

Simultaneous separation for the Kowalevski and Goryachev-Chaplygin gyrostats

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Abstract

In the special case of zero square integral the Kowalevski gyrostat and Goryachev-Chaplygin gyrostat share a simple separation of variables originated from the 4×4 Lax matrix.

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Introduction

The theory of *separation of variables* (SoV) is the theory of special canonical transformations and the theory of *quantum separation of variables* is the theory of corresponding integral transforms. The former was already understood in this sense starting from the development of the Hamilton-Jacobi approach for solving Liouville integrable systems. The real power of the latter, which by now is undoubted, requires further investigation and demonstration.

The *speciality* of a separating transformation stems from its definition as a transform resulting in new variables *being separated* or equations *being decoupled* from one another. Below we give a (working) definition of SoV in the context of finite-dimensional integrable Hamiltonian dynamics.

By *separation of variables* for an n -degrees-of-freedom integrable system having n independent Poisson commuting integrals $\{H_j(\mathbf{q}, \mathbf{p})\}_{j=1}^n$ we mean a canonical transformation from the (old) Darboux variables q_j, p_j , $j = 1, \dots, n$, to new Darboux variables u_j, v_j , $j = 1, \dots, n$, which satisfy the following *separation equations*:

$$\sum_{j=1}^n a_{ij}(u_i, v_i) H_j(\mathbf{q}, \mathbf{p}) = b_i(u_i, v_i), \quad i = 1, \dots, n. \quad (1)$$

In other words, being expressed in terms of the new variables, the integrals of motion H_j acquire the following ‘separated form’:

$$\mathbf{H} = \mathbf{A}^{-1}\mathbf{B}, \quad (\mathbf{A})_{ij} = a_{ij}, \quad (\mathbf{B})_i = b_i, \quad (\mathbf{H})_i = H_i. \quad (2)$$

The conditions that the functions a_{ij} and b_i in (1) depend on the new (separation) variables with the index i *only*, is crucial. It indeed means that the n equations in (1) are really separated from one another. Therefore, they are n equations, each of one degree of freedom, sharing only the common values of the Hamiltonians H_j ¹.

The above definition includes, as even more special canonical transform, the (classical) coordinate separation of variables when the new coordinates, say \mathbf{u} , are the functions of the old coordinates (\mathbf{q}) only, and they do not depend on the momenta (\mathbf{p}). See [18, 32, 33, 10] for some history of this sub-class of transformations and for many examples of such situation.

General separating canonical transforms, however, have new (separation) variables which are non-trivial functions of all $2n$ initial canonical variables,

$$u_j = u_j(\mathbf{q}, \mathbf{p}), \quad v_j = v_j(\mathbf{q}, \mathbf{p}), \quad j = 1, \dots, n, \quad (3)$$

and which satisfy the standard Poisson brackets

$$\{u_j, u_k\} = \{v_j, v_k\} = \{u_j, v_k\} = 0, \quad j \neq k, \quad \{v_j, u_j\} = 1, \quad j = 1, \dots, n. \quad (4)$$

Examples of such separating canonical transforms are usually much more sophisticated than those of the coordinate ones. As far as I know, the first explicit example was given by van Moerbeke in 1976 in [48] concerning the separation of variables for the periodic Toda lattice (see also [11]). In 1980 Kozlov [30] rewrote the classical results of Goryachev and Chaplygin on integration in quadratures of the Goryachev-Chaplygin top as a simple canonical transform. The method of SoV, viewed as a method of special canonical transformations, was

¹as one can see, this definition is similar to the one defining Stäckel systems

further developed by Komarov in a series of works on tops, including quantum separation of variables, see [20, 21, 22]. Many further examples have been produced since 1982, with the theory benefiting mostly from the developments of the algebraic geometric and r -matrix understanding of the method of separation of variables. This led to a rather satisfying picture of the present state-of-art of non-coordinate separation of variables. See [54] and [16, 17] for more details.

