# INITIAL-BOUNDARY VALUE PROBLEMS FOR LINEAR AND SOLITON PDEs

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#### Abstract

Evolution PDEs for dispersive waves are considered in both linear and nonlinear integrable cases, and initial-boundary value problems associated with them are formulated in spectral space. A method of solution is presented, which is based on the elimination of the unknown boundary values by proper restrictions of the functional space and of the spectral variable complex domain. Illustrative examples include the linear Schrödinger equation on compact and semicompact *n*-dimensional domains and the nonlinear Schrödinger equation on the semiline.

## 1 Introduction

Initial-Boundary Value (IBV) problems for Partial Differential Equations (PDEs) play an important role in applications to physics and, in general, to natural sciences.

It is well known that the basic difficulty associated with the study of IBV problems for linear and nonlinear integrable PDEs is the presence of unknown boundary values in the relevant equations of any method of solution. In this paper, after formulating the IBV problems for linear and nonlinear integrable PDEs in spectral space, we present a method of solution, the Elimination-by-Restriction (EbR) approach, which is based on a strategy of elimination of the unknown boundary values by proper restriction of the functional space and of the complex domain of definition of the involved spectral functions. This approach is inspired by the Green's function (GF) method, which, in the linear context, is essentially its counterpart in configuration space, and by our recent findings in the nonlinear context. The paper is organized as follows. In §2 we

deal with IBV problems for linear PDEs with constant coefficients. After introducing the proper Fourier Transform (FT) for that problem and establishing its analyticity properties, we first express the Fourier transform of the solution in terms of the Fourier transforms of known and unknown initial - boundary values using Green's formula. Then we present the EbR approach in which, using systematically a strategy of elimination of the unknown boundary values, one obtains the appropriate spectral representation of the solution whose support may eventually turn out to be discrete rather than continuous as in the general Fourier integral one starts from. We illustrate the power of the method solving IBV problems for the Schrödinger equation on an *n*-dimensional rectangular box and quadrant. In §3 we apply the EbR strategy to soliton equations. After defining the proper spectral transform S(k,t) for the given IBV problem, we apply the EbR procedure eliminating the unknown boundary values from the equations defining S(k,t) and characterizing S(k,t) via a nonlinear integral equation. This approach, presented on the prototype example of the nonlinear Schrödinger (NLS) equation on the semiline, can be in principle generalized to the segment case. Since the EbR technique operates on the spectral variables conjugated to the space ones, it works well either using the space-time transform or just the space transform. In the linear case we find it simpler to work with the space-time FT, while in the nonlinear case it seems more convenient to use the space transform, namely the well-known Inverse Scattering (Spectral) Transform (IST). We finally show the equivalence between IBV problems for soliton equations on the semiline and some specific forced initial value problems on the whole line. An important application of this equivalence is that, from the well-known asymptotics of soliton equations on the whole line, one can obtain immediately the asymptotic behaviour for IBV problems on the semiline with decaying boundary data. We have confined the relevant literature to §4.

The results contained in this paper are an expansion of part of the material presented by the authors at the Euroconference "NEEDS 2001" and at the workshop: "Boundary Value Problems", the first and last events of the Semester: *Integrable Systems*, held at the Isaac Newton Institute of Cambridge during the period July - December 2001.

# 2 The Elimination-by-Restriction Approach: The Linear Case

It is well -known that the Fourier Transform (FT) is the proper tool to solve initial - boundary value (IBV) problems for linear PDE's in  $\mathcal{R}^{n+1}$  with decaying boundary values:

$$\mathcal{L}(\nabla, \frac{\partial}{\partial t})u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad \mathbf{x} = (x_1, ..., x_n) \in \mathcal{R}^n, \quad t > 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u(\mathbf{x}, t) \to 0, \ |\mathbf{x}| \to \infty,$$
(1)

where  $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ ,  $\mathcal{L}$  is a constant coefficients partial differential operator,  $u(\mathbf{x}, t)$  is the unknown field,  $f(\mathbf{x}, t)$  is a given forcing and  $u_0(\mathbf{x})$  is the given initial condition.

In this section we present a very effective approach, in Fourier space, for solving more complicated IBV problems, defined on compact or semi - compact space domains V:

$$\mathcal{L}(\nabla, \frac{\partial}{\partial t})u(\mathbf{x}, t) = f(\mathbf{x}, t), \qquad \mathbf{x} \in V \subset \mathcal{R}^n, \quad t > 0,$$
(2)

with Dirichelet or Neumann or mixed boundary conditions on  $\partial V$ .

