \min_{(t_i) \in T} \sum_{i=1}^n t_i^2 = \sum_{i=1}^n x_i^2$ . Define  $u_r^{A_i, \Omega} := u_r^{A_i} + x_i (E^{A_i}/e^r - u_r^{A_i})$  and call it the *least-squares price of  $A_i$  in  $\Omega$*  for each  $1 \leq i \leq n$ . For each mixed game  $\sum_{i=1}^n k_i A_i \in \widehat{\Omega}$ , define  $u_r^{\sum_{i=1}^n k_i A_i, \Omega} := \sum_{i=1}^n k_i u_r^{A_i, \Omega}$  (see [7, Section 9.6]).

**Lemma 2.2.** *If  $(p_i) \in Q$  exists such that  $\sum_{i=1}^n p_i A_i$  is constant and  $p_k \neq 0$ , then  $u_r^{A_k, \Omega} = E^{A_k}/e^r$ .*

**Proof.** Write  $B := \sum_{i=1}^n p_i A_i$ ; then as  $B$  is constant,  $u_r^B = E^B/e^r$ . From Definition 2.1, we obtain  $u_r^{A_i} \leq u_r^{A_i, \Omega} \leq E^{A_i}/e^r$  and  $u_r^B \leq \sum_{i=1}^n p_i u_r^{A_i, \Omega} \leq \sum_{i=1}^n p_i E^{A_i}/e^r = u_r^B$ . Thus,  $u_r^{A_k, \Omega} = E^{A_k}/e^r$  if  $p_k \neq 0$ .  $\square$

**Lemma 2.3.** *If  $u_r^{\sum_{i=1}^n p_i A_i} = \sum_{i=1}^n p_i u_r^{A_i}$  for each  $(p_i) \in Q$ , then  $u_r^{A_i, \Omega} = u_r^{A_i}$  ( $1 \leq i \leq n$ ).*

**Proof.** By the above assumption, we obtain  $L((0)) = 1$  and  $v = 0$ , which implies the conclusion.  $\square$

**Theorem 2.4.** *The system of least-squares prices is arbitrage-free, and there is a mixed game that earns profit equal to the growth rate of  $e^r$ .*

**Proof.** As  $T \subset S$  and  $Q$  are compact, and  $u_r^{\sum_{i=1}^n p_i A_i}$  is continuous with respect to  $(p_i) \in Q$  (see Theorem 1.1),  $(x_i) \in T$  and  $(q_i) \in Q$  exist such that

$$(2.2) \quad L((x_i)) = \max_{(p_i) \in Q} \frac{u_r^{\sum_{i=1}^n p_i A_i}}{\sum_{i=1}^n p_i (u_r^{A_i} + x_i (E^{A_i}/e^r - u_r^{A_i}))} = \frac{u_r^{\sum_{i=1}^n q_i A_i}}{\sum_{i=1}^n q_i u_r^{A_i, \Omega}} = 1.$$

This shows that the mixed game  $\sum_{i=1}^n q_i A_i$  earns profit that is equal to the growth rate of  $e^r$ . On the other hand, for each nonzero mixed game  $\sum_{i=1}^n k_i A_i = k \sum_{i=1}^n p_i A_i$

$\in \widehat{\Omega}$  ( $k := \sum_{i=1}^n k_i$ ,  $k > 0$ ,  $p_i := k_i/k$ ,  $(p_i) \in Q$ ), by equation (2.2), we have  $u_r^{\sum_{i=1}^n k_i A_i} = k u_r^{\sum_{i=1}^n p_i A_i} \leq k \sum_{i=1}^n p_i u_r^{A_i, \Omega} = \sum_{i=1}^n k_i u_r^{A_i, \Omega}$ . Therefore, the game  $\sum_{i=1}^n k_i A_i$  earns profit that is equal to or less than the growth rate of  $e^r$ .  $\square$