Notice here that in the realm of the algebraic geometry usually associated with many Liouville integrable systems, the separation equations (1) appear as equations for a set of separating Darboux coordinates on the spectral curve of a Lax matrix $L(u)$, namely:

$$(1) \quad \Leftrightarrow \quad \det(L(u_i) - v_i) = 0, \quad i = 1, \dots, n. \quad (5)$$

SoV is not unique, so that in the situation when there exist several Lax matrices for the same integrable system, one should expect several separations. But even for the same Lax matrix there are many different separations given by different sections², cf. [36, 39, 43].

In the present paper we find a new canonical transform which simultaneously separates two classical tops: the Kowalevski top [29] and the Goryachev-Chaplygin top [13, 9]. The latter top is integrable only for the zero value of the square integral, $\ell = 0$ (which is one of the Casimirs), so that although the canonical transform will be defined for arbitrary value of ℓ , it will separate the Kowalevski top (and the other one) only for $\ell = 0$. SoV for the Kowalevski top for $\ell \neq 0$, which would generalize the found transform, remains an unsolved problem.

Sretenskii [55] discovered an integrable extension of the Goryachev-Chaplygin top adding the gyrostatic term to the Hamiltonian. Komarov [24] and, independently, Yehia [57] found the gyrostat extension of the Kowalevski top in 1987. The quantum Goryachev-Chaplygin gyrostat was treated in [23]. For quantum Kowalevski gyrostat see [24]. Here we will always include such gyrostatic terms (cf. the parameter c below).

In Section 1 we give definitions of the integrable systems concerned and their Lax matrices. In Section 2 we recall the separation method with main results presented in Section 3. In Section 4 we re-write the found separating transformation through the generating function. Solution of the inverse problem is given in Section 5. Finally, some concluding remarks can be found in the last Section.

1 Algebra $e(3)$, tops and Lax matrices

The Poisson brackets for the $e(3)$ generators $J_k, x_k, k = 1, 2, 3$, are defined in the standard way:

$$\{J_k, J_l\} = \varepsilon_{klm} J_m, \quad \{J_k, x_l\} = \varepsilon_{klm} x_m, \quad \{x_k, x_l\} = 0, \quad (6)$$

where ε_{klm} is the completely anti-symmetric tensor, $\varepsilon_{123} = 1$.

The Casimirs of the bracket (6) have the form

$$C_1 = x_1 J_1 + x_2 J_2 + x_3 J_3 = \ell, \quad C_2 = x_1^2 + x_2^2 + x_3^2 = 1. \quad (7)$$

The Kowalevski gyrostat has the Hamiltonian H ,

$$H = J^2 + (J_3 + c)^2 - 2bx_1, \quad (8)$$

²those usually differ by the number of non-moving poles of the Baker-Akhiezer function

and the second integral K ,

$$K = (J_3 + c)^2 J^2 + 2b(J_3 + c)(J_1 x_3 - J_3 x_1) - b^2 x_2^2 - 2blJ_1, \quad (9)$$

which are Poisson commuting:

$$\{H, K\} = 0. \quad (10)$$

Here we use the following notation for the square of the vector of angular momentum:

$$J^2 := J_1^2 + J_2^2 + J_3^2. \quad (11)$$

The two integrals of motion, H and K , define what is called Kowalevski gyrostat. It is a Liouville integrable system with two degrees of freedom.