### 2.1 The Fourier Transform and its properties

The natural FT associated with the space - time domain  $\mathcal{D} = V \otimes (0, \infty)$  (in short:  $FT_{\mathcal{D}}$ ) is defined by

$$\hat{F}(\mathbf{k},q) = \int_{\mathcal{D}} d\mathbf{x} dt e^{-i(\mathbf{k}\cdot\mathbf{x}+qt)} F(\mathbf{x},t)$$
(3)

for any smooth function  $F(\mathbf{x}, t)$ ,  $(\mathbf{x}, t) \in \mathcal{D}$ , assuming that  $F(\mathbf{x}, t) \to 0$ ,  $t \to \infty$ fast enough; here  $\mathbf{k} = (k_1, ..., k_n) \in \mathcal{R}^n$ ,  $q \in \mathcal{R}$  and  $\mathbf{k} \cdot \mathbf{x} = \sum_j k_j x_j$ . Its inverse:

$$F(\mathbf{x},t)\chi_{\mathcal{D}}(\mathbf{x},t) = \int_{\mathcal{R}^{n+1}} \frac{d\mathbf{k}dq}{(2\pi)^{n+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+qt)} \hat{F}(\mathbf{k},q)$$
(4)

reconstructs  $F(\mathbf{x}, t)$  in  $\mathcal{D}$  and zero outside, where  $\chi_{\mathcal{D}}(\mathbf{x}, t)$  is the characteristic function of the domain  $\mathcal{D}$ :  $\chi_{\mathcal{D}}(\mathbf{x}, t) = 1$ ,  $(\mathbf{x}, t) \in \mathcal{D}$ ,  $\chi_{\mathcal{D}}(\mathbf{x}, t) = 0$ ,  $(\mathbf{x}, t) \notin \mathcal{D}$ (therefore:  $\chi_{\mathcal{D}}(\mathbf{x}, t) = \chi_{V}(\mathbf{x})H(t)$ , where H(t) is the usual Heaviside (step) function).

If the space domain is the whole space:  $V = \mathcal{R}^n$ , the  $FT_{\mathcal{D}}$  (3) is defined in  $\mathcal{A} = \mathcal{R}^n \otimes \overline{\mathcal{I}}_q$ , where  $\overline{\mathcal{I}}_q$  is the closure of the lower half q-plane  $\mathcal{I}_q$ , analytic in  $q \in \mathcal{I}_q$ ,  $\forall \mathbf{k} \in \mathcal{R}^n$  and exhibits a proper asymptotic behaviour for large |q| in the analyticity region. If the space domain V is compact, the  $FT_{\mathcal{D}}$  acquires strong analyticity properties in all the Fourier variables: it is defined in  $\mathcal{A} = \mathcal{C}^n \otimes \overline{\mathcal{I}}_q$ , analytic in  $q \in \mathcal{I}_q$ ,  $\forall \mathbf{k} \in \mathcal{C}^n$ , entire in every complex  $k_j$ ,  $j = 1, ..., n \ \forall q \in \overline{\mathcal{I}}_q$  and exhibits a proper asymptotic behaviour, for large  $(\mathbf{k}, q)$ , in the analyticity regions. If the space domain is semi - compact, then the analyticity in the Fourier variables  $k_j$ , j = 1, ..., n is limited to open regions of the complex plane, depending on the geometric properties of the domain V.

To express the  $FT_{\mathcal{D}}$  of the solution in terms of the  $FT_{\mathcal{D}}$ 's of the forcing and of the IB conditions we make use of the well - known **Green's formula** (identity):

$$b\mathcal{L}a - a\mathcal{L}b = div \ J(\mathbf{x}, t),\tag{5}$$

and of its integral consequence, the celebrated Green's integral identity:

$$\int_{\mathcal{D}} (b\mathcal{L}a - a\tilde{\mathcal{L}}b) d\mathbf{x} dt = \int_{\partial \mathcal{D}} J(\mathbf{x}, t) \cdot \nu d\sigma, \tag{6}$$

