COUPLING THE SINE-GORDON FIELD THEORY TO A MECHANICAL SYSTEM AT THE BOUNDARY

P.Baseilhac *

Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502 Kyoto, Japan.

G.W.Delius [†] and A.George [‡]

Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK §

Abstract

We describe an integrable system consisting of the sine-Gordon field, restricted to the half line, and coupled to a non-linear oscillator at the boundary. By extension of the coupling constant to imaginary values we also outline the equivalent system for the sinh-Gordon field. We show how Sklyanin's formalism can be applied to situations with dynamic boundary conditions, and illustrate the method with the derivation of our example system.

1. Introduction

Recent work on integrable quantum fields in 1+1 dimensions has introduced a class of theories where the domain of the field is bounded. The boundaries restrict the field either to the half line or an interval and impose fixed boundary conditions upon it (see, for example, [1, 2, 3]). In the present work we introduce a new class of integrable theories where the field is coupled to a classical or quantum mechanical system at the boundary of its domain. Effectively we are allowing new degrees of freedom to exist on the boundary and so the field has dynamic boundary conditions.

^{*}pascal@yukawa.kyoto-u.ac.jp

[†]gwd2@york.ac.uk; delivered talk

[‡]ag160@york.ac.uk; author of paper

[§]http://www.york.ac.uk/mathematics/physics/

In section two we describe one example of such a system, where the sine-Gordon field is restricted to the half line bounded by a non-linear oscillator. Then by extension of the coupling constant into imaginary values we briefly consider the equivalent system for the sinh-Gordon field.

Section three contains the derivation of our example system, including details on how Sklyanin's technique has applied to a system with dynamic boundary K matrices. Section four comprises a discussion on this class of theory.

Much of the content of this paper follows the work of Baseilhac and Delius [4]. Due to restrictions on space we have omitted from the present work the discussion on dynamic solutions to the quantum reflection equation that appeared in [4, section 5].

2. The Sine-Gordon Field Coupled to Oscillator at Boundary

In this section we describe an integrable systems consisting of a sine-Gordon field restricted to the half line and coupled to an oscillator at the boundary.

The sine-Gordon model is a 1+1 dimensional theory describing a relativistic, self-interacting, massive, bosonic field. We introduce a boundary at the origin by restricting the field to the half line between $x=-\infty$ and x=0. The Hamiltonian for such a field is

$$H_{\rm sG} = \int_{-\infty}^{0} \left(\frac{1}{2} \pi^2 + \frac{1}{2} \left(\partial_x \phi \right)^2 - \frac{m^2}{\beta^2} \left(\cos \beta \phi - 1 \right) \right) dx,\tag{1}$$

where $\phi(x,t)$ is the sine-Gordon field and $\pi(x,t)$ is the conjugate momentum, with the usual Poisson brackets $\{\pi(x),\phi(y)\}=\delta(x-y)$. Both β , the sine-Gordon coupling constant, and m, the mass scale parameter, are real.

We now couple the field at the boundary to a classical system with the Hamiltonian

$$H_{\rm osc} = -\frac{2m}{\beta^2} \left(e^{-i\beta\phi(0)/2} \cos \hat{p} + e^{i\beta\phi(0)/2} \cos \hat{q} \right),\tag{2}$$

where $\phi(0)$ is the field at x = 0, and \hat{q} and \hat{p} are related to the position, q, and momentum, p, variables of the oscillator as

$$\hat{p} = \frac{\beta}{\sqrt{2Mm}}p$$
, and $\hat{q} = \frac{\beta\sqrt{Mm}}{2\sqrt{2}}q$, (3)

where M sets the mass of the oscillator. This becomes obvious after expansion of (2) for small p and q. Position and momentum variables

have the usual Poisson bracket $\{p,q\} = 1$ and our rescaled variables have the relation $\{\hat{p}, \hat{q}\} = \beta^2/4$.