Another integrable system closely related to the Kowalevski gyrostat is the Goryachev-Chaplygin gyrostat. It is integrable only when $\ell = 0$ and is defined by two Poisson commuting integrals of motion:

$$\hat{H} = J_1^2 + J_2^2 + (2J_3 + c)^2 - 4bx_1, \quad (12)$$

$$\hat{K} = (J_3 + c)(J_1^2 + J_2^2) + 2bx_3 J_1. \quad (13)$$

The Kowalevski top without the gyrostatic term ($c = 0$) was integrated by Kowalevski in 1889 [29]. See also [35] where a 2×2 Lax matrix was constructed for the Kowalevski top which is related to the separation of variables hidden in Kowalevski's integration. There is a large body of literature dedicated to the Kowalevski top, including the study of its geometry. See [20, 24, 26, 14, 51, 5, 15, 8, 42, 28, 56, 47, 44, 25, 45, 46] to name just a few references. The Goryachev-Chaplygin top and gyrostat also received much attention in the literature starting from the discovery of this integrable top by Goryachev and Chaplygin in 1900. We refer the reader to the works [13, 9, 30, 21, 22, 23, 52, 6, 12, 7, 27].

In parallel with the notations J_1, J_2 and x_1, x_2 we will be using their equivalent complex versions J_+, J_- and x_+, x_- :

$$J_{\pm} = J_1 \pm iJ_2, \quad x_{\pm} = x_1 \pm ix_2, \quad (14)$$

with the e(3) Poisson brackets (6) replaced, correspondingly, by

$$\{J_3, J_{\pm}\} = \mp iJ_{\pm}, \quad \{J_+, J_-\} = -2iJ_3, \quad \{J_3, x_{\pm}\} = \{x_3, J_{\pm}\} = \mp ix_{\pm}, \quad (15)$$

$$\{J_+, x_-\} = \{x_+, J_-\} = -2ix_3, \quad \{J_3, x_3\} = \{J_+, x_+\} = \{J_-, x_-\} = 0, \quad (16)$$

$$\{x_k, x_l\} = 0, \quad k, l = \pm, 3, \quad (17)$$

and the Casimirs (7) by

$$2C_1 = x_+ J_- + x_- J_+ + 2x_3 J_3 = 2\ell, \quad C_2 = x_+ x_- + x_3^2 = 1. \quad (18)$$

A 4×4 Lax matrix for the Kowalevski gyrostat was found in [51] and used in [8] to integrate the problem in terms of Prymian theta-functions. This integration is different from the one performed by Kowalevski. It is not obvious how to construct a separation of variables related to such Lax matrix. With the general case still being an interesting and challenging problem, we show here how to do it in the special case, when $\ell = 0$. That is, for this special case we construct a new separation of variables for the Kowalevski top (and

simultaneously for the Goryachev-Chaplygin top) with separation variables belonging to the spectral curve of the 4×4 Lax matrix from [51].

The required Lax matrix is as follows:

$$L(u) = -i \begin{pmatrix} \frac{c}{u} & -\frac{bx_-}{u^2} & \frac{J_-}{u} & -\frac{bx_3}{u^2} \\ \frac{bx_+}{u^2} & -\frac{c}{u} & \frac{bx_3}{u^2} & -\frac{J_+}{u} \\ \frac{J_+}{u} & -\frac{bx_3}{u^2} & \frac{2J_3+c}{u} & 2 + \frac{bx_+}{u^2} \\ \frac{bx_3}{u^2} & -\frac{J_-}{u} & -2 - \frac{bx_-}{u^2} & -\frac{2J_3+c}{u} \end{pmatrix}. \quad (19)$$

The spectral curve $\Gamma: \det(L(u) - v) = 0$, of the Lax matrix (19) has the form

$$\Gamma: \left(v^2 + \frac{H}{u^2} - \frac{b^2}{u^4} \right)^2 - 4 \frac{u^4 v^2 + K}{u^4} - 4 \frac{c^2 u^4 - b^2 \ell^2}{u^6} = 0, \quad (20)$$

or,

$$v^4 - 2v^2 \left(2 - \frac{H}{u^2} + \frac{b^2}{u^4} \right) - \frac{4c^2}{u^2} + \frac{H^2 - 4K}{u^4} + \frac{2b^2(2\ell^2 - H)}{u^6} + \frac{b^4}{u^8} = 0. \quad (21)$$