obtained by integrating (5) over the domain  $\mathcal{D}$  and by using the divergence theorem. In equation (5),  $\tilde{\mathcal{L}}$  is the formal adjoint of  $\mathcal{L}$ :  $\tilde{\mathcal{L}} = \mathcal{L}(-\nabla, -\frac{\partial}{\partial t}), J(\mathbf{x}, t)$ is an (n+1)-dimensional vector field, div is the (n+1)-dimensional divergence operator and  $a(\mathbf{x}, t)$  and  $b(\mathbf{x}, t)$  are arbitrary functions. In equation (6),  $d\sigma$  is the hypersurface element of the boundary and  $\nu$  is its outward unit normal. We remark that, given  $\mathcal{L}$ , its formal adjoint  $\tilde{\mathcal{L}}$  and two arbitrary functions a and b, an (n+1)-dimensional vector field  $J(\mathbf{x}, t)$  satisfying the Green's formula (5) always exists and can be algorithmically found to be a linear expression of a, band their partial derivatives of order up to N - 1, if  $\mathcal{L}$  is of order N.

The arbitrariness of a and b allows one to extract from (5) and (6) several important informations on the BV problem; with the particular choice

$$a = u(\mathbf{x}, t),$$
  $b = e^{-i(\mathbf{k} \cdot \mathbf{x} + qt)} / \mathcal{L}(i\mathbf{k}, iq),$  (7)

where  $\mathcal{L}(i\mathbf{k}, iq)$  is the eigenvalue of the operator  $\mathcal{L}$ , corresponding to the eigenfunction  $e^{i(\mathbf{k}\cdot\mathbf{x}+qt)}$ , the vector field J takes the following form:  $J = e^{-i(\mathbf{k}\cdot\mathbf{x}+qt)}J'(\mathbf{x}, t; \mathbf{k}, q)/\mathcal{L}(i\mathbf{k}, iq)$  and the Green's integral identity (6) gives the  $FT_{\mathcal{D}}$  of the solution in terms of the  $FT_{\mathcal{D}}$ 's (or, maybe, of generalized FT's) of the forcing and of all the initial - boundary values:

$$\hat{u}(\mathbf{k},q) = \frac{\hat{f}(\mathbf{k},q) - \int_{\partial \mathcal{D}} e^{-i(\mathbf{k}\cdot\mathbf{x}+qt)} J'(\mathbf{x},t;\mathbf{k},q) \cdot \nu d\sigma}{\mathcal{L}(i\mathbf{k},iq)} =: \frac{\hat{\mathcal{N}}(\mathbf{k},q)}{\mathcal{L}(i\mathbf{k},iq)}, \quad (\mathbf{k},q) \in \mathcal{A}.$$
(8)

In general,  $\mathcal{L}(i\mathbf{k}, iq)$ , the denominator of the above equation, is an entire and, most frequently, polynomial function of all its complex variables and its zeroes may lie on the real axis; therefore, before calculating the inverse FT, we must regularize it:

$$\mathcal{L}(i\mathbf{k}, iq) \to \mathcal{L}_{reg}(i\mathbf{k}, iq); \tag{9}$$

i.e., we must move a bit the singularities off the real axis, outside the domain  $\mathcal{A}$ .

Its inverse transform (4) gives the corresponding **Fourier representation** of the solution:

$$U(\mathbf{x},t) = u(\mathbf{x},t)\chi_{\mathcal{D}}(\mathbf{x},t) = \int_{\mathcal{R}^{n+1}} \frac{d\mathbf{k}dq}{(2\pi)^{n+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+qt)} \frac{\hat{\mathcal{N}}(\mathbf{k},q)}{\mathcal{L}_{reg}(i\mathbf{k},iq)}, \quad (\mathbf{x},t) \in \mathcal{R}^{n+1}$$
(10)

Clearly this is not the end of the story since, in general, the RHS of equation (8) depends on known and unknown boundary values.

### 2.2 Elimination-by-Restriction in Fourier space

The traditional ways in which IBV problems for linear PDE's are solved consist in finding convenient strategies for *eliminating the unknown boundary conditions* from the representation of the solution. On this idea is based the celebrated Green's function (GF) approach, in which:

i) one constructs the Green's integral representation

$$u(\mathbf{x},t) = \int_{\mathcal{D}} d\mathbf{x}' dt' \tilde{g}(\mathbf{x},t;\mathbf{x}',t') f(\mathbf{x}',t') - \int_{\partial \mathcal{D}} J(\mathbf{x},t;\mathbf{x}',t') \cdot \nu_{x'} d\sigma_{x'}, \quad (\mathbf{x},t) \in \mathcal{D}.$$
(11)

