A GLOBAL VIEW OF EQUIVARIANT VECTOR BUNDLES AND DIRAC OPERATORS ON SOME COMPACT HOMOGENEOUS SPACES

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Dedicated to the memory of George W. Mackey

ABSTRACT. In order to facilitate the comparison of Riemannian homogeneous spaces of compact Lie groups with noncommutative geometries ("quantizations") that approximate them, we develop here the basic facts concerning equivariant vector bundles and Dirac operators over them in a way that uses only global constructions and arguments. Our approach is quite algebraic, using primarily the modules of cross-sections of vector bundles. We carry the development through the construction of Hodge–Dirac operators. The inducing construction is ubiquitous.

1. Introduction

In the literature of theoretical high-energy physics one finds statements such as "matrix algebras converge to the sphere", and "these vector bundles on the matrix algebras are the monopole bundles that correspond to the monopole bundles on the sphere". I have provided suitable definitions and theorems [12, 13] that give a precise meaning to the first of these statements (and I am developing stronger versions); but I have not yet provided precise meaning to the second statement, though I laid much groundwork for doing this in [14]. To quantitatively compare vector bundles on an ordinary space, such as the sphere, with "vector bundles", i.e., projective modules, over a related noncommutative "space", it is technically desirable to have a description of the ordinary vector bundles that is as congenial to the methods of noncommutative geometry as possible. This means, for example, avoiding any use of local coordinates, and working with the modules of continuous cross-sections of a bundle rather than with the points of the bundle itself.

The purpose of this paper is to give such a congenial approach for the case of equivariant vector bundles over homogeneous spaces of compact connected Lie groups. (This includes the monopole bundles over the 2-sphere.) So this paper can be viewed as largely expository, with its novelty being primarily in our presentation of the known results, and the relative simplicity that our approach brings to this topic. (Compare with [11, 8, 16, 1].) We are able to work entirely just in terms of functions on the Lie group. The only differential geometry that we need is that which involves how elements of the Lie algebra give vector fields on

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²⁰⁰⁰ Mathematics Subject Classification. Primary 53C30; Secondary 46L87, 58J60, 53C05.

 $Key\ words\ and\ phrases.$ homogeneous spaces, equivariant vector bundles, connections, Dirac operators.

The research reported here was supported in part by National Science Foundation Grant DMS-0500501.

the group that can differentiate functions. Beyond this, our treatment is quite self-contained, and very algebraic in nature (somewhat along the lines found in Chapter 1 of [7] – see the comment near the top of page 86 of [7] as to why this kind of approach is appropriate). The main new result is Theorem 8.4 in which we give necessary and sufficient conditions for the Hodge-Dirac operator corresponding to a not-necessarily torsion-free connection to be formally self-adjoint. Notable is our use of "standard module frames" in various places where traditionally one would have used local arguments involving coordinate charts. Various steps of our development can be generalized in various directions, but for simplicity of exposition we do not explore these generalizations. I expect to use the material presented here for my study of the quantitative relations between ordinary and "quantum" vector bundles, along the line that can be inferred from [14].

In Section 2 of this paper we introduce equivariant vector bundles over homogeneous spaces and their modules of continuous cross-sections, while in Section 3 we give a concrete description of the algebra of module endomorphisms of a cross-section module. Section 4 is devoted to discussing the tangent bundle of a homogeneous space through its module of cross-sections. Sections 2 to 4 form in part a more leisurely version of section 13 and proposition 14.3 of [14]. In Section 5 we discuss connections on equivariant vector bundles, while in Section 6 we discuss the Levi–Civita connection for an invariant Riemannian metric on a homogeneous space. Finally, Section 7 is devoted to the Clifford bundle over the tangent space of a homogeneous space with Riemannian metric, in preparation for the discussion of the Hodge–Dirac operator that we give in Section 8. One notable aspect of Section 8 is that we give simple examples of non-torsion-free connections whose Hodge-Dirac operators are nevertheless formally self-adjoint. I have not seen this possibility discussed in the literature.

I have tried to put in enough detail so that this paper will be accessible to those who have not previously met vector bundles and Dirac operators.

Equivariant vector bundles are very closely related to the induced representations that were so central to much of the research of George Mackey. The inducing construction appears everywhere in this paper. As an undergraduate I had the pleasure of taking a year-long course on projective geometry taught by George Mackey, and a decade later his research on induced representations became of great importance for my own research.

I developed part of the material presented here during part of a ten-week visit at the Isaac Newton Institute in Cambridge, England, in the Fall of 2006. I am very appreciative of the quite stimulating and enjoyable conditions provided by the Newton Institute.

2. Induced vector bundles

In this section we assume that G is a compact group, and that K is a closed subgroup of G. Then G acts on the coset space G/K, which has its natural compact quotient topology from G. We let A = C(G/K), the C^* -algebra of continuous functions on G/K with pointwise operations and supremum norm $\|\cdot\|_{\infty}$. We will not specify whether the functions have values in \mathbb{R} or \mathbb{C} , as either it will be possible to infer which from the context, or it does not matter for what is being discussed. We will often view A as the subalgebra of C(G) consisting of functions f that satisfy f(xs) = f(x) for $x \in G$, $s \in K$. We will in general let λ denote the action of G by

left-translation on various types of functions on G (or G/K). In particular, we let λ denote the action of G on A, defined by $(\lambda_y f)(x) = f(y^{-1}x)$ for $f \in A$, $y, x \in G$.

When G is a Lie group, so that G/K is a smooth manifold [19], everything in this section has an evident version for smooth functions, but we will not state these versions, although we will use them in later sections.

Let (π, \mathcal{H}) be a finite-dimensional representation of K. Since K is compact we can equip the vector space \mathcal{H} with a π -invariant inner product. Thus we will assume throughout that (π, \mathcal{H}) is an orthogonal or unitary representation. We set

$$\Xi_{\pi} = \{ \xi \in C(G, \mathcal{H}) : \xi(xs) = \pi_s^{-1}(\xi(x)) \text{ for } x \in G, \ s \in K \}.$$

Here $C(G,\mathcal{H})$ denotes the vector space of continuous functions from G to \mathcal{H} . It is easily checked that Ξ_{π} is in fact a module over A for pointwise operations. In non-commutative geometry it is often most convenient to put module actions on the right, so that operators can be put on the left. We will follow this practice here for ordinary vector bundles, as done also in [6], since it is convenient here also. Thus we view Ξ_{π} as a right A-module. For consistency we must then write scalars on the right of elements of \mathcal{H} , and we have $(\xi f)(x) = \xi(x)f(x)$ for $\xi \in \Xi_{\pi}$, $f \in A$, $x \in G$.

The left action of G on itself gives an action of G on $C(G, \mathcal{H})$, and it is easily verified that this action carries Ξ_{π} into itself. We denote this action again by λ , so that $(\lambda_y \xi)(x) = \xi(y^{-1}x)$. Then we have the "covariance relation" $\lambda_y(\xi f) = (\lambda_y \xi)(\lambda_y f)$. With abuse of terminology we set:

Definition 2.1. The A-module Ξ_{π} with its G-action is called the equivariant vector bundle over G/K induced from (π, \mathcal{H}) .

We observe that for $\xi \in \Xi_{\pi}$ and $s \in K$ we have $(\lambda_s \xi)(e) = \xi(s^{-1}) = \pi_s(\xi(e))$, where e is the identity element of G. Thus as long as we remember how Ξ_{π} is a space of functions on G, we can in this way recover the original representation (π, \mathcal{H}) from Ξ_{π} with its G-action.

The inner product on \mathcal{H} determines a canonical bundle metric (often called a Riemannian or Hermitian metric) on Ξ_{π} , that is, an A-valued inner product [9], defined by

$$\langle \xi, \eta \rangle_A(x) = \langle \xi(x), \eta(x) \rangle_{\mathcal{H}}.$$

We take our inner product on \mathcal{H} to be linear in its second variable. It is easy to check that the A-valued inner product on Ξ_{π} is G-invariant in the sense that

$$\lambda_y(\langle \xi, \eta \rangle_A) = \langle \lambda_y \xi, \lambda_y \eta \rangle_A.$$

On G/K there is a unique G-invariant probability measure. Integrating functions on G/K against this measure is the same as viewing the functions as defined on G and integrating them against the Haar measure on G that gives G unit mass. Throughout this paper whenever we integrate over G/K or G it is with respect to these normalized measures.

On Ξ_{π} we can define an ordinary inner product, $\langle \cdot, \cdot \rangle$, by

$$\langle \xi, \eta \rangle = \int_{G/K} \langle \xi, \eta \rangle_A.$$

The action λ of G on Ξ_{π} preserves this inner product. The completion of Ξ_{π} for this inner product is the Hilbert space for Mackey's induced representation of G from the representation (π, \mathcal{H}) of K, with the representation of G just being the

extension of λ to the completion. The pointwise action of A on Ξ_{π} extends to an action on the completion, and this action can be viewed as Mackey's "system of imprimativity" for the induced representation.