As I noticed in [31] (cf. also [7]) this Lax matrix contains the 3×3 Lax matrix $\hat{L}(u)$ for the Goryachev-Chaplygin top as its $(1, 1)$ -minor:

$$\hat{L}(u) = -i \begin{pmatrix} -\frac{c}{u} & \frac{bx_3}{u^2} & -\frac{J_+}{u} \\ -\frac{bx_3}{u^2} & \frac{2J_3+c}{u} & 2 + \frac{bx_+}{u^2} \\ -\frac{J_-}{u} & -2 - \frac{bx_-}{u^2} & -\frac{2J_3+c}{u} \end{pmatrix}. \quad (22)$$

The spectral curve $\hat{\Gamma}: \det(\hat{L}(u) - v) = 0$, of the Lax matrix (22) has the form

$$v^3 - i \frac{cv^2}{u} - \left(4 - \frac{\hat{H}}{u^2} + \frac{b^2}{u^4} \right) v + \frac{4ic}{u} + i \frac{2\hat{K} - c\hat{H}}{u^3} + ib^2 \frac{c + 2x_3\ell}{u^5} = 0. \quad (23)$$

The Lax matrix (22) (and the curve (23)), for $\ell = 0$, was used in [7] to construct explicit theta-function formulas solving the dynamics of the Goryachev-Chaplygin top.

In the present paper I take one step further in studying a close connection between two problems and show that there exist separation variables u_j, v_j , $j = 1, 2$, which are canonical and which, for $\ell = 0$, belong to both curves, Γ and $\hat{\Gamma}$, simultaneously:

$$\begin{cases} \det(L(u_j) - v_j) = 0, \\ \det(\hat{L}(u_j) - v_j) = 0, \end{cases} \quad j = 1, 2, \quad (24)$$

$$\{u_j, u_k\} = \{v_j, v_k\} = \{u_j, v_k\} = 0, \quad j \neq k, \quad \{v_j, u_j\} = 1, \quad j = 1, 2. \quad (25)$$

Therefore, I construct a new separation of variables which is characterized by the property of being a *simultaneous separation for both tops*.

2 The method of SoV

Separation variables had been generally used to construct closed expressions for the action variables (in terms of abelian integrals) or to get a separated representation for the action function. Therefore, the SoV method had served for a long time an important but technical

role in solving Liouville integrable systems of classical mechanics. A new and much more exciting application of the method came with the development of quantum integrable systems. Because of the fact that quantization of the action variables seemed to be a rather formidable task, quantum separation of variables became an inevitable refuge. In fact, it has been successfully performed for many families of integrable systems (see, for instance, survey [54]).

Starting from about 1982 the method of separation of variables gets connected with the R -matrix formalism of the quantum inverse scattering method, developed during that time by the Leningrad School. It was noticed by Komarov (see [52] and [54]) that for the 2×2 L -operators (Lax matrices) the separation variables ought to be the zeros of the off-diagonal element of the L -operator. This observation was fully exploited by Sklyanin in [52, 53] who developed a beautiful (pure algebraic) setting for the method within the framework of the R -matrix technique. Since then this approach took off and led to separations for many families of integrable systems. The method was further generalized to include higher rank L -operators and non-standard normalizations, see the 1995 review [54] and the later developments in [19, 37, 38, 34, 36, 39, 40, 41].

An alternative, algebraic geometric approach, which dates back to Adler and van Moerbeke [3, 4] and Mumford [49] and includes many researchers, have been developed starting from about the same time (see, for instance, [50, 15, 1, 2, 56, 16, 17]). It is based on thorough studies of the geometry of lower genus (algebraic completely) integrable systems. It has also led to many important new separations for complicated systems and tops.

Below we recall the most important formulas of the SoV method, adopting the algebraic description and following mainly the work [36] (see also [54, 34]).