of the solution of the IBV problem (2) as another application of (6), corresponding to the choice:  $a(\mathbf{x},t) = u(\mathbf{x},t)$  and  $b(\mathbf{x},t) = \tilde{g}(\mathbf{x}',t';\mathbf{x},t)$ , where  $\tilde{g}$  is **any** Green's function of  $\hat{\mathcal{L}}_x$ :  $\tilde{\mathcal{L}}_x \tilde{g}(\mathbf{x}',t';\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ ,  $(\mathbf{x},t)$ ,  $(\mathbf{x}',t') \in \mathcal{D}$ . ii) One uses the arbitrariness of  $\tilde{g}$  and constructs that particular Green's function which allows one to eliminate contributions depending on unknown boundary values. On the elimination idea is also based the eigenfunction expansion method, essentially equivalent to the GF approach, in which one constructs a set of eigenfunctions of  $\mathcal{L}$  with proper boundary conditions which allow again to eliminate the unknown boundary values. Both approaches are of functional analytical nature.

In this section we shall show how the elimination strategy can be conveniently implemented working in the Fourier space defined by (3) (the elimination strategy in spectral space has been already investigated for soliton equations on the semiline (see  $\S4$ )).

Equation (8), defined in a proper domain  $\mathcal{A} \subset \mathcal{C}^{n+1}$ , usually exhibits several symmetry properties which are consequence of the structure of  $\mathcal{L}$  and of the geometry of the space domain V. The first and more critical part of the method consists in constructing a linear operator  $\mathcal{E}$  which, by exploiting systematically these symmetry properties in Fourier space, annihilates, in the RHS of (8), the contributions coming from the unknown boundary values:

$$\mathcal{E}\hat{u}(\mathbf{k},q) = \mathcal{E}(\frac{\hat{N}}{\mathcal{L}})(\mathbf{k},q) = \{ only \ known \ quantities \}, \quad (\mathbf{k},q) \in \mathcal{A}' \subset \mathcal{A}.$$
(12)

The elimination procedure, described by the linear operator  $\mathcal{E}$ , allows one to construct  $\mathcal{E}\hat{u}$  in a subspace  $\mathcal{A}'$  of the original space of definition of  $\hat{u}$ . It is necessary therefore to establish if this information is sufficient to reconstruct the solution u; i.e., if  $\mathcal{E}\hat{u}$  defines an invertible spectral transform in  $\mathcal{D}$ .

Using equation (3), this new spectral transform is defined by:

$$\begin{aligned}
\mathcal{E}\hat{u}(\mathbf{k},q) &= \int_{\mathcal{D}} d\mathbf{x} dt \tilde{\varphi}_{\mathbf{k},q}(\mathbf{x},t) u(\mathbf{x},t), & (\mathbf{k},q) \in \mathcal{A}' \\
\tilde{\varphi}_{\mathbf{k},q}(\mathbf{x},t) &:= \mathcal{E}(e^{-i(\mathbf{k}\cdot\mathbf{x}+qt)}),
\end{aligned}$$
(13)

and is invertible provided an eigenfunction  $\varphi_{\mathbf{k},q}(\mathbf{x},t)$  of  $\mathcal{L}$  exists with the prop-

erty of satisfying the completeness condition:

$$\sum_{(\mathbf{k},q)\in\mathcal{A}'}\varphi_{\mathbf{k},q}(\mathbf{x},t)\tilde{\varphi}_{\mathbf{k},q}(\mathbf{x}',t') = \delta(t-t')\delta(\mathbf{x}-\mathbf{x}'), \quad (\mathbf{x},t), (\mathbf{x}',t')\in\mathcal{D}.$$
(14)

In this case the inverse transform allows one to construct a function  $\tilde{U}(\mathbf{x}, t)$ , defined in general on the whole space-time, which coincides with the solution in  $\mathcal{D}$ :

$$\tilde{U}(\mathbf{x},t) = \sum_{\mathbf{k},q} \varphi_{\mathbf{k},q}(\mathbf{x},t) \mathcal{E}(\frac{\hat{N}}{\mathcal{L}})(\mathbf{k},q), \quad (\mathbf{x},t) \in \mathcal{R}^n \otimes (0,\infty), 
\tilde{U}(\mathbf{x},t) = u(\mathbf{x},t), \quad (\mathbf{x},t) \in \mathcal{D}.$$
(15)