The total Hamiltonian of the system can be expressed as the sum of the oscillator and bulk field Hamiltonians

$$H = H_{\rm sG} + H_{\rm osc}. (4)$$

We are now in a position to find Hamilton's equations for the bulk field, they are

$$\frac{d}{dt}\phi(x) = \{H, \phi(x)\} = \pi(x),\tag{5}$$

$$\frac{d}{dt}\pi(x) = \{H, \pi(x)\} = \partial_x^2 \phi(x) - \frac{m^2}{\beta} \sin \beta \phi(x) \tag{6}$$

$$-\delta(x)\left(\partial_x\phi(0) + \frac{im}{\beta}\left(e^{-i\beta\phi(0)/2}\cos\hat{p} - e^{i\beta\phi(0)/2}\cos\hat{q}\right)\right).$$

The two terms proportional to $\delta(x)$ in (6) come from two different sources. The $\partial_x \phi(0)$ term comes from the boundary contribution due to the partial integration of the sine-Gordon field. The other terms are due to the boundary oscillator containing $\phi(0)$. We require, for a physical solution, that $\pi(x)$ is continuous, i.e. the $\delta(x)$ term must be zero. This gives us the boundary condition

$$\partial_x \phi(0) = -\frac{im}{\beta} \left(e^{-i\beta\phi(0)/2} \cos \hat{p} - e^{i\beta\phi(0)/2} \cos \hat{q} \right). \tag{7}$$

By substituting (5) into (6) and applying the boundary condition (7) we recover the usual sine-Gordon equation of motion

$$\partial_t^2 \phi - \partial_x^2 \phi = -\frac{m^2}{\beta} \sin \beta \phi. \tag{8}$$

We can find Hamilton's equations for the boundary oscillator in a similar way

$$\frac{d}{dt}\hat{p} = \{H, \hat{p}\} = -\frac{m}{2}e^{i\beta\phi(0)/2}\sin\hat{q},\tag{9}$$

$$\frac{d}{dt}\hat{q} = \{H, \hat{q}\} = \frac{m}{2}e^{-i\beta\phi(0)/2}\sin\hat{p}.$$
 (10)

By combining these two equations we are able to find the equation of motion of the oscillator

$$\left(\ddot{\hat{q}} + \frac{i\beta}{2}\dot{\phi}(0)\dot{\hat{q}}\right)\mathcal{Z}\left(\frac{2}{m}e^{i\beta\phi(0)/2}\dot{\hat{q}}\right) = -\frac{m^2}{4}\sin\hat{q},\tag{11}$$

where dots denote derivatives with respect to time and the function $\mathcal Z$ is defined as

$$\mathcal{Z}(x) = \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$
 (12)

We note that the oscillator's Hamiltonian, and thus the boundary condition for the field, are complex. This generally leads to complex solutions, both in the bulk and for boundary states. This is not without precedent, for example soliton solutions in affine Toda theories are complex, however it has been shown that the energies of these solutions are nevertheless real [5, 6]. We hope that the energies of solutions to our system will also be real.

We note that our boundary condition (7) is similar in form to the previously known integrable boundary conditions [2, 1, 7]

$$\partial_x \phi(0) = -\frac{im}{\beta} \left(\eta_0 e^{-i\beta\phi(0)/2} - \eta_1 e^{i\beta\phi(0)/2} \right), \tag{13}$$

where η_0 and η_1 are parameters. In the classical case our boundary condition (7) reduces to equation (13) with $\eta_0 = \eta_1 = 1$ when p = q = 0. This will never occur in the quantum case as the position and momentum of the system can never be simultaneously known.

We also note that by requiring the coupling constant to be purely imaginary, $\beta = i\hat{\beta}$, we can obtain the equations for the equivalent integrable system for the sinh-Gordon equation. For instance the Hamiltonian becomes

$$H_{ShG} = \int_{-\infty}^{0} \left(\frac{1}{2} \pi^2 + \frac{1}{2} \left(\partial_x \phi \right)^2 + \frac{m^2}{\hat{\beta}^2} \left(\cosh \hat{\beta} \phi - 1 \right) \right) dx$$

$$+ \frac{2m}{\hat{\beta}^2} \left(e^{-\hat{\beta} \phi(0)/2} \cosh \left(\frac{\hat{\beta}}{\sqrt{2Mm}} p \right) + e^{\hat{\beta} \phi(0)/2} \cosh \left(\frac{\hat{\beta} \sqrt{Mm}}{2\sqrt{2}} q \right) \right).$$

$$(14)$$

We note that the Hamiltonian is real, which leads to real solutions.

3. Derivation of Sine-Gordon Example

In this section we describe Sklyanin's formalism [2] and how we have applied this technique to dynamic boundary conditions. We illustrate our method with the derivation of the example system presented in section two.