It can be shown that Ξ_{π} is the space of continuous cross-sections of the usual equivariant vector bundle induced from (π, \mathcal{H}) . But we do not need this fact, though our development can be useful in proving this fact. A theorem of Swan [17, 6] says that the module of continuous cross-sections will always be a projective module (finitely generated, but in this paper whenever we say "projective" we always also mean "and finitely generated"), and conversely. We include here a direct proof, taken from [14] but with antecedents in [10], that Ξ_{π} is a projective module. One important feature of the proof is that, as we will explain, it provides a way of obtaining a projection p in a matrix algebra $M_n(A)$ for some n, such that Ξ_{π} is isomorphic to the right A-module pA^n . Such projections are crucial for quantifying the relation between vector bundles on compact metric spaces that are close together for Gromov–Hausdorff distance, as seen in [14].

Proposition 2.2. For G, K and (π, \mathcal{H}) as above, the induced module Ξ_{π} is a projective A-module.

Proof. Find a finite-dimensional orthogonal or unitary representation $(\tilde{\pi}, \tilde{\mathcal{H}})$ of G such that \mathcal{H} is a subspace of $\tilde{\mathcal{H}}$ and the restriction of $\tilde{\pi}$ to K, acting on \mathcal{H} , is π . Such representations $(\tilde{\pi}, \tilde{\mathcal{H}})$ exist according to the Frobenius reciprocity theorem, which has an elementary proof in our context [2]. Note that $C(G/K, \tilde{\mathcal{H}})$ is a free A-module, with basis coming from any basis for $\tilde{\mathcal{H}}$. Define Φ from Ξ_{π} to $C(G/K, \tilde{\mathcal{H}})$ by

$$(\Phi \xi)(x) = \tilde{\pi}_x(\xi(x))$$

for $x \in G$. Clearly Φ is an injective A-module homomorphism. Let P be the orthogonal projection from $\tilde{\mathcal{H}}$ onto \mathcal{H} , and define a function p from G to $\mathcal{L}(\tilde{\mathcal{H}})$, the algebra of linear operators on $\tilde{\mathcal{H}}$, by

$$p(x) = \tilde{\pi}_x P \tilde{\pi}_x^*.$$

It is easily seen that p is a projection in the algebra $C(G/K, \mathcal{L}(\tilde{\mathcal{H}}))$. Furthermore, this algebra acts as endomorphisms on the free A-module $C(G/K, \tilde{\mathcal{H}})$ in the evident pointwise way, and it is easy to check [14] that p is the projection onto the range of Φ . Thus the range of Φ , and so Ξ_{π} , is projective.

We remark that the actual vector bundle corresponding to Ξ_{π} can be viewed as assigning to each point \dot{x} of G/K the range subspace of p(x). Notice that for a given (π, \mathcal{H}) there may be many choices for the representation $(\tilde{\pi}, \tilde{\mathcal{H}})$ above, and so many choices of the projection p.

In case G is a Lie group, it is known that finite-dimensional representations (as homomorphisms between Lie groups) are real-analytic, so smooth. Consequently the projection p of the above proof is smooth, and this shows that the subspace Ξ_{π}^{∞} of smooth elements of Ξ_{π} is a projective module over $C^{\infty}(G/K)$.

Let A be any unital C^* -algebra, and let Ξ be a right A-module. Assume that Ξ is equipped with an A-valued inner product [9]. A finite sequence $\{\eta_j\}$ of elements of Ξ is said to be a *standard module frame* for Ξ if the "reproducing formula"

$$\xi = \sum \eta_j \langle \eta_j, \xi \rangle_A$$

holds for all $\xi \in \Xi$. (It is easily seen, p. 46 of [14], that if Ξ has a standard module frame, $\{\eta_j\}_{j=1}^n$, then Ξ is a projective A-module, and the matrix $p = \{\langle \eta_j, \eta_k \rangle_A\}$ in $M_n(A)$ is a projection such that $\Xi \cong pA^n$.) We can construct standard module frames for the induced vector bundles Ξ_{π} as follows. For $(\tilde{\pi}, \tilde{\mathcal{H}})$ as used above, let $\{e_j\}$ be an orthonormal basis for $\tilde{\mathcal{H}}$. For each j define η_j by $\eta_j(x) = P\tilde{\pi}_x^{-1}e_j$ for $x \in G$. Each η_j is easily seen to be in Ξ_{π} . Then for $\xi \in \Xi_{\pi}$ we have

$$(\sum \eta_j(x)\langle \eta_j, \xi \rangle_A)(x) = \sum P\tilde{\pi}_x^{-1} e_j \langle P\tilde{\pi}_x^{-1} e_j, \xi(x) \rangle_{\mathcal{H}}$$
$$= P\sum (\tilde{\pi}_x^{-1} e_j) \langle (\tilde{\pi}_x^{-1} e_j), \xi(x) \rangle_{\mathcal{H}} = P\xi(x) = \xi(x),$$

where we have used that $\{\tilde{\pi}_x^{-1}e_j\}$ is equally well an orthonormal basis for $\tilde{\mathcal{H}}$. Thus $\{\eta_j\}$ has the reproducing property, and so is a standard module frame for Ξ_{π} . (We remark that it may happen that $\eta_j = 0$ for certain j's.) We will make good use of standard module frames in some of the next sections.

3. Endomorphism bundles

In this section we will give a description of the endomorphism bundles of induced bundles. This description will be useful in our later discussion of connections. We will continue to work just with continuous functions and cross-sections, and we will leave it to the reader to notice that when G is a Lie group everything said in this section has a smooth version. We will need these smooth versions in later sections.

As before, let (π, \mathcal{H}) be a finite-dimensional orthogonal or unitary representation of K, and let Ξ_{π} be the induced bundle. We want to describe $\operatorname{End}_{A}(\Xi_{\pi})$ in terms of (π, \mathcal{H}) . Let $\{\eta_{j}\}$ be a standard module frame for Ξ_{π} . For any $T \in \operatorname{End}_{A}(\Xi_{\pi})$ and any $\xi \in \Xi_{\pi}$ we have

$$T\xi = T\left(\sum \eta_j \langle \eta_j, \xi \rangle_A\right) = \sum (T\eta_j) \langle \eta_j, \xi \rangle_A$$
$$= \left(\sum \langle T\eta_j, \eta_j \rangle_E\right) \xi,$$

where for $\zeta, \eta \in \Xi_{\pi}$ we let $\langle \zeta, \eta \rangle_E$ denote the "rank-one" operator on Ξ_{π} defined by $\langle \zeta, \eta \rangle_E \xi = \zeta \langle \eta, \xi \rangle_A$. Then $\langle \cdot, \cdot \rangle_E$ is an inner product on Ξ_{π} with values in $E = \operatorname{End}_A(\Xi_{\pi})$. It is linear in the first variable, and elements of E pull out of the first variable as if they were scalars [9]; we view Ξ_{π} as a left E-module. The calculation above shows that for any $T \in E$ we have $T = \sum \langle T \eta_j, \eta_j \rangle_E$. Thus E is spanned by the "rank-one" operators.

For any $\zeta, \eta, \xi \in \Xi_{\pi}$ we have

$$(\langle \zeta, \eta \rangle_E \xi)(x) = \zeta(x) \langle \eta(x), \xi(x) \rangle_{\mathcal{H}} = \langle \zeta(x), \eta(x) \rangle_0(\xi(x)),$$

where now $\langle \cdot, \cdot \rangle_0$ denotes the usual ordinary rank-one operator on \mathcal{H} given by two vectors. Since we just saw that the "rank-one" operators $\langle \zeta, \eta \rangle_E$ span E, we see now that every operator in E is given by a function in $C(G, \mathcal{L}(\mathcal{H}))$, as we would expect. Furthermore, for any $x \in G$ and $s \in K$ we have

$$\langle \zeta(xs), \eta(xs) \rangle_0 = \langle \pi_s^{-1}(\zeta(x)), \pi_s^{-1}(\eta(x)) \rangle_0 = \pi_s^{-1} \langle \zeta(x), \eta(x) \rangle_0 \pi_s.$$

It follows that if $T \in C(G, \mathcal{L}(\mathcal{H}))$ represents an element of E then $T(xs) = \pi_s^{-1}T(x)\pi_s$. But a simple direct check shows that any T satisfying this property gives an element of E. Thus we have obtained

Proposition 3.1. We can identify $\operatorname{End}_A(\Xi_{\pi})$ with

$$\mathcal{E}_{\pi} = \{ T \in C(G, \mathcal{L}(\mathcal{H})) : T(xs) = \pi_s^{-1} T(x) \pi_s \text{ for } x \in G, \ s \in K \},$$

where the action of such a T on $\xi \in \Xi_{\pi}$ is given just by $(T\xi)(x) = T(x)(\xi(x))$.

There is an evident action, say α , of K on $\mathcal{L}(\mathcal{H})$, given by $\alpha_s(\tau) = \pi_s \tau \pi_s^{-1}$, and we see that \mathcal{E}_{π} is just the "induced algebra over G/K for the action α ". As such, \mathcal{E}_{π} is an A-module. But it is also an A-module just from the fact that A is commutative so that A is contained in the center of \mathcal{E}_{π} , and we notice that these two A-module structures coincide.