In equations (14) and (15)  $\sum_{\mathbf{k},q}$  indicates a sum and/or an integral, depending on the nature of the domain  $\mathcal{A}'$  of definition of the new spectral transform. Summarizing:

Although the direct transform we started with is the Fourier transform  $\hat{u}$  defined in (3), the elimination-by-restriction procedure performed on it leads to a new direct transform  $\mathcal{E}\hat{u}$  (13), whose inversion usually differs considerably from (4). This implies that the reconstructed function  $\tilde{U}(\mathbf{x},t)$  coincides with the solution  $u(\mathbf{x},t)$  in  $\mathcal{D}$  but is not zero outside  $\mathcal{D}$ , inheriting the symmetry properties of the eigenfunction  $\varphi_{\mathbf{k},q}(\mathbf{x},t)$ .

### 2.3 Illustrative Example

In this section we apply the EbR approach to the IBV problem (2) corresponding to

$$\mathcal{L} = i\frac{\partial}{\partial t} + \Delta, \qquad \Delta = \nabla \cdot \nabla = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \qquad (16)$$
$$V = \{ \mathbf{x} : \ 0 \le x_j \le L_j, \ j = 1, ..., n \};$$

i.e., we solve IBV problems for the (n + 1)-dimensional Schrödinger equation in an *n*-dimensional rectangular box.

Then:

$$\widetilde{\mathcal{L}} = -i\frac{\partial}{\partial t} + \Delta, \qquad J = (iab, b \bigtriangledown a - a \bigtriangledown b), \\
\mathcal{L}(i\mathbf{k}, iq) = -(q + k^2) \quad \Rightarrow \quad \mathcal{L}_{reg}(i\mathbf{k}, iq) = -(q + k^2 - i0),$$
(17)

where  $k^2 = \mathbf{k} \cdot \mathbf{k}$ . Equations (8) and (17) give the following expression of the Fourier transform of the solution in terms of the Fourier transforms of the forcing and of all the initial - boundary values:

$$\hat{u}(\mathbf{k},q) = -\frac{\hat{\mathcal{N}}(\mathbf{k},q)}{q+k^2-i0}, \qquad (\mathbf{k},q) \in \mathcal{A}, \\ \hat{\mathcal{N}}(\mathbf{k},q) := \hat{f}(\mathbf{k},q) + i\hat{u}_0(\mathbf{k}) + \sum_{j=1}^n \{ [\hat{w}_{0j}(\mathbf{k}_j,q) + ik_j\hat{v}_{0j}(\mathbf{k}_j,q)] - (18) \\ e^{-ik_j L_j} [\hat{w}_{Lj}(\mathbf{k}_j,q) + ik_j\hat{v}_{Lj}(\mathbf{k}_j,q)] \},$$

in the definition domain  $\mathcal{A} = \mathcal{C}^n \otimes \overline{\mathcal{I}}_q$ , where  $\hat{f}$ ,  $\hat{u}_0$ ,  $\hat{v}_{0j}$ ,  $\hat{v}_{Lj}$ ,  $\hat{w}_{0j}$ ,  $\hat{w}_{Lj}$  are the FTs of the forcing and of the following IB values:

$$u_{0}(\mathbf{x}) = u(\mathbf{x},t)|_{t=0}, \quad v_{0j}(\mathbf{x}_{j},t) = u(\mathbf{x},t)|_{x_{j}=0}, \quad v_{Lj}(\mathbf{x}_{j},t) = u(\mathbf{x},t)|_{x_{j}=L_{j}}, w_{0j}(\mathbf{x}_{j},t) = \frac{\partial u}{\partial x_{j}}(\mathbf{x},t)|_{x_{j}=0}, \quad w_{Lj}(\mathbf{x}_{j},t) = \frac{\partial u}{\partial x_{i}}(\mathbf{x},t)|_{x_{j}=L_{j}};$$
(19)

i.e., for instance:

$$\hat{u}_0(\mathbf{k}) = \int\limits_V d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} u_0(\mathbf{x}), \quad \hat{v}_{0j}(\mathbf{k}_j, q) = \int\limits_0^\infty dt \int\limits_{V_j} d\mathbf{x}_j e^{-i(\mathbf{k}_j\cdot\mathbf{x}_j+qt)} v_{0j}(\mathbf{x}_j, t).$$
(20)

In equations (18)-(20), as well as in the following,  $\mathbf{x}_j = (x_1, ..., \check{x}_j, ..., x_n) \in \mathcal{R}^{n-1}$ ,  $\mathbf{k}_j = (k_1, ..., \check{k}_j, ..., k_n) \in \mathcal{R}^{n-1}$ ,  $\int_{V_j} d\mathbf{x}_j = \int_0^{L_1} dx_1 \cdot \cdot (\int_0^{L_j} dx_j) \cdot \cdot \int_0^{L_n} dx_n$  with the understanding that the superscript  $\check{}$  indicates that the quantity underneath is removed.

