# Generalized dimensions of Feigenbaum's attractor from renormalization-group functional equations

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## Abstract

A method is suggested for the computation of the generalized dimensions of fractal attractors at the period-doubling transition to chaos. The approach is based on an eigenvalue problem formulated in terms of functional equations, with a coefficient expressed in terms of Feigenbaum's universal fixed-point function. The accuracy of the results is determined only by precision of the representation of the universal function.

PACS numbers: 05.45.Df, 05.45.-a, 05.10.Cc

The multifractal or thermodynamic formalism is an important tool for description of strange sets arising in dynamical systems in different contexts. Its basic ideas have been clearly formulated e.g. in the paper of Halsey et al. [1]. Some of the examples presented by these and other authors relate to the fractal attractors that occur at the onset of chaos via period doubling and quasiperiodicity [2]–[7]. The multifractal analysis reveals global scaling properties of these attractors, such as the generalized dimensions and the $f(\alpha)$ spectra. They are of principal interest because of their universality for systems of different nature. Moreover, they allow a measurement in physical experiments [7].

One of the well-studied multifractal objects is the Feigenbaum attractor, which occurs at the period-doubling transition to chaos in unimodal one-dimensional maps with quadratic extremum and in a wide class of more general nonlinear dissipative systems [8, 2, 9]. Beside the original procedure of Halsey et al. (namely the construction and analysis of the partition functions defined as sums over some natural covering of the attractor), several other approaches to the computation of the multifractal characteristics have been developed. Bensimon et al. [3] used a method based on a break up of a partition sum into two components with subsequent use of the scaling property. Kovács [4] suggested a procedure of extracting the dimensions from the eigenvalue problem for the Frobenius-Perron operator. Christiansen et al. [5] exploited the idea of approximating the strange sets by periodic orbits and expressed the desired quantities in terms of cycle expansions. (To our knowledge, the calculation of the Hausdorff dimension of the Feigenbaum attractor in Ref. [5] remains the most precise to date.)

In some sense, the *global* description of scaling properties in the multifractal formalism seems opposite to the *local* description in terms of the Feigenbaum renormalization group approach [8]. The latter is based on the solution of the functional fixed-point equation and associated with scaling relations for the evolution operators in a neighborhood of the extremum of the map under consideration.

In this note we present a novel method for precise computation of the multifractal characteristics: the problem may be presented in terms of the Feigenbaum renormalization transformation applied to some auxiliary function. The desired quantities, such as the generalized dimensions and the $f(\alpha)$ spectrum, can be extracted from an eigenvalue problem, formulated as a functional equation involving Feigenbaum's universal fixed-point function. An analogous approach was previously suggested in application to the problem of the effect of noise onto the period-doubling transition [10, 2]. That problem appears to be linked with one special generalized dimension, as noted e.g. in Refs. [2, 11], and this circumstance obviously supports a possibility of the generalization we undertake here. A similar idea was used in Ref. [12] for a study of scaling regularities in the Fourier spectrum and response function of the quadratic map at the period-doubling transition to chaos.

Using the representation of the Feigenbaum function from Ref. [13] we obtain the generalized dimensions with high precision in excellent agreement with the previously known data. Also we present accurate results for generalized dimensions of the multifractal attractors at the onset of chaos in the unimodal maps of degrees 4, 6, and 8, for which the universal functions $g(x)$ are available in the literature in the form of numerically found polynomial expansions [14].

To estimate the multifractal characteristics for the Feigenbaum attractor by the standard approach the generalized partition functions $\Gamma_k(q,\,\tau) = \sum_{i=1}^{2^k} p_i^q/l_i^\tau$ are exploited. Here $q$ and $\tau$ are some real parameters, $p_i = 2^{-k}$, $l_i = |x_i - x_{i+2^k}|$, and the sequence $x_i$ results from iterations of the unimodal map at the limit point of the period-doubling accumulation, starting from the extremum point. Obviously, $\Gamma_k(q,\,\tau) = 2^{-qk} S_{2^k}(\tau)$, where $S_{2^k}(\tau) = \sum_{i=1}^{2^k} l_i^{-\tau}$. For each given $q$ an appropriate value of $\tau = \tau(q)$ may be found that ensures an asymptotic equality $\Gamma_{k+1}(q,\,\tau) = \Gamma_k(q,\,\tau)$ as $k \to \infty$. Vice versa, for a given $\tau$ we can select a respective value of $q = q(\tau)$. This relation of $q$ and $\tau$ is used then to obtain the generalized dimensions and $f(\alpha)$ spectrum.