4. The tangent bundle

We now assume that G is a compact connected Lie group with Lie algebra \mathfrak{g} . We assume again that K is a closed subgroup of G, which need not be connected, and we denote its Lie algebra (for its connected component of the identity element e) by \mathfrak{k} . Then G/K is a compact smooth manifold [19]. We seek an attractive realization of its tangent bundle, as a projective A-module, where now we set $A = C^{\infty}(G/K)$, with functions being real-valued. We use below some standard facts [7, 19] about Lie groups and their Lie algebras. This section and the next two have a number of points of contact with section 5 of [1].

Much as before, we view $C^{\infty}(G/K)$ as a subalgebra of $C^{\infty}(G)$. The action λ of G on $C^{\infty}(G)$ and $C^{\infty}(G/K)$ gives an infinitesimal action of \mathfrak{g} , defined for $X \in \mathfrak{g}$ by $(\lambda_X f)(x) = (Xf)(x) = D_0^t(f(\exp(-tX)x))$, where we write D_0^t for $(d/dt)|_{t=0}$. This gives a Lie algebra homomorphism from \mathfrak{g} into the Lie algebra $\mathrm{Der}(C^{\infty}(G))$ of derivations of the algebra $C^{\infty}(G)$, and so into $\mathrm{Der}(C^{\infty}(G/K))$. (See proposition 2.4 of chapter 0 of [18].) The derivations λ_X of $C^{\infty}(G/K)$ are often called the fundamental vector fields for $C^{\infty}(G/K)$. Since $C^{\infty}(G)$ is commutative, $\mathrm{Der}(C^{\infty}(G))$ is a module over $C^{\infty}(G)$, and a basis for \mathfrak{g} gives a module basis for $\mathrm{Der}(C^{\infty}(G))$, that is, $\mathrm{Der}(C^{\infty}(G))$ is a free $C^{\infty}(G)$ -module. We can thus realize the elements of $\mathrm{Der}(C^{\infty}(G))$, which is the module of smooth cross-sections of the tangent bundle of G, as elements of $C^{\infty}(G,\mathfrak{g})$, where the action of $W \in C^{\infty}(G,\mathfrak{g})$ on $C^{\infty}(G)$ is given by

$$(\lambda_W f)(x) = D_0^t(f(\exp(-tW(x))x)).$$

Accordingly, we denote $C^{\infty}(G,\mathfrak{g})$ by $\mathcal{T}(G)$, for "tangent space".

Now for $f \in A = C^{\infty}(G/K)$ and $W \in \mathcal{T}(G)$ we need to have W(xs) = W(x) for $x \in G$ and $s \in K$ if we want $\lambda_W f \in A$. Furthermore, if for some $x_1 \in G$ we have $W(x_1) \in \mathrm{Ad}_{x_1}(\mathfrak{k})$, then $\mathrm{Ad}_{x_1}^{-1}(W(x_1)) \in \mathfrak{k}$, so that

$$(\lambda_W f)(x_1) = D_0^t(f(\exp(-tW(x_1))x_1))$$

= $D_0^t(f(x_1 \exp(-t\operatorname{Ad}_{x_1}^{-1}(W(x_1)))) = D_0^t(f(x_1)) = 0.$

Conversely, if $(\lambda_W f)(x_1) = 0$ for all $f \in A$, then $W(x_1) \in \mathrm{Ad}_{x_1}(\mathfrak{k})$.

Choose an Ad-invariant inner product, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, on \mathfrak{g} , which we fix for the rest of this paper. Let \mathfrak{m} denote the orthogonal complement to \mathfrak{k} in \mathfrak{g} . Let P be the orthogonal projection of \mathfrak{g} onto \mathfrak{m} . For any $W \in \mathcal{T}(G)$ define \tilde{W} by $\tilde{W}(x) = \mathrm{Ad}_x(P(\mathrm{Ad}_x^{-1}(W(x))))$. Then $W(x) - \tilde{W}(x) \in \mathrm{Ad}_x(\mathfrak{k})$ for all x, and so $W - \tilde{W}$ acts on A as the 0-derivation. This and the earlier calculations suggest that we consider (4.1)

$$\{W \in C^{\infty}(G, \mathfrak{g}) : W(x) \in \mathrm{Ad}_x(\mathfrak{m}) \text{ and } W(xs) = W(x) \text{ for all } x \in G, s \in K\}.$$

And indeed it is easily seen that any such W, acting as derivations of $C^{\infty}(G)$ by the earlier formula, carries A into itself, and that every smooth vector field on G/K is represented in this way. However, it is inconvenient that the range space of the W's is not constant on G. But

$$\exp(W(x))x = x \exp(\operatorname{Ad}_x^{-1}(W(x))),$$

and if W is as in 4.1 then $\operatorname{Ad}_x^{-1}(W(x)) \in \mathfrak{m}$ for all x. When we also take into account the effect on the right K-invariance, we are led to:

Notation 4.2. With notation as above, set

$$\mathcal{T}(G/K) = \{ W \in C^{\infty}(G, \mathfrak{m}) : W(xs) = \operatorname{Ad}_{s^{-1}}(W(x)) \text{ for } x \in G, \ s \in K \}.$$

We let elements of $\mathcal{T}(G/K)$ act as derivations of $C^{\infty}(G/K)$ by

$$(\delta_W f)(x) = D_0^t(f(x \exp(tW(x)))),$$

where we have also left behind the earlier minus sign. It is clear that $\mathcal{T}(G/K)$ is a module over A for pointwise operations. We recognize $\mathcal{T}(G/K)$ as just the induced bundle for the representation Ad restricted to K on \mathfrak{m} . It is easy to check that $\mathcal{T}(G/K)$ does give derivatives in all tangent directions at any point of G/K. We can then to use the fact that $\mathcal{T}(G/K)$ is an A-module to show that it does contain all the smooth vector fields on G/K. Thus it does represent the space of smooth cross-sections of the tangent bundle of G/K, though we will not explicitly need this fact. This description of the tangent bundle can be found, for example, in [16, 5, 1].

For $X \in \mathfrak{g}$ the corresponding fundamental vector field on G/K is given, in the form used for (4.1), by

$$X(x) = \operatorname{Ad}_x(P\operatorname{Ad}_x^{-1}(X)).$$

Then in the form used for the definition of $\mathcal{T}(G/K)$ this fundamental vector field, which we now denote by \hat{X} , is given by

$$\hat{X}(x) = -P \operatorname{Ad}_x^{-1}(X).$$

(See section 0.3 of [16].) Since the map from elements of $\mathfrak g$ to vector fields is a Lie algebra homomorphism, we have

$$[\hat{X}, \hat{Y}](x) = -P \operatorname{Ad}_x^{-1}([X, Y]),$$

where the brackets on the left denote the commutator of \hat{X} and \hat{Y} as operators on A. Also, a quick calculation shows that $\lambda_y \hat{X} = (\operatorname{Ad}_y X)^{\wedge}$ for $y \in G$. We remark that \hat{X} may well take value 0 at some points of G/K — we are confronting the fact that the tangent bundle of G/K may well not be a trivial bundle. It may even happen that for certain X's in \mathfrak{g} we have $\hat{X} \equiv 0$. We warn the reader that for general $V, W \in \mathcal{T}(G/K)$ the commutator [V, W] as derivations of A is again an element of $\mathcal{T}(G/K)$ but it is usually not given by any pointwise formula. The torsion-free condition shows how to express this commutator in terms of the Levi-Civita connection or other torsion-free connections as we will see later.

The restriction to \mathfrak{m} of our chosen inner product on \mathfrak{g} determines a chosen G-invariant Riemannian metric on G/K, specializing what was done in Section 2 . If $\{X_j\}$ is an orthonormal basis for \mathfrak{g} , then one can chase through the discussion at the end of the previous section to see that $\{\hat{X}_j\}$ is a standard module frame for

 $\mathcal{T}(G/K)$ for this Riemannian metric. But this is also easy to verify directly: For any $W \in \mathcal{T}(G/K)$ and $x \in G$ we have

$$\left(\sum \hat{X}_{j}\langle \hat{X}_{j}, W \rangle_{A}\right)(x) = \sum -(P \operatorname{Ad}_{x}^{-1} X_{j})\langle -P \operatorname{Ad}_{x}^{-1} X_{j}, W(x) \rangle_{\mathfrak{g}}$$
$$= P \sum (\operatorname{Ad}_{x}^{-1} X_{j})\langle \operatorname{Ad}_{x}^{-1} X_{j}, W(x) \rangle_{\mathfrak{g}} = PW(x) = W(x),$$

since $\{\operatorname{Ad}_x^{-1} X_j\}$ is equally well an orthonormal basis for \mathfrak{g} . Of course, again we may have $\hat{X}_j \equiv 0$ for some j's.