Of course, if all the above boundary values were known, the solution u would be given by the formula (10):

$$U(\mathbf{x},t) = u(\mathbf{x},t)\chi_{V}(\mathbf{x})H(t) = -\int_{\mathcal{R}^{n+1}} \frac{d\mathbf{k}dq}{(2\pi)^{n+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+qt)} \frac{\mathcal{N}(\mathbf{k},q)}{q+k^{2}-i0} = -\int_{\mathcal{R}^{n+1}} \frac{dqd\mathbf{k}}{(2\pi)^{n+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+qt)} \frac{\hat{f}(\mathbf{k},q)}{q+k^{2}-i0} + H(t) \int_{\mathcal{R}^{n}} \frac{d\mathbf{k}}{(2\pi)^{n}} e^{i(\mathbf{k}\cdot\mathbf{x}-k^{2}t)} \hat{u}_{0}(\mathbf{k}) + \sum_{j=1}^{n} \int_{\mathcal{R}^{n-1}} \frac{d\mathbf{k}_{j}}{(2\pi)^{n-1}} e^{i\mathbf{k}_{j}\cdot\mathbf{x}_{j}} \int_{\gamma} \frac{dk_{j}}{2\pi i} \{e^{i(k_{j}|x_{j}|-k^{2}t)} [\hat{w}_{0j}(\mathbf{k}_{j},-k^{2}) + isign(x_{j})k_{j}\hat{v}_{0j}(\mathbf{k}_{j},-k^{2})] - e^{i(k_{j}|x_{j}-L_{j}|-k^{2}t)} [\hat{w}_{Lj}(\mathbf{k}_{j},-k^{2}) + isign(x_{j}-L_{j})k_{j}\hat{v}_{Lj}(\mathbf{k}_{j},-k^{2})]\},$$

$$(21)$$

where  $d\mathbf{k}_j = dk_1..dk_j..dk_n$  and  $\gamma = (i\infty, 0) \cup (0, \infty)$ .

In view of the distinguished parity properties of the Fourier transforms in (18), in the following we shall make use of the parity operators:

$$\Delta_{\pm} = \prod_{l=1}^{n} (1 \pm \hat{\sigma}_l), \quad \Delta_{\pm}^{(j)} = \prod_{\substack{l=1\\l \neq j}}^{n} (1 \pm \hat{\sigma}_l), \tag{22}$$

where  $\hat{\sigma}_j$  is the involution  $\hat{\sigma}_j : k_j \rightarrow -k_j$ .

Suppose we are interested in solving the Dirichelet problem; applying the parity operator  $\Delta_{-}$  to (18) eliminates all  $\hat{w}_{0}$ 's:

$$\Delta_{-}\hat{u}(\mathbf{k},q) = -\left(\Delta_{-}[\hat{f}(\mathbf{k},q) + i\hat{u}_{0}(\mathbf{k})] + 2i\sum_{j=1}^{n}k_{j}\Delta_{-}^{(j)}\hat{v}_{0j}(\mathbf{k}_{j},q) + 2i\sum_{j=1}^{n}[\sin(k_{j}L_{j})\Delta_{-}^{(j)}\hat{w}_{Lj}(\mathbf{k}_{j},q) - k_{j}\cos(k_{j}L_{j})\Delta_{-}^{(j)}\hat{v}_{Lj}(\mathbf{k}_{j},q)]\right) / (q+k^{2}-i0), \ (\mathbf{k},q) \in \mathcal{A}.$$

$$(23)$$

To eliminate also the  $\hat{w_L}'s$ , the values of  $k_j$  must be restricted to the discrete set  $k_j = h_j := \frac{\pi m_j}{L_j}$ ,  $m_j \in \mathcal{Z}$ , so that the original domain  $\mathcal{A}$  is finally restricted

$$(\mathbf{k},q) \in \mathcal{A}' = \{(\mathbf{h},q); q \in \bar{\mathcal{I}}_q, \mathbf{h} = (h_1,..,h_n), h_j = \frac{\pi m_j}{L_j}, m_j \in \mathcal{Z}, j = 1,..,n\}$$
  