As required by Sklyanin's formalism there exist two matrix valued functions $a_x(\theta,x)$ and $a_t(\theta,x)$ which depend upon the sine-Gordon field, its conjugate momentum, and on the spectral parameter $\theta \in \mathbb{C}$ such that the classical equation of motion for the field (8) can be written as the Lax pair equation

$$[\partial_x - a_x(\theta, x), \partial_t - a_t(\theta, x)] = 0 \quad \forall \theta.$$
 (15)

These are

$$a_x(\theta, x) = \frac{\beta}{4i} \frac{\partial \phi}{\partial t} \sigma_3 + \frac{m}{4i} \left(e^{\theta} + e^{-\theta} \right) \sin \left(\frac{\beta \phi}{2} \right) \sigma_1 + \frac{m}{4i} \left(e^{\theta} - e^{-\theta} \right) \cos \left(\frac{\beta \phi}{2} \right) \sigma_2, \tag{16}$$

$$a_{t}(\theta, x) = \frac{\beta}{4i} \frac{\partial \phi}{\partial x} \sigma_{3} + \frac{m}{4i} \left(e^{\theta} - e^{-\theta} \right) \sin \left(\frac{\beta \phi}{2} \right) \sigma_{1} + \frac{m}{4i} \left(e^{\theta} + e^{-\theta} \right) \cos \left(\frac{\beta \phi}{2} \right) \sigma_{2}, \tag{17}$$

where σ_k are the Pauli matrices, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Equation (15) is the compatability condition for the overdetermined set of equations

$$\frac{\partial T}{\partial x_{+}} = a_{x}(\theta, x_{+})T,$$

$$\frac{\partial T}{\partial t} = a_{t}(\theta, x_{+})T - Ta_{t}(\theta, x_{-}).$$
(18)

The transition matrix $T \equiv T(x_+, x_-, \theta)$ is defined as the solution to (18) with initial condition $T(x_-, x_-, \theta) = I$. T can be expressed as a path ordered exponential

$$T(x_{+}, x_{-}, \theta) = \mathcal{P} \exp\left(\int_{x}^{x_{+}} a_{x}(\theta, x) dx\right), \tag{19}$$

such that the operators are ordered with those at points nearest x_+ furthest to the left. This gives the matrix the inversion property

$$T^{-1}(x_+, x_-, \theta) = T(x_-, x_+, \theta).$$
 (20)

The Poisson brackets for the matrices $a_x(\theta, x)$ can be written in the form

$$\{\hat{a}_{x}(\theta, x), \hat{a}_{x}(\theta', y)\} = \delta(x - y)[r(\theta - \theta'), \hat{a}_{x}(\theta, x) + \hat{a}_{x}(\theta', y)],$$
(21)

which leads to Poisson brackets for the transition matrix

$$\{ \stackrel{1}{T} (x_{+}, x_{-}, \theta), \stackrel{2}{T} (x_{+}, x_{-}, \theta') \} = [r(\theta - \theta'), \stackrel{1}{T} (x_{+}, x_{-}, \theta) \stackrel{2}{T} (x_{+}, x_{-}, \theta')],$$
(22)

where $\stackrel{1}{A}=A\otimes I$ and $\stackrel{2}{A}=I\otimes A$. This gives us the r-matrix, which is independent of both the field and its conjugate momentum.

For the derivation of the sine-Gordon example we redefine the Poisson brackets such that the range of the integral extends only over the space occupied by the field. So at fixed time the Poisson brackets for any observable \mathcal{O}_i is defined as

$$\{\mathcal{O}_{1}, \mathcal{O}_{2}\} = \int_{x_{-}}^{x_{+}} \left(\frac{\partial \mathcal{O}_{1}}{\partial \pi(x)} \frac{\partial \mathcal{O}_{2}}{\partial \phi(x)} - \frac{\partial \mathcal{O}_{1}}{\partial \phi(x)} \frac{\partial \mathcal{O}_{2}}{\partial \pi(x)} \right) dx + \frac{\partial \mathcal{O}_{1}}{\partial p} \frac{\partial \mathcal{O}_{2}}{\partial q} - \frac{\partial \mathcal{O}_{1}}{\partial q} \frac{\partial \mathcal{O}_{2}}{\partial p}.$$
(23)

By calculating these Poisson brackets for $a_x(\theta, x)$ we can find the r-matrix for the sine-Gordon system,

$$r_{\rm sg}(\theta) = \frac{\beta^2 \cosh(\theta)}{16 \sinh(\theta)} (I \otimes I - \sigma_3 \otimes \sigma_3) - \frac{\beta^2}{16 \sinh(\theta)} (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)$$
(24)