5. Connections

In this section we will give a description of all G-invariant connections on an induced bundle. By a connection (or covariant derivative) on the space Ξ of smooth cross-sections of a vector bundle on a manifold M we mean [7, 6] a linear transformation ∇ from the space $\mathcal{T}(M)$ of smooth tangent-vector fields on M to linear transformations on Ξ such that for $W \in \mathcal{T}(M)$, $\xi \in \Xi$ and $f \in C^{\infty}(M)$ we have the Leibniz rule

$$\nabla_W(\xi f) = (\nabla_W \xi) f + \xi(\delta_W f)$$

and the rule

$$\nabla_{Wf}(\xi) = (\nabla_W \xi) f.$$

For the case we have been considering, in which M = G/K and Ξ is an induced vector bundle Ξ_{π} , there is a canonical connection, ∇^{0} , defined by

$$(\nabla_W^0 \xi)(x) = D_0^t(\xi(x \exp(tW(x)))).$$

To see that $\nabla_W^0 \xi$ is indeed in Ξ_{π} we calculate that for $x \in G$ and $s \in K$

$$\begin{array}{lcl} (\nabla_W^0 \xi)(xs) & = & D_0^t(\xi(xs\exp(tW(xs)))) = D_0^t(\xi(xs\exp(t\operatorname{Ad}_s^{-1}(W(x))))) \\ & = & D_0^t(\xi(x\exp(t(W(x)))s)) = \pi_s^{-1}((\nabla_W^0 \xi)(x)), \end{array}$$

as needed. The other properties stated above for the definition of a connection are easily verified by similar calculations.

There is a standard definition [7] of what it means for a connection ∇ on an equivariant vector bundle to be invariant for the group action, and when this definition is applied to our induced bundles Ξ_{π} over G/K, it requires that

$$\lambda_{y}(\nabla_{W}\xi) = \nabla_{\lambda_{y}W}(\lambda_{y}\xi)$$

for all $y \in G$, $W \in \mathcal{T}(G/K)$ and $\xi \in \Xi_{\pi}$. It is straightforward to check that the canonical connection ∇^0 is G-invariant.

Let ∇ be any connection on Ξ_{π} , and set $L = \nabla - \nabla^0$. It is easily checked that $L_W \in \operatorname{End}_A(\Xi_{\pi})$ for each $W \in \mathcal{T}(G/K)$, and so L_W can be represented as a function in \mathcal{E}_{π} according to Proposition 3.1. We mentioned in Section 3 that \mathcal{E}_{π} is an A-module because A is commutative. It is easily checked that L is an A-module homomorphism from $\mathcal{T}(G/K)$ to \mathcal{E}_{π} , and that, conversely, if L is any A-module homomorphism from $\mathcal{T}(G/K)$ to \mathcal{E}_{π} then $\nabla^0 + L$ is a connection on Ξ_{π} . This is just the well-known fact that the set of connections forms an affine space over $\operatorname{Hom}_A(\mathcal{T}(G/K),\operatorname{End}_A(\Xi_{\pi}))$. If ∇ is G-invariant, then, because ∇^0 also is G-invariant, so is L, in the sense that $\lambda_y(L_W\xi) = L_{\lambda_yW}(\lambda_y\xi)$ for all $y \in G$, $W \in \mathcal{T}(G/K)$ and $\xi \in \Xi_{\pi}$. When we view L_W as a function in \mathcal{E}_{π} , invariance gives

$$L_W(y^{-1}x)\xi(y^{-1}x) = (\lambda_y(L_W\xi))(x) = (L_{\lambda_y W}(\lambda_y \xi))(x) = L_{\lambda_y W}(x)\xi(y^{-1}x).$$

Since this is true for all ξ , we see that for all $y \in G$ we have

$$\lambda_y L_W = L_{\lambda_y W}$$

as functions on G. In particular $L_W(y^{-1}) = (\lambda_y L_W)(e) = L_{\lambda_y W}(e)$, so that L is determined once we know $L_W(e)$ for all $W \in \mathcal{T}(G/K)$. Note further that since $L_W \in \mathcal{E}_{\pi}$, we have by invariance

$$L_{\lambda_s W}(e) = L_W(es^{-1}) = \pi_s L_W(e) \pi_s^{-1}$$

for $s \in K$.

Suppose now that $\{X_j\}$ is an orthonormal basis for \mathfrak{g} , so that $\{\hat{X}_j\}$ is a standard module frame for $\mathcal{T}(G/K)$, as seen near the end of Section 4. Then

$$L_W(e) = \sum L_{\hat{X}_j}(e) \langle \hat{X}_j, W \rangle_A(e),$$

so that $L_W(e)$ is determined once we know $L_{\hat{X}_j}(e)$ for all j. But $\hat{X}(e) = 0$ if $X \in \mathfrak{k}$. Thus in the above expression we only need to sum over a basis for \mathfrak{m} . Equivalently, $L_W(e)$ is determined once we know $L_{\hat{X}}(e)$ for all $X \in \mathfrak{m}$. Notice that on \mathfrak{m} the map $X \mapsto \hat{X}$ is injective. Define γ_L on \mathfrak{m} by $\gamma_L(X) = L_{\hat{X}}(e)$, so that L is determined by γ_L . Then γ_L is a real-linear transformation from \mathfrak{m} to $\mathcal{L}(\mathcal{H})$. Recall that $\lambda_y(\hat{X}) = (\mathrm{Ad}_y(X))^{\hat{}}$. Then from the property of $L_W(e)$ obtained at the end of the previous paragraph we have

$$\gamma_L(\mathrm{Ad}_s(X)) = \pi_s \gamma_L(X) \pi_s^{-1}$$

for $s \in K$ and $X \in \mathfrak{m}$. The following theorem has its roots at least back in Nomizu [11]. See also section X.2 of [8] and section 0 of [16].

Theorem 5.1. With notation as above, the map $L \mapsto \gamma_L$ gives a bijection between the set of G-invariant connections on Ξ_{π} and the set of linear operators γ from \mathfrak{m} to $\mathcal{L}(\mathcal{H})$ with the property that

$$\gamma(\mathrm{Ad}_s(X)) = \pi_s \gamma(X) \pi_s^{-1}$$

for all $s \in K$ and $X \in \mathfrak{m}$.

Proof. We have shown above that every invariant connection ∇ gives rise to $L = \nabla - \nabla^0$, and L gives rise to γ_L , which in turn determines L and so ∇ . We must show, conversely, that any γ as in the statement of the theorem gives rise to an $L \in \operatorname{Hom}_A(\mathcal{T}(G/K), \mathcal{E}_{\pi})$ such that $\gamma = \gamma_L$, and so gives rise to the connection $\nabla = \nabla^0 + L$, which is G-invariant. We first define L on each \hat{X} for $X \in \mathfrak{g}$ by

$$L_{\hat{X}}(x) = \gamma(P(\operatorname{Ad}_{x}^{-1}(X))).$$

It is easy to check that $L_{\hat{X}} \in \mathcal{E}_{\pi}$, that is, that

$$L_{\hat{X}}(xs) = \pi_s^{-1} L_{\hat{X}}(x) \pi_s$$

for $x \in G$ and $s \in K$. We then choose a standard module frame $\{\hat{X}_j\}$ for $\mathcal{T}(G/K)$, and set $L_W = \sum L_{\hat{X}_j} \langle \hat{X}_j, W \rangle_A$. It is then easy to check that this L has the desired properties, and that $\gamma_L = \gamma$.

In the presence of a bundle metric on a vector bundle, a connection is said to be compatible with the bundle metric if the Leibniz rule

$$\delta_W(\langle \xi, \eta \rangle_A) = \langle \nabla_W \xi, \eta \rangle_A + \langle \xi, \nabla_W \eta \rangle_A$$

holds. It is easy to verify that the canonical connection ∇^0 on Ξ_{π} is compatible with the bundle metric that we have been using. If ∇ is another connection on Ξ_{π} that is compatible, and if we set $L = \nabla - \nabla^0$, then we see that L must satisfy

$$0 = \langle L_W \xi, \eta \rangle_A + \langle \xi, L_W \eta \rangle_A.$$

This says that L, as a function with values in $\mathcal{L}(\mathcal{H})$, must in fact have its values in the subspace $\mathcal{L}^{sk}(\mathcal{H})$ of skew-symmetric or skew-Hermitian operators on \mathcal{H} . When L is G-invariant and we define γ_L as above by $\gamma_L(X) = L_{\hat{X}}(e)$, this then says exactly that γ_L must have its values in $\mathcal{L}^{sk}(\mathcal{H})$. Thus we obtain:

Corollary 5.2. With notation as above, the map $L \mapsto \gamma_L$ gives a bijection between the set of G-invariant compatible connections on Ξ_{π} and the set of linear operators γ from \mathfrak{m} to $\mathcal{L}^{sk}(\mathcal{H})$ with the property that

$$\gamma(\mathrm{Ad}_s(X)) = \pi_s \gamma(X) \pi_s^{-1}$$

for all $s \in K$ and $X \in \mathfrak{m}$.

6. The Levi-Civita connection

For a connection ∇ on a tangent bundle itself it makes sense to talk about its torsion [7, 6], which for a connection ∇ on $\mathcal{T}(G/K)$ is the bilinear form T_{∇} defined by

$$T_{\nabla}(V, W) = \nabla_{V}(W) - \nabla_{W}(V) - [V, W]$$

for $V,W\in\mathcal{T}(G/K)$, with values in $\mathcal{T}(G/K)$. Then the Levi–Civita connection associated to a Riemannian metric on the tangent space is by definition, for our case of $\mathcal{T}(G/K)$, the (necessarily unique) connection ∇ which is compatible with the Riemannian metric and has $T_{\nabla}\equiv 0$. Since our Riemannian metric is G-invariant, we can expect its Levi–Civita connection to be G-invariant also.