**Theorem 2.5.** *The system of least-squares prices is minimal in order to be arbitrage-free.*

**Proof.** We prove this by using reduction to absurdity. Assuming that a set of prices  $\{R_i\}$  of  $\{A_i\}$  exists such that  $R_i \leq u_r^{A_i, \Omega}$  ( $1 \leq i \leq n$ ) and  $R_k < u_r^{A_k, \Omega}$  for some  $k$ . If  $R_j < u_r^{A_j}$  for some  $j$ , then the game  $A_j$  earns profit exceeding the growth rate of  $e^r$ . Thus, we can assume that  $u_r^{A_i} \leq R_i$  ( $1 \leq i \leq n$ ). Therefore,  $(s_i) \in S$  exists such that  $R_i = u_r^{A_i} + s_i(E^{A_i}/e^r - u_r^{A_i})$ , where  $s_j := 0$  is chosen if  $u_r^{A_j} = E^{A_j}/e^r$ . It is easy to verify that  $s_i \leq x_i$  ( $1 \leq i \leq n$ ) and  $s_k < x_k$ . From the above statement, we have  $\sum_{i=1}^n s_i^2 < \sum_{i=1}^n x_i^2$ , which implies  $(s_i) \notin T$ , and thus  $L((s_i)) > 1$ . Therefore, a point  $(q_i) \in Q$  exists such that  $\sum_{i=1}^n q_i R_i < u_r^{\sum_{i=1}^n q_i A_i}$ , that is, the mixed game  $\sum_{i=1}^n q_i A_i$  earns profit exceeding the growth rate of  $e^r$ .  $\square$

It is not difficult to verify that if  $\Omega = \{A_i : 1 \leq i \leq n\}$  and  $\Omega' = \{B_j : 1 \leq j \leq s\}$  are the bases of the convex cone  $\widehat{\Psi}$ , then  $n = s$ ,  $A_i = v_i B_i$ , and  $u_r^{A_i, \Omega} = v_i u_r^{B_i, \Omega'}$  ( $v_i > 0$ ,  $1 \leq i \leq n$ ) after the permutations. Therefore,  $u_r^{A, \Omega} = u_r^{A, \Omega'}$  for each  $A \in \widehat{\Psi}$ , and we can define  $u_r^{A, \Psi} := u_r^{A, \Omega}$ .

### 3. Remarks

**Remark 3.1.** Consider a random variable  $X$  with nonnegative bounded profit  $a(x)$  and distribution  $dF(x)$ . In the case where  $\exp(\int \log a(x) dF(x))/e^r \leq 1/\int 1/a(x) dF(x)$ , the price is given by  $u_r^X = \exp(\int \log a(x) dF(x))/e^r$ , and the optimal proportion of investment is 1. Otherwise, the price  $u = u_r^X$  and the optimal proportion of investment  $t$  are determined by the simultaneous equations  $\exp(\int \log(a(x)t/u - t + 1) dF(x)) = e^r$  and  $\int (a(x) - u)/(a(x)t - ut + u) dF(x) = 0$  (see [3, Corollaries 5.1, 5.3, and Section 6]).

**Remark 3.2.** Remark 3.1 can be generalized to the nonnegative unbounded case where  $\int_{a(x) > 1} a(x)^\nu dF < \infty$  for some  $\nu > 0$ . For example, because  $\sum_{j=1}^{\infty} (2^j)^{1/2}/2^j = 2/(2 - \sqrt{2})$  and  $\exp(\sum_{j=1}^{\infty} (\log 2^j)/2^j)/e^r = 4/e^{0.05} > 1/\sum_{j=1}^{\infty} 1/4^j = 3$ , the St. Petersburg game {profit  $2^j$  with probability  $1/2^j$ ,  $j = 1, 2, \dots$ } is priced at 4.816 with the optimal proportion of investment 0.204.

**Remark 3.3.** In Section 2, the value of  $n$  is 1 or 2. However, when the reader challenges to study dice games, the value of  $n$  may be 36. To generalize this theory to the convex cone  $\widehat{\Psi}$  with a finite basis  $\Omega$ , we need the fact that  $u_r^{\sum_{i=1}^n p_i A_i}$  is concave and continuous with respect to  $(p_i) \in Q$  for any positive integer  $n$ . This can be achieved using [2, Theorems 185 and 214] and [9, Theorems 10.1, 10.3 and 20.5] with tedious discussions. Therefore, in Definition 2.1,  $\sup_{(p_i) \in Q}$  can be replaced by  $\max_{(p_i) \in Q}$  because of Berge's maximum theorem (see [10, Theorem 2.1]).