(24)

Therefore the EbR operator  $\mathcal{E}$  of this example reads:

$$\mathcal{E} \cdot = \int_{\mathcal{R}^n} d\mathbf{k} \delta(\mathbf{k} - \mathbf{h}) \Delta_- \cdot$$
 (25)

and its application to  $\hat{u}$  leads to the wanted result:

$$\mathcal{E}\hat{u}(\mathbf{k},q) = (\Delta_{-}\hat{u})(\mathbf{h},q) = -\{\Delta_{-}[\hat{f}(\mathbf{h},q) + i\hat{u}_{0}(\mathbf{h})] + 2i\sum_{j=1}^{n} h_{j}\Delta_{-}^{(j)}[\hat{v}_{0j}(\mathbf{h}_{j},q) - (-)^{m_{j}}\hat{v}_{Lj}(\mathbf{h}_{j},q)]\}/(q+h^{2}-i0),$$
(26)

where  $\mathbf{h}_j = (h_1, ..., \check{h}_j, ..., h_n)$  and  $h^2 = \mathbf{h} \cdot \mathbf{h}$ . The transform  $\mathcal{E}\hat{u}(\mathbf{k}, q)$  generated by the EbR procedure is the well-known multidimensional **discrete** sine transform:

$$\Delta_{-}\hat{u}(\mathbf{h},q) = \int_{\mathcal{D}} dt d\mathbf{x} \tilde{\varphi}_{\mathbf{h},q}(\mathbf{x},t) u(\mathbf{x},t),$$
  
$$\tilde{\varphi}_{\mathbf{h},q}(\mathbf{x},t) := \Delta_{-} (e^{-i(\mathbf{h}\cdot\mathbf{x}+qt)}) = (-2i)^{n} e^{-iqt} \prod_{l=1}^{n} \sin(h_{l}x_{l}).$$
(27)

For its inversion we use:

$$\varphi_{\mathbf{h},q}(\mathbf{x},t) = \frac{i^n}{L_1 \cdot L_n} \frac{e^{iqt}}{2\pi} \prod_{l=1}^n \sin(h_l x_l), \qquad \sum_{\mathbf{k},q} = \int_{\mathcal{R}} dq \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty , \quad (28)$$

so that (15) yields the function  $\tilde{U}(\mathbf{x}, t)$ , defined in the whole space time, which coincides with the solution  $u(\mathbf{x}, t)$  of the IBV problem under scrutiny for  $(\mathbf{x}, t) \in \mathcal{D}$ :

$$\tilde{U}(\mathbf{x},t) = \frac{i^n}{L_1 \cdots L_n} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \prod_{l=1}^n \sin(h_l x_l) \int_{\mathcal{R}} \frac{dq}{2\pi} e^{iqt} \Delta_- \left(\frac{\hat{N}(\mathbf{h},q)}{\mathcal{L}(i\mathbf{h},iq)}\right), \quad (\mathbf{x},t) \in \mathcal{R}^{n+1}, \quad (29)$$

Equation (29) implies, due to the symmetry properties of  $\varphi_{\mathbf{h},q}(\mathbf{x},t)$ , that  $\tilde{U}$  is the odd (2*L*)-periodic extension of the solution outside  $\mathcal{D}$  and provides the discrete sine-Fourier representation of the solution:

$$u(\mathbf{x},t) = -\frac{i^{n}}{L_{1}\cdots L_{n}} \{ \sum_{\mathbf{h},S} \prod_{l=1}^{n} \sin(h_{l}x_{l}) [\int_{\mathcal{R}} \frac{dq}{2\pi} e^{iqt} \frac{\Delta_{-}\hat{f}(\mathbf{h},q)}{q+h^{2}-i0} - e^{-ih^{2}t} \Delta_{-}\hat{u}_{0}(\mathbf{h})] + \sum_{j=1}^{n} L_{j} \sum_{\mathbf{h}_{j},S} \prod_{l\neq j} \sin(h_{l}x_{l}) \int_{\gamma} \frac{dk_{j}}{\pi} \frac{ik_{j}e^{-i(k_{j}^{2}+\mathbf{h}_{j}\cdot\mathbf{h}_{j})t}}{\sin(k_{j}L_{j})} \Delta_{-}^{(j)} [\sin k_{j}(L_{j}-x_{j})\hat{v}_{0j}(\mathbf{h}_{j},-k_{j}^{2}-\mathbf{h}_{j}\cdot\mathbf{h}_{j}) + (30) \\ \sin(k_{j}x_{j})\hat{v}_{Lj}(\mathbf{h}_{j},-k_{j}^{2}-\mathbf{h}_{j}\cdot\mathbf{h}_{j})] \}, \qquad (\mathbf{x},t) \in \mathcal{D},$$