We now introduce two matrix valued functions of θ , $K_{\pm}(\theta)$, which are independent of the field and describe the nature of the boundaries at $x = x_{\pm}$. It is at this point that we deviate from previous applications of Sklyanin's method. We wish to describe a dynamic boundary so require that its K matrix be dependent upon the position and momentum variables of the boundary system. We further require that the K matrices satisfy the classical reflection equation

$$\{ \overset{1}{K_{\pm}} (\theta), \overset{2}{K_{\pm}} (\theta') \} = [r(\theta - \theta'), \overset{1}{K_{\pm}} (\theta) \overset{2}{K_{\pm}} (\theta')]$$

$$+ \overset{1}{K_{\pm}} (\theta) r(\theta + \theta') \overset{2}{K_{\pm}} (\theta') - \overset{2}{K_{\pm}} (\theta') r(\theta + \theta') \overset{1}{K_{\pm}} (\theta).$$
(25)

As we want the two boundaries to be independent of each other, we also require that the relation

$$\{ \overset{1}{K_{\pm}} (\theta), \overset{2}{K_{\mp}} (\theta') \} = 0$$
 (26)

be satisfied.

As we are deriving a system on the half line we choose to place x_{-} at $-\infty$ and have K_{-} take the trivial solution to the classical reflection equation (25), $K_{-}(\theta) = I$. We impose $\phi(x_{-},t) = 0$, $\pi(x_{-},t) = 0$ when $x_{-} = -\infty$. We note the independence condition (26) is automatically satisfied due to our choice of K_{-} . We will take [8]

$$K_{+}(\theta) = (27)$$

$$2 \begin{pmatrix} \cosh(\tilde{p} + \tilde{q})e^{\theta} - \cosh(\tilde{p} - \tilde{q})e^{-\theta} & 2\sinh^{2}(\theta) - 2\sinh^{2}(\tilde{p}) \\ 2\sinh^{2}(\tilde{q}) - 2\sinh^{2}(\theta) & \cosh(\tilde{p} - \tilde{q})e^{\theta} - \cosh(\tilde{p} + \tilde{q})e^{-\theta} \end{pmatrix},$$

where $\{\tilde{p}, \tilde{q}\} = \beta^2/8$. This is a solution to (25) with $r(\theta) = r_{\rm sg}(\theta)$ defined in (24). We will position this boundary at the origin, $x_+ = 0$.

Following Sklyanin's technique we now define the generating function to be

$$\tau(\theta) = \text{tr}\left(K_{+}(\theta)T(x_{+}, x_{-}, \theta)K_{-}(\theta)T^{-1}(x_{+}, x_{-}, -\theta)\right). \tag{28}$$

These generating functions are in involution, i.e. $\{\tau(\theta), \tau(\theta')\} = 0$, and can be expanded around any singularities in $T(x_+, x_-, \theta)$. Such an expansion will produce an infinite number of conserved quantities I_n , that are also in involution with each other. One of these quantities will be identifiable as the Hamiltonian of the system, implying that all I_n are time conserved.

Applying the inversion relation (20) and the fact that we chose $K_{-} = I$ to (28) gives us an expression for the generating function for the sine-Gordon example

$$\tau_{\rm sg}(\theta) = \operatorname{tr}\left(K_{+}(\theta)T(0, -\infty, \theta)T(-\infty, 0, -\theta)\right). \tag{29}$$

We note from (16) that the matrix $a_x(\theta, x)$ has singularities at $|\theta| = \pm \infty$. Thus the transition matrix T also has two singularities. Performing a Laurent expansion on $\tau(\theta)$ around either of these points gives a Laurent series, the coefficients of which give us an infinite number of conserved quantities I_n [9],

$$\ln \tau(\theta) - 2\theta = \sum_{n=-1}^{\infty} I_n e^{-n\theta}.$$
 (30)

The first non-trivial quantity is

$$I_1 = -\frac{i\beta^2}{2m}H + \text{const},\tag{31}$$

where

$$H = \int_{-\infty}^{0} \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\partial_x \phi)^2 - \frac{m^2}{\beta^2} (\cos(\beta \phi) - 1) \right) dx + \frac{2m}{\beta^2} \left(e^{i\beta\phi(0)/2} \cosh(\tilde{p} + \tilde{q}) + e^{-i\beta\phi(0)/2} \cosh(\tilde{p} - \tilde{q}) \right),$$
(32)

which, under the canonical transformation

$$\tilde{p} + \tilde{q} \to i\hat{q}, \quad \tilde{p} - \tilde{q} \to -i\hat{p}, \quad \phi(x) \to \phi(x) + 2\pi/\beta,$$
 (33)