A Baker-Akhiezer function f is the eigenvector of the Lax matrix for an integrable system

$$L(u) f = v f, \quad (26)$$

considered as a function on the spectral curve $\Gamma : \det(L(u) - v) = 0$. The inverse scattering method pins down the separation variables as poles of this function (see the first example of this general fact in [48] and [11]). There is, however, a large freedom of similarity transformations of the Lax matrix,

$$L(u) \mapsto VL(u)V^{-1}, \quad (27)$$

which do not change the spectrum of $L(u)$ but change the divisor of poles of f . This freedom can be characterized, and therefore fixed, by introducing a *normalization* of the Baker-Akhiezer function,

$$\vec{\alpha} \cdot f \equiv \sum_{i=1}^N \alpha_i f_i = 1, \quad (f \equiv (f_1, \dots, f_N)^t), \quad (28)$$

which is given by a *normalization* (row-) *vector* $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$. In other words, one considers a *section of the line-bundle* (cf. [16, 17, 43]). A proper (separating) normalization/section should give a divisor $\mathcal{D} = \sum_{j=1}^n (u_j, v_j)$ of moving poles of f , consisting of n (independent) points on the curve Γ whose coordinates are canonical variables. The canonicity of the separation variables is usually checked by a calculation involving a r -matrix, but in lower genus situations it can be proved in a direct calculation. For a non-separating

normalization/section the divisor \mathcal{D} will usually have more points than needed, which will not give canonical variables.

Many families of Lax matrices have a simple separating normalization when one of the components of the Baker-Akhiezer vector is put 1 (and, hence, one looks at the common poles of the other components), which corresponds to the following vector $\vec{\alpha}$ ³:

$$\vec{\alpha}_0 \equiv (1, 0, \dots, 0, 0). \quad (29)$$

We will call such normalization the *standard normalization*. There are examples of non-standard (dynamical) separating normalizations for the systems with elliptic r -matrix [54, 17], Calogero-Moser systems [36] and the D -type Toda lattice [34]. See also [43] where non-standard dynamical separating normalizations were used to construct Bäcklund transformations. Generally, Lax matrices with extra symmetries require non-standard separating normalizations.

Now, assuming that we know a separating normalization $\vec{\alpha}$ for an integrable system with the Lax matrix $L(u)$, let us derive the equations for the separation variables (u_j, v_j) , $j = 1, \dots, n$.

From the linear problem (26) and normalization (28) we derive that $\vec{\alpha} \cdot L^k f = v^k$, $k = 0, \dots, N-1$, hence,

$$f = \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha} \cdot L(u) \\ \vdots \\ \vec{\alpha} \cdot L^{N-1}(u) \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{N-1} \end{pmatrix}. \quad (30)$$

Another useful representation for the eigenvector f , which can be verified directly, is as follows:

$$f_j = \frac{(L(u) - v)_{jk}^\wedge}{(\vec{\alpha} \cdot (L(u) - v)^\wedge)_k}, \quad \forall k = 1, \dots, N, \quad (31)$$

where the wedge denotes the adjoint matrix. It follows from it that the poles of f are the common zeros of the vector equation:

$$\vec{\alpha} \cdot (L(u) - v)^\wedge = 0. \quad (32)$$

Eliminating v from these equations, one can get a single equation for the u -components of the separating variables as zeros of the following determinant:

$$B(u) = \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha} \cdot L(u) \\ \vdots \\ \vec{\alpha} \cdot L^{N-1}(u) \end{pmatrix} = 0. \quad (33)$$

Notice that this is exactly the denominator in the representation (30).

Also, from the equations (32) we can obtain explicit formulas for the v -components of the separation variables in the form

$$v = A(u), \quad (34)$$

with $A(u)$ being rational functions of the entries of $L(u)$. Let us derive those formulas.