For large $k$ the lengths of the intervals $l_i$ are small, and they may be estimated via the derivatives as

$$l_i \cong |dx_i/dx_1|l_1. \tag{1}$$

In this approximation we can compute them together with the sums $S$ step by step via simultaneous iterations of the relations

$$\begin{aligned} x_{i+1} &= f(x_i), \\ l_{i+1} &= |f'(x_i)|l_i, \\ S_{i+1} &= S_i + l_{i+1}^{-\tau}\Psi(x_i). \end{aligned} \tag{2}$$

We have introduced here an auxiliary function $\Psi(x)$ which at this moment is supposed to be identically equal to 1.

By twofold application of Eqs. (2) we obtain

$$\begin{aligned} x_{i+2} &= f(f(x_i)), \\ l_{i+2} &= |f'(f(x_i))f'(x_i)|l_i, \\ S_{i+2} &= S_i + l_{i+2}^{-\tau}\left[|f'(f(x_i))|^\tau \Psi(x_i) + \Psi(f(x_i))\right]. \end{aligned} \tag{3}$$

Now we perform Feigenbaum's scale change $x \mapsto x/\alpha$, $l \mapsto l/|\alpha|$ ($\alpha$ is the Feigenbaum constant) and arrive at the equations, which are of the same form as (2), but with new functions $f_{\text{new}}(x) = \alpha f(f(x/\alpha))$, $\Psi_{\text{new}}(x) = L_{f,\tau}\Psi(x)$ where $L_{f,\tau}$ is the linear operator

$$L_{f,\tau} : \Psi(x) \mapsto |\alpha|^\tau \left[|f'(f(x/\alpha))|^\tau \Psi(x/\alpha) + \Psi(f(x/\alpha))\right]. \tag{4}$$

This transformation may be repeated again and again. Asymptotically, $f(x)$ converges to the fixed-point function satisfying the Feigenbaum-Cvitanović equation $g(x) = \alpha g(g(x/\alpha))$, and the sequence $\Psi_k(x)$ will follow the recursion $\Psi_{k+1}(x) = L_{g,\tau}\Psi_k(x)$.

Thus, as $k \to \infty$, the function $\Psi_k$ tends to the eigenfunction associated with the largest eigenvalue of the linear operator $L_{g,\tau}$, the eigenproblem being

$$\nu(\tau)\Psi(x) = |\alpha|^\tau \left[ |g'(g(x/\alpha))|^\tau \Psi(x/\alpha) + \Psi(g(x/\alpha)) \right]. \tag{5}$$

(A particular case of this equation for $\tau = 2$ appears in the theory of effect of noise onto the period-doubling transition. A possibility of computation of the noise scaling constant via sums of the derivatives over the Feigenbaum attractor was noted e.g. in Refs. [15, 11].)

¿From the construction, we see that the eigenvalue $\nu(\tau)$ indicates a rate of growth or decrease of the sums $S$:

$$S_{2^k}(\tau) \propto \nu(\tau)^k. \tag{6}$$

To have $\Gamma_k \to \text{const}$ as $k \to \infty$, we must have

$$\nu(\tau) = 2^q, \text{ or } q = \log_2 \nu(\tau). \tag{7}$$

Then, in accordance with the multifractal formalism, we can obtain the generalized dimensions

$$D_q = \frac{\tau}{q-1}, \tag{8}$$

and the $f(\alpha)$ spectrum as an implicitly defined relation between the variables

$$\alpha = \frac{d\tau}{dq} \text{ and } f = q\frac{d\tau}{dq} - \tau. \tag{9}$$

Although our argumentation starts from the approximate relation (1), we believe that the final Eq. (5) is exact. The data from the numerical computations presented below strongly supports this conjecture: the generalized dimensions are in excellent agreement with the best known numerical results, up to all reliable digits. Apparently, in the asymptotics of $k \to \infty$ the approximate nature of (1) becomes inessential. One might hope that a rigorous proof can be found.