reproduces the Hamiltonian of our example sine-Gordon system presented in section two (equations (1) and (2)). \hat{p} and \hat{q} are defined in equation (3). The next non-trivial quantity after the Hamiltonian gives the spin 3 charge, applying the same canonical transformation,

$$I_{3} = \int_{-\infty}^{0} \frac{\beta^{4}}{16m^{3}} \left(\pi^{4} + 6\pi^{2} (\partial_{x}\phi)^{2} + (\partial_{x}\phi)^{4} \right) - \frac{\beta^{2}}{m^{3}} \left((\partial_{x}\pi)^{2} + (\partial_{x}^{2}\phi)^{2} \right)$$

$$- \frac{\beta^{2}}{4m} \left(\pi^{2} + 5(\partial_{x}\phi)^{2} \right) \cos(\beta\phi) + \frac{m}{8} (\cos(2\beta\phi) - 1) dx$$

$$+ e^{3i\beta\phi(0)/2} \left(\frac{1}{2} \cos \hat{q} + \frac{1}{6} \cos^{3} \hat{q} \right)$$

$$- e^{i\beta\phi(0)/2} \left(\frac{3}{2} \cos \hat{p} - \frac{1}{2} \cos^{2} \hat{q} \cos \hat{p} + \frac{\beta^{2}}{2m^{2}} \pi^{2} \cos \hat{q} \right)$$

$$+ e^{-3i\beta\phi(0)/2} \left(\frac{1}{2} \cos \hat{p} + \frac{1}{6} \cos^{3} \hat{p} \right)$$

$$- e^{-i\beta\phi(0)/2} \left(\frac{3}{2} \cos \hat{q} - \frac{1}{2} \cos^{2} \hat{p} \cos \hat{q} + \frac{\beta^{2}}{2m^{2}} \pi^{2} \cos \hat{p} \right)$$

$$- \frac{2i\beta}{m} \sin \hat{q} \sin \hat{p}. \tag{34}$$

In principle, similar expressions can be obtained for any of the infinite number of conserved charges of our system. The existence of these higher order local integrals of motion shows that our system is classically integrable.

4. Discussion

In this paper we have described in detail one example of an integrable field theory coupled to a dynamic boundary system; this being the sine-Gordon field, restricted to the half line, and coupled to a non-linear oscillator at the boundary.

We have detailed how Sklyanin's formalism may be applied to systems with dynamic K matrices, and illustrated this with the derivation of our example system.

Work on the sinh-Gordon and sine-Gordon systems presented here will continue, treating the systems both classically and in the quantum case.

Acknowledgments

We would like to thank Evgueni Sklyanin, Vadim Kuznetsov and Peter Bowcock for helpful discussions. GWD is supported by an EPSRC advanced fellowship and AG by a PPARC studentship. PB's work was supported by Marie Curie fellowship HPMF-CT-1999-00094 and is currently supported by a JSPS fellowship. GWD thanks the organizers for a most enjoyable conference.

References

- S Ghoshal and A B Zamolodchikov, Boundary S Matrix and Boundary State in Two-Dimensional Integrable Field Theory, Int. J. Mod. Phys. 9 (1994) 3841, preprint hep-th/9306002
- [2] E K Sklyanin, Boundary Conditions for Integrable Equations, Funct. Anal. Appl. 21 (1987) 164
- [3] E K Sklyanin, Boundary Conditions for Integrable Quantum Systems, J. Phys. A 21 (1988) 2375
- [4] P Baseihac and G W Delius, Coupling Integrable Field Theories to Mechanical Systems at the Boundary, J. Phys. A 34 (2001) 8259, preprint hep-th/0106275
- [5] T Hollowood, Solitons in Affine Toda Field Theories, Nucl. Phys. B 384 (1992) 523
- [6] D I Olive, N Turok and J W R Underwood, Solitons and the Energy-Momentum Tensor for Affine Toda Theory, Nucl. Phys. B 401 (1993) 663
- [7] A MacIntyre, Integrable Boundary Conditions for the Classical Sine-Gordon Theory, J. Phys. A 28 (1995) 1089, preprint hep-th/9410026
- [8] B Suris Yu, Discrete Time Generalized Toda Lattices: Complete Integrability and Relation with Relativistic Toda Lattices, Phys. Lett. A 145 (1990) 113
- [9] L D Faddeev and L A Takhtajan, Hamiltonian Methods in the Theory of Solitons, (Berlin: Springer) 1987