³or to any other similar vector with the 1 elsewhere

Define the matrices $L^{(p)}$, $p = 1, \dots, N$, with the following entries:

$$L_{ij}^{(p)} := \sum_{i_1=1}^N \cdots \sum_{i_{p-1}=1}^N \begin{vmatrix} L_{i,j} & L_{i,i_1} & \cdots & L_{i,i_{p-1}} \\ L_{i_1,j} & L_{i_1,i_1} & \cdots & L_{i_1,i_{p-1}} \\ \vdots & \vdots & \ddots & \vdots \\ L_{i_{p-1},j} & L_{i_{p-1},i_1} & \cdots & L_{i_{p-1},i_{p-1}} \end{vmatrix}, \quad p = 2, 3, \dots, N, \quad (35)$$

and put $L^{(1)} \equiv L$. These matrices satisfy the recursion relation of the form

$$L^{(p)} = L \left(\text{tr } L^{(p-1)} \right) - (p-1) L^{(p-1)} L. \quad (36)$$

Introduce the matrix $\mathcal{B}(u)$ by the formula

$$\mathcal{B}(u) := \begin{pmatrix} \vec{\alpha} \cdot L^{(1)}(u) L^{-1}(u) \\ \vec{\alpha} \cdot L^{(2)}(u) L^{-1}(u) \\ \frac{1}{2} \vec{\alpha} \cdot L^{(3)}(u) L^{-1}(u) \\ \vdots \\ \frac{1}{(N-1)!} \vec{\alpha} \cdot L^{(N)}(u) L^{-1}(u) \end{pmatrix}. \quad (37)$$

With the help of this matrix we can represent the system of equations (32) as a system of linear homogeneous equations for the vector of powers of $(-v)$:

$$\vec{\alpha} \cdot (L(u) - v)^\wedge \equiv ((-v)^{N-1}, (-v)^{N-2}, \dots, 1) \cdot \mathcal{B}(u) = 0, \quad (38)$$

from which we derive that

$$(-v)^{j-i} = \frac{(\mathcal{B}^\wedge(u))_{ki}}{(\mathcal{B}^\wedge(u))_{kj}}, \quad \forall k. \quad (39)$$

The formulas (39) give plenty of representations for the rational functions $A(u)$ in (34), all of them being compatible on the separation variables since, because of the equality

$$B(u) = (-1)^{[N/2]} \det(\mathcal{B}(u)), \quad (40)$$

the matrix $\mathcal{B}^\wedge(u_j)$ has rank 1. For more details see [36].

3 A new separation of variables

Consider the standard normalization vector

$$\alpha_0 = (1, 0, 0, 0) \quad (41)$$

for the 4×4 Lax matrix (19). Then, for $\ell = 0$, the defining equations,

$$(L(u) - v)_{1k}^\wedge = 0, \quad k = 1, 2, 3, 4, \quad (42)$$

give the following polynomial $B(u)$ for the separation variables u_1 and u_2 :

$$B(u) = u^4 + B_2 u^2 + B_0 = (u^2 - u_1^2)(u^2 - u_2^2), \quad (43)$$

$$B_2 = \frac{2b(J_3 + c)(x_3 J_- - x_- J_3) + b^2(x_-^2 + 2x_3^2)}{J_-^2 + 2bx_-} + \frac{2b^2 x_3 x_- (cJ_- + bx_3)}{J_-^2 (J_-^2 + 2bx_-)}, \quad (44)$$

$$B_0 = b^2 \frac{((J_3 + c)J_- + bx_3)^2}{J_-^2 (J_-^2 + 2bx_-)}. \quad (45)$$

Note also the following useful formula for the variable B_2 :

$$B_2 = \frac{b^2 x_-^2}{J_-^2 + 2bx_-} - \frac{2\sqrt{B_0}}{aJ_- \sqrt{J_-^2 + 2bx_-}} \left(x_- J_- J_3 - (J_-^2 + bx_-) x_3 \right), \quad (46)$$

which is a linear expression in terms of J_3 and x_3 .