We have performed numerical solutions of the eigenvalue problem (5) for the classic Feigenbaum attractor of the quadratic map and for unimodal maps of even integer degrees $d = 4, 6,$ and 8. In principle, the achievable precision of the results is determined only by accuracy of the approximation of the universal functions.

With the known polynomial approximation of $g(x)$ and value of the scaling constant $\alpha$ we have numerically performed the functional transformation defined by the right-hand side of Eq. (5). The unknown function $\Psi(x)$ is represented by a table of its values at the nodes of a one-dimension grid on the interval $[0, 1]$ and by an interpolation scheme between the nodes. In actual computations it was convenient to use a grid of constant step along the axis of variable $y = |x|^d$ and a fourth-order interpolation in terms of $y$. Given the input table for $\Psi(x)$ the program yields an analogous table as output.

Suppose we fix $\tau$ and wish to estimate $q$. We define an initial condition as $\Psi(x) \equiv 1$, perform the functional transformation, and normalize the resulting function as $\Psi^0(x) = \Psi(x)/\Psi(0)$. The new function is taken as the initial condition and so on. This operation is repeated many times, until the form of the function $\Psi(x)$ stabilizes. Then, the value of $\Psi(0)$ before the normalization is taken to be $\nu(\tau)$, and we finally set $q(\tau) = \log_2 \nu(\tau)$.

To find $\tau$ for a given $q$ we use the above procedure together with a simple iteration scheme for the numerical solution of the algebraic equation $q(\tau) = q$. We may then calculate $D_q = \tau/(q-1)$ at $q \neq 1$. In particular, $D_0$ is the Hausdorff dimension, and $D_2$ is the correlation dimension.

To obtain the information dimension $D_1$ it is necessary to determine the limit as $q \to 1$, that is, as $\tau \to 0$. Formally, this follows from L'Hôpital's rule: $D_1 = \lim\limits_{q \to 1} \frac{\tau(q)}{q-1} = \left(\frac{d\tau}{dq}\right)_{q=1} = \left(\frac{dq}{d\tau}\right)_{\tau=0}^{-1}$. To compute this without loss of accuracy we use the following algorithm. For $\tau \ll 1$ let us write $\Psi_k(x) = 2^k |\alpha|^{k\tau}[1 + \tau h_k(x)]$ and substitute this expression into Eq. (??). To first order we have

$$h_{k+1}(x) = \tfrac{1}{2}\left[h_k(x/\alpha) + h_k(g(x/\alpha))\right] + \tfrac{1}{2}\ln|g'(g(x/\alpha))|. \tag{10}$$

Numerically, representing $h_k(x)$ by a table of its values and performing a large number of steps of the transformation one can observe that $h_{k+1}(x) - h_k(x) \to \theta = \text{const}$ as $k \to \infty$. This implies that $\Psi_k \propto |\alpha|^{k\tau} 2^k e^{k\theta\tau}$. On the other hand, $\Psi_k \propto 2^{kq(\tau)} \cong 2^{k(q + \tau dq/d\tau)} = 2^{k(1+\tau/D_1)}$. Hence,

$$D_1 = \frac{\ln 2}{\ln|\alpha| + \theta}. \tag{11}$$

For quadratic maps we have performed the computations based on a polynomial representation of $g(x)$ with coefficients taken from the paper of Lanford [13]. His data are of very high precision, but in our calculations the accuracy is limited due to a use of standard double-precision arithmetic. As a result, we get not more than 14 true digits in the generalized dimensions. These data are presented in the first column of Table 1. Note excellent agreement of the Hausdorff dimension (up to the last decimal digit!) with the result of Christiansen et al. [5]. Other dimensions for the Feigenbaum attractor were presented by Kovács [4], and they coincide with our results up to the 10-th digit, the accuracy achieved in that work.