**Remark 3.4.** Let  $S$  denote the stock price which is a nonnegative random variable. Define  $P := \max(K - S, 0)$  and  $C := \max(S - K, 0)$  for the strike price  $K$ . Applying Lemma 2.2 with  $\Omega = \{P, C, S - C\}$ , the equalities  $P + (S - C) = K$

and  $C + (S - C) = S$  imply Put-call parity  $u_r^{C, \Omega} - u_r^{P, \Omega} + K/e^r = u_r^{S, \Omega}$  (see [7, Sections 12.3 and 13.2]).

**Remark 3.5.** In this remark, we assume that the risk-free interest rate  $r = 0.02$  is simple (not continuously compound). Consider two independent coin-flipping games,  $X = (50, 1)$  and  $Y = (30.6191, 14)$ , where the variances are  $v_X = 600.25$  and  $v_Y = 69.0486$ , respectively. Assume that the rates of mean return ([7, section 6.4]) are  $r_X = 0.233546$  and  $r_Y = 0.079211$ , respectively. Thus, from the one-fund theorem ([7, section 6.9]), we have the weight

$$w_X = \frac{\frac{r_X - r}{v_X}}{\frac{r_X - r}{v_X} + \frac{r_Y - r}{v_Y}} = 0.2932,$$

which implies that the single fund of risky assets is

$$w_X X + (1 - w_X) Y = (36.3016, 24.5552, 21.9348, 10.1884).$$

The four values of this fund occur with the same probability of  $1/4$ . The price of this fund is 21.3995 according to Remark 3.1, where  $e^r$  is replaced by  $1 + r$ .

However, the price of  $wX + (1 - w)Y$  ( $0 \leq w \leq 1$ ) reaches its maximum value of 21.4134 when  $w = 0.3514$ , that is, the fund

$$0.3514X + 0.6486Y = (37.4295, 26.6504, 20.2109, 9.4318)$$

is more valuable than the single fund  $w_X X + (1 - w_X) Y$  because  $21.3995 < 21.4134$ .

It should be noted that by using  $1 + r$  instead of  $e^r$ , Theorem 1.1 gives the prices of  $X$  and  $Y$  as  $u_X = 20.6721$  and  $u_Y = 20.6721$ , respectively. Thus, the corresponding rates of mean return are  $r_X = 25.5/u_X - 1 = 0.233546$  or  $r_Y = 22.30955/u_Y - 1 = 0.07921$ .

Moreover, Remark 3.1 gives us the optimal proportion  $t = 0.4222$  for the risky fund. Thus, the best proportions of investment to  $X$ , to  $Y$ , and the risk-free asset are  $tw = 0.1484$ ,  $t(1 - w) = 0.2738$ , and  $1 - t = 0.5778$ , respectively. The mean-variance portfolio theory cannot provide a proportion of 0.5778 for the risk-free asset (see [7, section 7.1]).

**Remark 3.6.** Let  $Y := \{Y_t\}_{0 \leq t \leq T < \infty}$  be a measurable stochastic process with a filtration. Put  $\Psi := \{\tau ; \tau \text{ is a stopping time such that } \tau \leq T\}$ . We define the price  $\bar{u}^Y$  by  $\sup_{\tau \in \Psi} u_{rE(\tau)}^{Y_\tau}$ , where  $u_{rE(\tau)}^{Y_\tau}$  is the price of the random variable  $Y_{\tau(\omega)}(\omega)$  (see [Karatzas et al. (1998)]) with respect to the growth rate  $e^{rE(\tau)}$ :

$$\sup_{z \leq \text{ess inf}_\omega \frac{Y_{\tau(\omega)}(\omega)}{z/u+1}} \int_\Omega \log(Y_{\tau(\omega)}(\omega)z/u - z + 1) d\omega = rE(\tau).$$

The geometric price of  $Y := \{Y_t\}_{0 \leq t < \infty}$  is defined by  $\sup_{0 < T < \infty} \bar{u}^{\{Y_t\}_{0 \leq t \leq T}}$ .

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