Let us calculate the torsion of the canonical connection ∇^0 on $\mathcal{T}(G/K)$. Now for any connection ∇ it is not difficult to verify that T_{∇} is A-bilinear. (See §8 of chapter 1 of [7].) Thus for the reasons seen earlier, it is sufficient to calculate with fundamental vector fields. Now to begin with, for $X, Y \in \mathfrak{g}$ we have

$$\begin{split} (\nabla^0_{\hat{X}}(\hat{Y}))(x) &= D_0^t \hat{Y}(x \exp(-tP \operatorname{Ad}_x^{-1}(X))) \\ &= D_0^t (-P(\operatorname{Ad}_{\exp(tP \operatorname{Ad}_x^{-1}(X)} \operatorname{Ad}_x^{-1}(Y))) = -P([P \operatorname{Ad}_x^{-1}(X), \operatorname{Ad}_x^{-1}(Y)]). \end{split}$$

If a connection ∇ is G-invariant, then it is easily seen that T_{∇} is also, in the sense that

$$\lambda_y(T_{\nabla}(V,W)) = T_{\nabla}(\lambda_y V, \lambda_y W).$$

Then it suffices to calculate at e. Accordingly

$$T_{\nabla^0}(\hat{X}, \hat{Y})(e) = -P([PX, Y]) + P([PY, X]) - [\hat{X}, \hat{Y}](e).$$

Since $X \to \hat{X}$ is a Lie algebra homomorphism, $[\hat{X}, \hat{Y}](e) = [X, Y]^{\hat{}}(e) = -P([X, Y])$, and so if we let Q = I - P,

$$T_{\nabla^0}(\hat{X}, \hat{Y})(e) = P([PY, X] - [PX, Y] + [X, Y]) = P([PY, X] + [QX, Y])$$

= $P(-[X, PY] + [QX, PY] + [QX, QY]) = -P([PX, PY]),$

since $[QX,QY] \in \mathfrak{k}$ so that P([QX,QY]) = 0. From this we find easily that for $V,W \in \mathcal{T}(G/K)$ we have

$$T_{\nabla^0}(V, W)(x) = -P[V(x), W(x)]$$
.

We thus see that $T_{\nabla^0} \equiv 0$ exactly if $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{k}$, which is exactly the condition for G/K to be a symmetric space [7, 5], since the involutive transformation on \mathfrak{g} which is the identity on \mathfrak{k} and the negative of the identity on \mathfrak{m} is then a Lie algebra homomorphism. We thus find the well-known fact (see section 3.5 of [5]) that ∇^0 is the Levi–Civita connection exactly if G/K is a symmetric space.

If G/K is not a symmetric space, then the Levi–Civita connection will be of the form $\nabla^0 + L$. We can again work at e, and one soon sees that if we define L^0 at e by

$$(L^0_{\hat{X}}(\hat{Y}))(e) = (1/2)P([PX, PY])$$

and extend L^0 to G/K by

$$(L_{\hat{X}}^{0}(\hat{Y}))(x) = (L_{\lambda_{x}^{-1}\hat{X}}^{0}(\lambda_{x}^{-1}\hat{Y}))(e),$$

noting that $\lambda_x^{-1}(\hat{X}) = (\operatorname{Ad}_x^{-1}(X))^{\hat{}}$, and finally extend L^0 to $\mathcal{T}(G/K)$ by using a standard module frame of fundamental vector fields as done earlier, then $L^0 \in \operatorname{Hom}_A(\mathcal{T}(G/K), \operatorname{End}_A(\mathcal{T}(G/K)))$, and

$$(T_{\nabla^0 + L^0}(\hat{X}, \hat{Y})(e) = (T_{\nabla^0}(\hat{X}, \hat{Y}))(e) + (L_{\hat{X}}^0(\hat{Y}))(e) - (L_{\hat{Y}}^0(\hat{X}))(e)$$

= $-P([PX, PY]) + (1/2)P([PX, PY]) - (1/2)P([PY, PX]) = 0.$

Thus $\nabla^0 + L^0$ has torsion 0. We see that the γ for L^0 as in Theorem 5.1 is defined by

$$\gamma_X = (1/2)P \circ \operatorname{ad}_X$$

for all $X \in \mathfrak{m}$ as an operator on \mathfrak{m} . (Compare with theorem X.2.10 of [8] and lemma 0.4.3 of [16].) It is easily seen that $\gamma_X \in \mathcal{L}^{sk}(\mathfrak{m})$ because the inner product on \mathfrak{g} was chosen to be Ad-invariant, so that ad_Z is a skew-symmetric operator on \mathfrak{g} for every $Z \in \mathfrak{g}$. From Corollary 5.2 it follows that $\nabla^0 + L^0$ is compatible with the canonical Riemannian metric on $\mathcal{T}(G/K)$, so that $\nabla^0 + L^0$ is the Levi–Civita connection for that Riemannian metric. When one carries through the calculations with the fundamental vector fields one finds that L^0 is given for general $V, W \in \mathcal{T}(G/K)$ by

$$(L_V^0 W)(x) = (1/2)P([V(x), W(x)]).$$

One easily checks directly that $L^0 \in \operatorname{Hom}_A(\mathcal{T}(G/K), \operatorname{End}_A(\mathcal{T}(G/K)))$ when L^0 is defined in this way, that L^0 is G-invariant, and is skew-symmetric. But it does not seem so easy to check directly that $\nabla^0 + L^0$ has 0 torsion, or even to guess that the above formula is the correct one for L^0 , without working with the fundamental vector fields. In summary:

Theorem 6.1. With notation as above, the Levi-Civita connection for the canonical metric on G/K is $\nabla^0 + L^0$ where ∇^0 is the canonical connection on $\mathcal{T}(G/K)$ and L^0 is defined by

$$(L_V^0 W)(x) = (1/2)P([V(x), W(x)]).$$

If $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$, then $L^0 \equiv 0$.

7. The Clifford-Algebra bundle

We can form the Clifford algebra, $\operatorname{Clif}(\mathfrak{m})$, over \mathfrak{m} with respect to the inner product on \mathfrak{m} (coming from that on \mathfrak{g}). By definition [5, 6] \mathfrak{m} sits as a vector subspace of $\operatorname{Clif}(\mathfrak{m})$ and generates $\operatorname{Clif}(\mathfrak{m})$ as a unital algebra. We follow the common convention in Riemannian geometry that the defining relation for elements of \mathfrak{m} is

$$X \cdot Y + Y \cdot X = -2\langle X, Y \rangle_{\mathfrak{m}} 1,$$

where we denote the product in $\mathrm{Clif}(\mathfrak{m})$ by ".". One can view $\mathrm{Clif}(\mathfrak{m})$ as a deformation of the exterior algebra over \mathfrak{m} in the direction of the inner product. Any isometric operator on \mathfrak{m} extends uniquely to an algebra automorphism of $\mathrm{Clif}(\mathfrak{m})$. Since the action Ad of K on \mathfrak{m} is by isometries, it extends to an action of K as algebra automorphisms of $\mathrm{Clif}(\mathfrak{m})$. We denote this action again by Ad.

The tangent space at each point of G/K is isomorphic to \mathfrak{m} , and we can form the smooth cross-section algebra of the bundle of the corresponding Clifford algebras. We denote it by $\mathrm{Clif}(\mathcal{T}(G/K))$. We seek an explicit description of this algebra in terms of our explicit description of $\mathcal{T}(G/K)$. Since $\mathrm{Clif}(\mathcal{T}(G/K))$ should be an A-module algebra and contain $\mathcal{T}(G/K)$ as a submodule that generates it, we are led to set

$$\operatorname{Clif}(\mathcal{T}(G/K)) = \{ \varphi \in C^{\infty}(G, \operatorname{Clif}(\mathfrak{m})) : \varphi(xs) = \operatorname{Ad}_{\mathfrak{s}}^{-1}(\varphi(x)) \text{ for } x \in G, \ s \in K \}.$$

This is, of course, yet another "induced" algebra. It contains $A = C^{\infty}(G/K)$ in its center in the evident way, it contains $\mathcal{T}(G/K)$, and it is an algebra generated by $\mathcal{T}(G/K)$ with the expected Clifford-algebra relations, namely

$$V \cdot W + W \cdot V = -2\langle V, W \rangle_{A}$$

Since our canonical connection is compatible with the Riemannian metric on $\mathcal{T}(G/K)$, it extends to a connection on $\mathrm{Clif}(\mathcal{T}(G/K))$, which we denote again by ∇^0 . It can be defined directly by

$$(\nabla_W^0 \varphi)(x) = D_0^t(\varphi(x \exp(tW(x))).$$

This clearly satisfies the Leibniz rule

$$\nabla^0_W(\varphi \cdot \psi) = (\nabla^0_W \varphi) \cdot \psi + \varphi \cdot (\nabla^0_W \psi)$$

for $\varphi, \psi \in \text{Clif}(\mathcal{T}(G/K))$, that is, ∇_W^0 is a derivation of $\text{Clif}(\mathcal{T}(G/K))$ for each $W \in \mathcal{T}(G/K)$. Note that when this ∇_W^0 is restricted to $A \subset \text{Clif}(\mathcal{T}(G/K))$ it gives δ_W . Since the Riemannian metric on $\mathcal{T}(G/K)$ is invariant for the left action λ of G on $\mathcal{T}(G/K)$ by translation, λ extends to an action of G by algebra automorphisms on $\text{Clif}(\mathcal{T}(G))$, which we again denote by λ . It is of course given by $(\lambda_y \varphi)(x) = \varphi(y^{-1}x)$, and when restricted to $A \subset \text{Clif}(\mathcal{T}(G/K))$ it gives the original action of G on A.