In the remaining three columns of the Table 1 we present results for the generalized dimensions of the multifractal attractors at the onset of chaos in unimodal maps of degrees 4, 6, and 8 obtained using the universal functions given e.g. in Ref.[14].

As an alternative to the traditional definition of the generalized dimensions $D_q$ one might consider a family of dimensions enumerated by the index $\tau$. Let us designate them as $D^\tau$: $D^\tau = D_{q(\tau)} = \tau/(q(\tau) - 1)$. As mentioned, for $\tau = 2$ the equation (5) is of a form studied in the theory of noise effect onto the period-doubling transition [10]; the noise scaling constant is defined as $\gamma = \sqrt{\nu(2)}$. Hence, the dimension $D^2$ is linked with the effect of noise. The scaling factor $\gamma$ is expressed via the dimension $D^2$ as $\gamma = 2^{1/D^2 + 1/2}$. In Table 2 we present high-precision data for values of $q(2)$, dimensions $D^2$, and noise scaling factors obtained from the numerical solution of the eigenproblem (5) for maps of degree 2, 4, 6, and 8. The data for the factors $\gamma$ improve the previously known results [16]. (Observe that value of $q$ depends on the degree, so one cannot speak on a definite dimension from the family $D_q$ associated with the noise effect!)

For the problem of unimodal maps, our method of calculation of the generalized dimensions does not have obvious computational advantages over those of Refs. [3, 4, 5], but it does represent the problem in a new light and indicates novel links between global and local descriptions of the scaling regularities. An analogous use of renormalization-group equations will be feasible in the calculation of multifractal properties in many other situations at the onset of chaos, e.g. in bimodal one-dimensional maps [17], asymmetric

one-dimensional maps [18], two-dimensional period-doubling maps [19], quasiperiodically forced maps [20], and complex analytic maps [21]. Such an approach will be useful especially for situations where computations based on the traditional partition-function approach are difficult.

**Acknowledgements**

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Table 1: Generalized dimensions for fractal attractors at the onset of chaos in unimodal maps of degree $d$

|          | $d = 2$          | $d = 4$       | $d = 6$      | $d = 8$      |
|----------|------------------|---------------|--------------|--------------|
| $D_5$    | 0.45392270234470 | 0.407695571   | 0.373232166  | 0.351475400  |
| $D_4$    | 0.46615155691823 | 0.426832904   | 0.392400635  | 0.370142909  |
| $D_3$    | 0.48077684940009 | 0.454569793   | 0.421638052  | 0.399231039  |
| $D_2$    | 0.49783645928917 | 0.495316676   | 0.468035066  | 0.447019466  |
| $D_1$    | 0.51709757255124 | 0.555181822   | 0.544847134  | 0.531111008  |
| $D_0$    | 0.53804514358055 | 0.642575065   | 0.683433256  | 0.707102082  |
| $D_{-1}$ | 0.55991291016494 | 0.763919555   | 0.946229117  | 1.146118382  |
| $D_{-2}$ | 0.58173600034603 | 0.894257449   | 1.205507002  | 1.510079742  |
| $D_{-3}$ | 0.60247817187829 | 0.992066238   | 1.354808070  | 1.698747772  |
| $D_{-4}$ | 0.62126594260209 | 1.056616863   | 1.445090859  | 1.811996998  |
| $D_{-5}$ | 0.63760518368338 | 1.100453275   | 1.505301852  | 1.887496869  |

Table 2: Generalized dimensions $D^2$ and noise scaling factors for unimodal maps of degree $d$

|                     | $d = 2$          | $d = 4$     | $d = 6$      | $d = 8$      |
|---------------------|------------------|-------------|--------------|--------------|
| $q(2)$              | 5.45324245756108 | 6.086657808 | 6.654767241  | 7.070578662  |
| $D^2 = D_{q(2)}$    | 0.44911096107158 | 0.393185481 | 0.353683877  | 0.329457884  |
| $\gamma$            | 6.61903651081803 | 8.243910853 | 10.037886410 | 11.59386214  |