The corresponding rational function $A(u)$ can be chosen as follows:

$$A(u) = \frac{A_{-1}}{u} + \frac{A_{-3}}{u^3}, \quad (47)$$

$$A_{-1} = ic + \frac{ix_3(J_-^2 + 2bx_-)}{x_- J_-}, \quad A_{-3} = ib \frac{(J_3 + c)J_- + bx_3}{x_- J_-}. \quad (48)$$

Now, it is easy to check the following Poisson brackets between the two functions:

$$\begin{aligned} \{A(u), A(v)\} &= \{B(u), B(v)\} = 0, \\ \{A(u), B(v)\} &= \frac{2}{u^3(u^2 - v^2)} (u^4 B(v) - v^4 B(u)), \end{aligned} \quad \forall u, v \in \mathbb{C}. \quad (49)$$

Using these formulas one derives that the variables u_1, u_2 , defined by (43)–(46), and their conjugated counterparts,

$$v_j = A(u_j), \quad j = 1, 2, \quad (50)$$

are indeed canonical (Darboux) variables:

$$\{u_j, u_k\} = \{v_j, v_k\} = \{u_j, v_k\} = 0, \quad j \neq k, \quad \{v_j, u_j\} = 1, \quad j = 1, 2. \quad (51)$$

These variables by construction satisfy (for $\ell = 0$) to (42) and, therefore, to (24).

Notice here that the definitions of the separation variables do not depend on the value of ℓ and also that the brackets (51) are true *for any* ℓ .

4 Generating function

Let us fix a special representation of the underlying $e(3)$ algebra which will allow us to write down the found canonical transformation explicitly through the generating function. It is a kind of holomorphic representation in terms of the Darboux variables $q_j, p_j, j = 1, 2$:⁴

$$J_- = q_1, \quad x_- = q_2, \quad (52)$$

$$J_3 = i(q_1 p_1 + q_2 p_2) + \ell, \quad x_3 = iq_2 p_1 + 1, \quad (53)$$

$$J_+ = q_1 p_1^2 + 2q_2 p_1 p_2 - 2i\ell p_1 - 2ip_2, \quad x_+ = q_2 p_1^2 - 2ip_1. \quad (54)$$

⁴cf. a holomorphic representation of $\mathfrak{sl}(2)$

The meaning of this realization is in the fact that the variables J_- and x_- do not depend on the momenta \mathbf{p} and the variables J_3 and x_3 are linear in momenta. This together with linearity of the variables u_1u_2 and $u_1^2 + u_2^2$ in terms of J_3 and x_3 make it possible to integrate the equations explicitly in terms of elementary functions.

Finally, the canonical transformation defined in the previous Section is given by the following generating function $F(\mathbf{u}|\mathbf{q})$:

$$F(\mathbf{u}|\mathbf{q}) = \frac{iq_1}{q_2} + \frac{ic}{2} \log(q_1^2 + 2bq_2) + i\ell \log(q_2) + \frac{ib^2q_2}{2u_1u_2\sqrt{q_1^2 + 2bq_2}} + \frac{i}{2} \sqrt{q_1^2 + 2bq_2} \left(2 \frac{u_1u_2}{b} - \frac{u_1^2 + u_2^2}{q_2u_1u_2} \right). \quad (55)$$

Therefore, the equations of the change of variables $(\mathbf{q}, \mathbf{p}) \leftrightarrow (\mathbf{u}, \mathbf{v})$ are written in the form

$$p_j = \frac{\partial F(\mathbf{u}|\mathbf{q})}{\partial q_j}, \quad v_j = -\frac{\partial F(\mathbf{u}|\mathbf{q})}{\partial u_j}, \quad j = 1, 2. \quad (56)$$