The standard Dirac operator for a Riemannian manifold is defined in terms of the Levi–Civita connection, and so if G/K is not a symmetric space, we should consider instead the connection $\nabla^0 + L^0$ defined in the previous section. Recall that $L_W^0(x)$ is a skew-symmetric operator on \mathfrak{m} for each $W \in \mathcal{T}(G/K)$ and $x \in G$. Now any skew-symmetric operator, say R, on \mathfrak{m} , as the generator of a one-parameter group of isometries of \mathfrak{m} , and so of automorphisms of $\mathrm{Clif}(\mathfrak{m})$, determines a derivation of $\mathrm{Clif}(\mathfrak{m})$, which just extends the action of R on \mathfrak{m} . Then L_W^0 defines a derivation of $\mathrm{Clif}(\mathcal{T}(G/K))$, which we denote again by L_W^0 . It is defined by

$$(L_W^0\varphi)(x) = L_W^0(x)(\varphi(x)),$$

where $L_W^0(x)$ here denotes the extension of $L_W^0(x)$ on \mathfrak{m} to a derivation of $\mathrm{Clif}(\mathfrak{m})$. Since the sum of two derivations is a derivation, $\nabla_W^0 + L_W^0$ acts as a derivation on $\mathrm{Clif}(\mathcal{T}(G/K))$. It is easily checked that $\nabla^0 + L^0$ is then an A-linear map from $\mathcal{T}(G/K)$ into the algebra of derivations of $\mathrm{Clif}(\mathcal{T}(G/K))$. This is the "Levi–Civita Clifford connection" that is used when trying to define the Dirac operator for the λ -invariant Riemannian metric on G/K.

We note that for any L which is skew-symmetric the above construction works for L in place of L^0 , that is, the construction works for any connection on $\mathcal{T}(G/K)$ that is compatible with the Riemannian metric. We have not yet used the torsion = 0 condition.

8. The Hodge-Dirac operator

To define the usual Dirac operator on G/K for its canonical Riemannian metric (which depends on the Ad-invariant inner product on \mathfrak{g} that has been chosen), one must deal with the issues of whether G/K is spin or spin^c , and with the bookkeeping details coming from whether G/K is of even or odd dimension. We will not pursue these aspects in this paper (so see [5, 6, 16, 1, 15]). But one always has the generalized Dirac operator that is often called the Hodge–Dirac operator. (See 9.B of [6], except that here we work over \mathbb{R} rather than \mathbb{C} .)

We work first with smooth functions, before putting on a Hilbert-space structure. We need a representation of $\text{Clif}(\mathcal{T}(G/K))$ to serve as "spinors". We take the left-regular representation of $\text{Clif}(\mathcal{T}(G/K))$ on itself. So our Dirac operator will be an operator on $\mathcal{S} = \text{Clif}(\mathcal{T}(G/K))$. We recall that we have required the sections of $\text{Clif}(\mathcal{T}(G/K))$ to be smooth.

Let ∇ be any connection on $\mathcal{T}(G/K)$ compatible with the Riemannian metric on G/K, and extend it to $\mathrm{Clif}(\mathcal{T}(G/K))$ as done in the previous section. For $\varphi \in \mathcal{S}$ define $d\varphi$ by $d\varphi(W) = \nabla_W \varphi$ for $W \in \mathcal{T}(G/K)$. Thus we can view $d\varphi$ as an element of $\mathcal{S} \otimes \mathcal{T}^*(G/K)$, where $\mathcal{T}^*(G/K)$ denotes the cross-section module of the cotangent bundle. But by means of the Riemannian metric we can identify $\mathcal{T}^*(G/K)$ with $\mathcal{T}(G/K)$. When $d\varphi$ is viewed as an element of $\mathcal{S} \otimes \mathcal{T}(G/K)$, we denote it, with some abuse of notation, by grad_{φ} . Let c denote the product on the algebra $\mathcal{S} = \mathrm{Clif}(\mathcal{T}(G/K))$, viewed as a linear map from $\mathcal{S} \otimes \mathcal{S}$ to \mathcal{S} . We view $\mathcal{T}(G/K)$ as a subspace of \mathcal{S} , and so we view $\mathcal{S} \otimes \mathcal{T}(G/K)$ as a subspace of $\mathcal{S} \otimes \mathcal{S}$. In this way we view grad_{φ} as an element of $\mathcal{S} \otimes \mathcal{S}$, to which we can apply c. We can then define an operator, D, on \mathcal{S} by

$$(8.1) D\varphi = c(\operatorname{grad}_{\varphi})$$

for all $\varphi \in \mathcal{S}$. When ∇ is the Levi–Civita connection, this will be our Hodge–Dirac operator, but we do not yet assume that ∇ has torsion 0.

Let us obtain a more explicit formula for $\operatorname{grad}_{\varphi}$, and so for D. Consider a standard module frame $\{W_j\}$ for $\mathcal{T}(G/K)$, for example $\{\hat{X}_j\}$ where $\{X_j\}$ is an orthonormal basis for \mathfrak{g} . Then for any $V \in \mathcal{T}(G/K)$ we have

$$d\varphi(V) = \nabla_V \varphi = \nabla_{\sum W_j \langle W_j, V \rangle_A} \varphi = \sum (\nabla_{W_j} \varphi) \langle W_j, V \rangle_A,$$

and so we have

$$\operatorname{grad}_{\varphi} = \sum (\nabla_{W_j} \varphi) \otimes W_j.$$

When we apply the Clifford product c to this formula for $\operatorname{grad}_{\varphi}$, but also use our earlier "dot" notation for the Clifford product, we find that

(8.2)
$$D\varphi = \sum (\nabla_{W_j}\varphi) \cdot W_j.$$

We emphasize that D is independent of the choice of standard module frame, as can easily be seen directly form the fact that $(U, V) \mapsto (\nabla_U \varphi) \cdot V$ is clearly A-bilinear. Thus we can choose different frames at our convenience to do computations. A first instance of this appears in the next paragraph.

Suppose that the connection ∇ is λ -invariant. We saw in the previous section that this implies that its extension to \mathcal{S} satisfies $\lambda_y(\nabla_W\varphi) = \nabla_{\lambda_y W}(\lambda_y \varphi)$. Let us see that this implies that D commutes with λ . For a given standard module frame $\{W_j\}$ we have

$$D(\lambda_y \varphi) = \sum \nabla_{W_j} (\lambda_y \varphi) \cdot W_j = \sum (\lambda_y (\nabla_{\lambda_y^{-1} W_j} \varphi)) \cdot \lambda_y (\lambda_y^{-1} W_j)$$
$$= \lambda_y \left(\sum (\nabla_{\lambda_y^{-1} W_j} \varphi) \right) \cdot (\lambda_y^{-1} W_j) = \lambda_y (D\varphi),$$

where we have used the easily verified fact that $\{\lambda_y^{-1}W_j\}$ is again a standard module frame. Thus

$$D\lambda_u = \lambda_u D$$

for all $y \in G$.

The elements of $A = C^{\infty}(G/K)$ act as pointwise multiplication operators on \mathcal{S} , and it is important to calculate their commutators with D. For $f \in A$ let M_f denote the corresponding operator on \mathcal{S} . Then by the Leibniz rule for ∇

$$[D, M_f](\varphi) = \sum (\nabla_{W_j}(\varphi f)) \cdot W_j - (\sum (\nabla_{W_j} \varphi) \cdot W_j) f$$

= $\varphi \cdot (\sum W_j(\delta_{W_j} f))$.

But much as above

$$df(V) = \delta_V f = \delta_{\sum W_j \langle W_j, V \rangle_A} f$$

= $\sum (\delta_{W_j} f) \langle W_j, V \rangle_A = \left\langle \sum W_j (\delta_{W_j} f), V \right\rangle_A$,

so that $\sum W_i(\delta_{W_i}f)$ is the usual gradient, grad, of f. That is:

Proposition 8.3. For $f \in A$ and $\varphi \in S$ we have

$$[D, M_f]\varphi = \varphi \cdot \operatorname{grad}_f,$$

the product being that in $Clif(\mathcal{T}(G/K)) = \mathcal{S}$.

We now want a Hilbert space structure on \mathcal{S} . As on any Clifford algebra, there is a canonical normalized trace, τ , on $\mathrm{Clif}(\mathfrak{m})$, determined by the properties that $\tau(1)=1$ and that if $\{Y_j\}_{j=1}^q$ is a set of mutually orthogonal elements of \mathfrak{m} then $\tau(Y_1\cdot\ldots\cdot Y_q)=0$. Note in particular that $\tau(X\cdot Y)=-\langle X,Y\rangle_{\mathfrak{m}}$ for any $X,Y\in\mathfrak{m}$. (One way to see the existence of τ is to consider an orthonormal basis for \mathfrak{m} and to decree that τ is 0 on all of the corresponding basis elements [5,6] for $\mathrm{Clif}(\mathfrak{m})$ except 1.) On $\mathrm{Clif}(\mathfrak{m})$ there is the standard involutory automorphism carrying X to -X for $X\in\mathfrak{m}$, and the standard involutory anti-automorphism that reverses the order of products. We let * denote the composition of these two, so that * is an anti-automorphism of $\mathrm{Clif}(\mathfrak{m})$ that carries X to -X for $X\in\mathfrak{m}$. It is involutory

since its square is an automorphism that is the identity on $\mathfrak{m} \subset \mathrm{Clif}(\mathfrak{m})$. On $\mathrm{Clif}(\mathfrak{m})$ we define an ordinary inner product, $\langle \cdot, \cdot \rangle_c$, by

$$\langle \varphi, \psi \rangle_c = \tau(\varphi^* \cdot \psi)$$

for $\varphi, \psi \in \text{Clif}(\mathfrak{m})$. It is easy to verify that for any orthonormal basis for \mathfrak{m} the corresponding basis for $\text{Clif}(\mathfrak{m})$ is orthonormal for this inner product, and that the inner product is, in fact, positive definite. Both the left and right regular representations of $\text{Clif}(\mathfrak{m})$ on itself are easily seen to be *-representations for this inner product. In particular, elements of \mathfrak{m} act, on left and right, as skew-symmetric operators.

We apply all of the above structures to $S = \text{Clif}(\mathcal{T}(G/K))$. We obtain an involution on S defined by $(\varphi^*)(x) = (\varphi(x))^*$, and we obtain an A-valued inner product, $\langle \cdot, \cdot \rangle_A$, on S defined by

$$\langle \varphi, \psi \rangle_A(x) = \langle \varphi(x), \psi(x) \rangle_c$$

for $\varphi, \psi \in \mathcal{S}$. The left and right regular representations of $\mathrm{Clif}(\mathcal{T}(G/K))$ on \mathcal{S} are then "*-representations", that is, for any $\theta \in \mathcal{S}$ we have

$$\langle \theta \cdot \varphi, \ \psi \rangle_A = \langle \varphi, \ \theta^* \cdot \psi \rangle_A,$$

and similarly for θ acting on the right. Finally, we can define an ordinary inner product on S by

$$\langle \varphi, \psi \rangle = \int_{G/K} \langle \varphi, \psi \rangle_A(x) dx.$$

When S is completed for this inner product we obtain our Hilbert space of "spinors" for the corresponding "Hodge-Dirac" operator. We denote this Hilbert space by $L^2(S,\tau)$. We can now view the Hodge-Dirac operator as an unbounded operator on $L^2(S,\tau)$ with domain S.

One reason for the importance of the torsion = 0 condition, or at least a weak version of it, is in determining whether D is formally self-adjoint, that is,

$$\langle D\varphi, \ \psi \rangle = \langle \varphi, \ D\psi \rangle$$

for all $\varphi, \psi \in \mathcal{S}$. For any $U \in \mathcal{T}(G/K)$ let T^U_{∇} be the A-endomorphism of $\mathcal{T}(G/K)$ defined by $T^U_{\nabla}(V) = T_{\nabla}(U, V)$. Then T^U_{∇} is given by a function on G who values are operators on \mathfrak{m} , according to Proposition 3.1. Thus we can define $\operatorname{trace}(T^U_{\nabla})$ pointwise as a function in A. Equivalently, $\operatorname{trace}(T^U_{\nabla}) = \sum_j \langle T_{\nabla}(U, W_j), W_j \rangle_A$ for one (hence every) standard module frame $\{W_j\}$ for $\mathcal{T}(G/K)$.

Theorem 8.4. Let ∇ be any G-invariant connection on $\mathcal{T}(G/K)$ compatible with our chosen Riemannian metric on G/K. Let D be the Hodge-Dirac operator defined as above for ∇ , viewed as an unbounded operator on $L^2(\mathcal{S}, \tau)$ with domain \mathcal{S} . Then D is formally self-adjoint if and only if

$$\operatorname{trace}(T^U_\nabla) \ = \ 0$$

for all $U \in \mathcal{T}(G/K)$, where T_{∇} is the torsion of ∇ .

Proof. As one might suspect, the proof is a complicated version of "integration by parts" or the divergence theorem. We somewhat follow the pattern of the proof in section 3.2 of [5] or of proposition 9.13 of [6]. Let $\{W_j\}$ be a standard module frame for $\mathcal{T}(G/K)$. We use first the Leibniz rule for ∇ extended to \mathcal{S} , and then the

fact that ∇ is compatible with the Riemannian metric, in order to calculate that for $\varphi, \psi \in \mathcal{S}$ we have

$$\langle D\varphi, \ \psi \rangle_{A} - \langle \varphi, \ D\psi \rangle_{A} = \sum_{j} (\langle (\nabla_{W_{j}}\varphi) \cdot W_{j}, \ \psi \rangle_{A} - \langle \varphi, \ (\nabla_{W_{j}}\psi) \cdot W_{j} \rangle_{A})$$

$$= \sum_{j} (-\langle \nabla_{W_{j}}\varphi, \ \psi \cdot W_{j} \rangle_{A} - \langle \varphi, \ \nabla_{W_{j}}(\psi \cdot W_{j}) - \psi \cdot (\nabla_{W_{j}}W_{j}) \rangle_{A})$$

$$= \sum_{j} (-\delta_{W_{j}}(\langle \varphi, \ \psi \cdot W_{j} \rangle_{A}) + \langle \varphi, \ \psi \cdot (\nabla_{W_{j}}W_{j}) \rangle_{A}).$$

For given φ and ψ the function $V \mapsto \langle \varphi, \psi \cdot V \rangle_A$ is A-linear, and so by the self-duality of $\mathcal{T}(G/K)$ for its Riemannian metric there is a $U \in \mathcal{T}(G/K)$ such that $\langle \varphi, \psi \cdot V \rangle_A = \langle U, V \rangle_A$ for all $V \in \mathcal{T}(G/K)$. By letting $\psi = 1$ and $\varphi = U$ (viewed as elements in $\text{Clif}(\mathcal{T}(G/K))$) we see that any U arises in this way. The above displayed expression is then equal to

$$\sum_{j} -\delta_{W_{j}}(\langle U, W_{j} \rangle_{A}) + \langle U, \nabla_{W_{j}} W_{j} \rangle_{A} = -\sum_{j} \langle \nabla_{W_{j}} U, W_{j} \rangle_{A} .$$

We thus see that D is formally self-adjoint exactly if $\int \sum_{j} \langle \nabla_{W_{j}} U, W_{j} \rangle_{A} = 0$ for all $U \in \mathcal{T}(G/K)$ and one, hence every, standard module frame $\{W_{j}\}$.

Now by the definition of T_{∇}

$$\langle \nabla_{W_j} U, W_j \rangle_A = \langle \nabla_U W_j - T_{\nabla}(U, W_j) - [U, W_j], W_j \rangle_A.$$

Notice then that, by the A-bilinearity of the inner product, $\sum \langle W_j, W_j \rangle_A$ is independent of the choice of standard module frame, and that $\{W_j(x)\}$ is a frame for \mathfrak{m} for any $x \in G$, so that we can evaluate the sum by using a frame for \mathfrak{m} that consists of an orthonormal basis for \mathfrak{m} . From this we see that $\sum \langle W_j, W_j \rangle_A \equiv \dim(\mathfrak{m})$. Consequently, by the compatibility of ∇ with the Riemannian metric, for any $U \in \mathcal{T}(G/K)$ we have

$$0 = \delta_U(\sum_j \langle W_j, W_j \rangle_A) = \sum_j \langle \nabla_U W_j, W_j \rangle_A + \langle W_j, \nabla_U W_j \rangle_A = 2 \sum_j \langle \nabla_U W_j, W_j \rangle_A.$$

Consequently $\sum_j \langle \nabla_U W_j, W_j \rangle_A = 0$. (We remark that this fact depends on the pointwise argument just above, and that the analogous argument can fail for modules over a non-commutative A that contains proper isometries.) Thus

$$\sum_{j} \langle \nabla_{W_j} U, W_j \rangle_A = -\sum_{j} \langle T_{\nabla}(U, W_j), W_j \rangle_A - \sum_{j} \langle [U, W_j], W_j \rangle_A .$$

Let $\nabla^t = \nabla^0 + L^0$, the Levi-Civata connection. We can apply the above equation to ∇^t and use that ∇^t is torsion-free to get an espression for the last term above. In this way we find that

$$\langle \nabla_{W_j} U, W_j \rangle_A = -\sum_j \langle T_{\nabla}(U, W_j), W_j \rangle_A + \sum_j \langle \nabla_{W_j}^t U, W_j \rangle_A .$$

We will show shortly that

$$\int_{G/K} \sum_{j} \langle \nabla_{W_j}^t U, W_j \rangle_A = 0$$

for all U. This will show that D is formally self-adjoint if and only if

$$0 = \int \sum_{j} \langle T_{\nabla}(U, W_{j}), W_{j} \rangle_{A} = \int \operatorname{trace}(T_{\nabla}^{U})$$

for all U. But the integrand of these latter integrals is clearly A-linear in U, so when we replace U by fU for any $f \in A$ the f comes outside the inner product. Since f is arbitrary, this means that the integral is 0 for all U exactly if the integrand itself is 0 for all U, and that is the condition in the statement of the theorem.