5 Inverse problem

The inverse problem, i.e. finding expressions for the initial e(3) variables in terms of separation variables, also has an explicit solution which is given below:

$$q_1^2 = -b^2 \frac{u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2)}{4u_1^6u_2^6(u_1v_1 - u_2v_2)^2} \left(u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2) + 4u_1^2u_2^2(u_1^2 - u_2^2) \right), \quad (57)$$

$$J_- = q_1, \quad x_- = \frac{b(u_1^2 - u_2^2) (u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2))}{2u_1^4u_2^4(u_1v_1 - u_2v_2)^2}, \quad (58)$$

$$J_3 = \frac{1}{2u_1^2u_2^2(u_1v_1 - u_2v_2) (u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2))} \times \\ \times \left(ib^4(u_1^2 - u_2^2)^2 + 2cu_1^4u_2^4(u_1v_1 - u_2v_2) (u_1^2(v_1^2 - 2) - u_2^2(v_2^2 - 2)) \right. \\ \left. + 2cb^2u_1^2u_2^2(u_1v_1 - u_2v_2)(u_1^2 - u_2^2) + 2ib^2u_1^2u_2^2(u_1^2v_1^2 - u_2^2v_2^2)(u_1^2 - u_2^2) \right. \\ \left. + iu_1^4u_2^4(u_1v_1 - u_2v_2) (v_1u_1^3(v_1^2 - 4) - v_2u_2^3(v_2^2 - 4) + u_1u_2v_1v_2(u_1v_1 - u_2v_2)) \right), \quad (59)$$

$$x_3 = \frac{2iq_1u_1^2u_2^2 (u_1^2(u_1v_1 - ic) - u_2^2(u_2v_2 - ic))}{b (u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2))}, \quad (60)$$

$$J_+ = -\frac{4q_1u_1^4u_2^4(u_1v_1 - u_2v_2)}{b^2 (u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2))^2} \times \\ \times \left(u_1^3u_2^3v_1v_2(u_1v_1 - u_2v_2) + b^2(u_1^3v_1 - u_2^3v_2) - icb^2(u_1^2 - u_2^2) \right. \\ \left. - icu_1^2u_2^2(u_1^2v_1^2 - u_2^2v_2^2) - c^2u_1^2u_2^2(u_1v_1 - u_2v_2) \right), \quad (61)$$

$$x_+ = \frac{1}{b (u_1^2(u_2^4v_2^2 - b^2) - u_2^2(u_1^4v_1^2 - b^2))^2} \times$$

$$\begin{aligned}
& \times \left(-4icb^2u_1^2u_2^2(u_1^2 - u_2^2)(u_1^3v_1 - u_2^3v_2) + 2b^2u_1^2u_2^2(u_1^2 - u_2^2)(u_1^4v_1^2 - u_2^4v_2^2) \right. \\
& \quad - 2b^2c^2u_1^2u_2^2(u_1^2 - u_2^2)^2 - 2c^2u_1^4u_2^4(u_1^2 - u_2^2) \left(u_1^2(v_1^2 - 4) - u_2^2(v_2^2 - 4) \right) \\
& \quad - 4icu_1^4u_2^4(u_1^3v_1 - u_2^3v_2) \left(u_1^2(v_1^2 - 4) - u_2^2(v_2^2 - 4) \right) \\
& \quad \left. + 2u_1^4u_2^4 \left((u_1^2v_1^2 - u_2^2v_2^2)(u_1^4v_1^2 - u_2^4v_2^2) - 4(u_1^3v_1 - u_2^3v_2)^2 \right) \right). \tag{62}
\end{aligned}$$

6 Concluding remarks

A separation of variables for the Kowalevski gyrostat, with $\ell = 0$, is found for the first time. It appeared to separate the Goryachev-Chaplygin gyrostat as well, thereby giving another separation for this problem.

If we put $c = 0$, i.e. switch off the gyrostatic term, we can compare our results with the original Kowalevski's and Goryachev-Chaplygin's separations for the respective tops (in the $\ell = 0$ case). It is easy to see that the new separation is as complicated as the (simple) Goryachev-Chaplygin separation and it is *much simpler* than the Kowalevski separation.

Hence, the new separation stands a good chance to be quantized.

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