Thus we have basically reduced the proof of the theorem to treating ∇^t . We remark that, because ∇^t is the Levi-Civita connection, $\sum_j \langle \nabla^t_{W_j} U, W_j \rangle_A = \operatorname{div}(U)$, so that in effect we need to prove the divergence theorem $\int \operatorname{div}(U) = 0$. Since ∇^t is compatible with the Riemannian metric, we have

$$\langle \nabla_{W_j}^t U, W_j \rangle_A = \delta_{W_j} (\langle U, W_j \rangle_A) - \langle U, \nabla_{W_j}^t W_j \rangle_A.$$

Now for any $X \in \mathfrak{g}$ and any $f \in A$ we have $\delta_{\hat{X}}(f) = \lambda_X f$, and

$$\int_{G} (\lambda_X f)(x) dx = 0$$

because $\lambda_X f$ is the uniform limit of the quotients

$$(\lambda_{\exp(-tX)}f - f)/t$$

as t converges to 0 (see lemma 1.7 of chapter 0 of [18], where one needs \mathcal{L} to be invariant), and the integral over G of each of these quotients is clearly 0 by the left-invariance of Haar measure. Thus if we choose our standard module frame to be $W_j = \hat{X}_j$ for an orthonormal basis $\{X_j\}$ for \mathfrak{g} , we see that for each j

$$\int_{G/K} \delta_{\hat{X}_j}(\langle U, \hat{X}_j \rangle_A) = 0.$$

Consequently

$$\int_{G/K} \sum_{j} \langle \nabla_{\hat{X}_{j}}^{t} U, \hat{X}_{j} \rangle_{A} = \int_{G/K} \sum_{j} \langle U, \nabla_{\hat{X}_{j}}^{t} \hat{X}_{j} \rangle_{A} .$$

Since we want this to be 0 for all U, it is clear that we need to show that

$$\sum_j \nabla^t_{\hat{X}_j} \hat{X}_j = 0 .$$

Now $\nabla^t = \nabla^0 + L^0$, and for any $W \in \mathcal{T}(G/K)$ we have

$$(L_W^0 W)(x) = (1/2)P[W(x), W(x)] = 0.$$

Thus we only need to show that $\sum_{j} \nabla^{0}_{\hat{X}_{j}} \hat{X}_{j} = 0$. But

$$(\sum_{j} \nabla^{0}_{\hat{X}_{j}} \hat{X}_{j})(x) = -\sum_{j} P[P \operatorname{Ad}_{x}^{-1} X_{j}, \operatorname{Ad}_{x}^{-1} X_{j}].$$

Note that $\{\operatorname{Ad}_x^{-1} X_j\}$ is an orthonormal basis for $\mathfrak g$ for each x. Now $(X,Y)\mapsto P[PX,Y]$ is bilinear, and so it is easily seen that the value of $\sum_j P[PX_j,X_j]$ does not depend on the choice of orthonormal basis. We can then choose our basis such that $X_1,\ldots X_p$ is a basis for $\mathfrak m$ while X_{p+1},\ldots ,X_n is a basis for $\mathfrak k$. Then for $j\leq p$ we have $PX_j=X_j$ so that the corresponding terms in the sum are 0, while for j>p we have $PX_j=0$ so that again the corresponding terms in the sum are 0.

We can use this to show that even though the canonical connection is often not torsion-free, we have:

Corollary 8.5. The Hodge-Dirac operator for the canonical connection ∇^0 is formally self-adjoint.

Proof. We apply the criterion of Theorem 8.4 to the canonical connection. From our calculation of T_{∇^0} in Section 6 we see that for any $U \in \mathcal{T}(G/K)$ we have $T_{\nabla^0}^U(x) = -P \circ \mathrm{ad}_{U(x)}$. We can use an orthonormal basis X_1, \ldots, X_p for \mathfrak{m} in calculating the trace. Extend this basis by an orthonormal basis X_{p+1}, \ldots, X_n for \mathfrak{k} . Then for any $U(x) = Y \in \mathfrak{m}$ we have

$$\operatorname{trace}(P \circ \operatorname{ad}_Y) = \sum^p \langle P[Y,X_j], X_j \rangle_{\mathfrak{m}} = \sum^n \langle [Y,X_j], X_j \rangle_{\mathfrak{g}} = \operatorname{trace}(\operatorname{ad}_Y \operatorname{on} \, \mathfrak{g})$$

since $[Y,X_j] \in \mathfrak{m}$ if $X_j \in \mathfrak{k}$ so they are orthogonal. But for each j we have $\langle [Y,X_j],X_j\rangle = \langle Y,[X_j,X_j]\rangle = 0$, and so trace(ad_Y on \mathfrak{g}) = 0. Thus ∇^0 satisfies the criterion of Theorem 8.4, and so its Hodge-Dirac operator is formally self-adjoint.

I have not seen mentioned in the literature the possibility that some non-torsionfree connections can nevertheless have formally self-adjoint Dirac operators.

A simple further calculation using the fact that L_U is a skew-adjoint operator and so has trace 0, gives:

Corollary 8.6. Let ∇ be a connection compatible with our chosen Riemannian metric, and let $L = \nabla - \nabla^0$. Then the Hodge-Dirac operator for ∇ is formally self-adjoint if and only if for one (hence every) standard module frame $\{W_j\}$ for $\mathcal{T}(G/K)$ we have

$$\sum_{j} L_{W_j} W_j = 0.$$

For the essential self-adjointness of Dirac operators see, for example, section 9.4 of [6] and section 4.1 of [5].

Let us now consider the operator norm of $[D, M_f]$ for $f \in A$. We saw earlier that $[D, M_f]\varphi = \varphi \cdot \operatorname{grad}_f$ for $\varphi \in \mathcal{S}$, and that $\operatorname{grad}_f \in \mathcal{T}(G/K)$. Let R denote the right-regular representation of \mathcal{S} on itself and so on $L^2(\mathcal{S}, \tau)$. Now the norm of any bounded operator, T, on an inner-product space satisfies the C^* -condition $||T||^2 = ||T^*T||$, so for any $\psi \in \mathcal{S}$ we have $||R_\psi||^2 = ||R_\psi^*R_\psi||$. When we use this for $\psi = V \in \mathcal{T}(G/K)$, and recall that elements of $\mathcal{T}(G/K)$ act as skew-symmetric operators for the Clifford product, we see that

$$||R_V||^2 = ||R_V^*R_V|| = ||-R_{V\cdot V}|| = ||R_{\langle V,V\rangle_A}||.$$

But simple arguments show that for any $g \in A$ we have $||R_g|| = ||g||_{\infty}$. When we apply all of this for $V = \operatorname{grad}_f$ we obtain

$$||R_{\operatorname{grad}_f}||^2 = ||\langle \operatorname{grad}_f, \operatorname{grad}_f \rangle_A||_{\infty},$$

so that $||R_{\operatorname{grad}_f}|| = ||\operatorname{grad}_f||_{\infty}$ for the evident meaning of this last term. But a standard argument (e.g., following definition 9.13 of [6]) shows that if we denote by ρ the ordinary metric on a Riemannian manifold M coming from its Riemannian metric, then for any two points p and q of M we have

$$\rho(p,q) = \sup\{|f(p) - f(q)| : \|\operatorname{grad}_f\|_{\infty} \le 1\}.$$

On applying this to G/K, using what we found above for the Dirac operator, we obtain, for ρ now the ordinary metric on G/K,

$$\rho(p,q) = \sup\{|f(p) - f(q)| : ||[D, M_f]|| \le 1\}.$$

This is the formula on which Connes focused for general Riemannian manifolds [3, 4] as it shows that the Dirac operator contains all the metric information (and, in fact, much more) for the manifold. This is his motivation for advocating that metric data for "non-commutative spaces" be encoded by providing them with a "Dirac operator".

We showed earlier that our D commutes with the action λ of G. This is exactly the manifestation in terms of D of the fact that the action of G on G/K is by isometries for the Riemannian metric and its ordinary metric.

It would be interesting to understand how all of the above relates to Connes' action principle for selecting *the* Dirac operator from among all of the spectral triples that give a specified Riemannnian [4]. (See also theorem 11.2 and section 11.4 of [6].) Of course, on the face of it Connes' action principle is just for spin-manifolds while many homogeneous spaces are not spin.

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