to:

where

$$\sum_{\mathbf{h},S} = \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty}, \qquad \sum_{\mathbf{h}_j,S} = \sum_{m_1=1}^{\infty} \cdots \left(\sum_{m_j=1}^{\check{\infty}}\right) \cdots \sum_{m_n=1}^{\infty}$$
(31)

and the integral  $\int_{\gamma} dk_j$  is regularized moving the singularities  $k_j = (\pi m_j/L_j), m_j \in \mathcal{Z}^+$  a bit off the first Quadrant.

The Neumann problem can be treated similarly, leading to the discrete cosine transform:

$$\mathcal{E} \cdot = \int_{\mathcal{R}^n} d\mathbf{k} \delta(\mathbf{k} - \mathbf{h}) \Delta_+ \cdot \\ \tilde{\varphi}_{\mathbf{h},q}(\mathbf{x}, t) := \Delta_+ (e^{-i(\mathbf{h} \cdot \mathbf{x} + qt)}) = 2^n e^{-iqt} \prod_{l=1}^n \cos(h_l x_l), \qquad (32)$$
$$\varphi_{\mathbf{h},q}(\mathbf{x}, t) = \frac{1}{2^n L_1 \cdot L_n} \frac{e^{iqt}}{2\pi} \prod_{l=1}^n \cos(h_l x_l), \qquad \sum_{\mathbf{k},q} = \int_{\mathcal{R}} dq \sum_{m_1 = -\infty}^\infty \cdot \cdot \sum_{m_n = -\infty}^\infty$$

and to the following discrete cosine representation of the solution:

$$u(\mathbf{x},t) = -\frac{1}{2^{n}L_{1}\cdots L_{n}} \{ \sum_{\mathbf{h},C} \prod_{l=1}^{n} \cos(h_{l}x_{l}) [ \int_{\mathcal{R}} \frac{dq}{2\pi} e^{iqt} \frac{\Delta_{+}\hat{f}(\mathbf{h},q)}{q+h^{2}-i0} - e^{-ih^{2}t} \Delta_{+}\hat{u}_{0}(\mathbf{h}) ] - 2 \sum_{j=1}^{n} L_{j} \sum_{\mathbf{h}_{j},C} \prod_{l\neq j} \cos(h_{l}x_{l}) \int_{\gamma} \frac{dk_{j}}{\pi} \frac{e^{-i(k_{j}^{2}+\mathbf{h}_{j}\cdot\mathbf{h}_{j})t}}{\sin(k_{j}L_{j})} \Delta_{+}^{(j)} [\cos k_{j}(L_{j}-x_{j})\hat{w}_{0j}(\mathbf{h}_{j},-k_{j}^{2}-\mathbf{h}_{j}\cdot\mathbf{h}_{j}) - \cos(k_{j}x_{j})\hat{w}_{Lj}(\mathbf{h}_{j},-k_{j}^{2}-\mathbf{h}_{j}\cdot\mathbf{h}_{j}) ] \}, \qquad (\mathbf{x},t) \in \mathcal{D},$$

where

$$\sum_{\mathbf{h},C} = \sum_{m_1 = -\infty}^{\infty} \cdots \sum_{m_n = -\infty}^{\infty}, \qquad \sum_{\mathbf{h}_j,C} = \sum_{m_1 = -\infty}^{\infty} \cdots \left(\sum_{m_j = -\infty}^{\check{\infty}}\right) \cdots \sum_{m_n = -\infty}^{\infty} (34)$$

and the integral  $\int_{\gamma} dk_j$  is regularized as in (30).

Using the convolution theorem, one immediately recovers from equations (30) and (33) the Green's integral representation (11) of the solution, corresponding respectively to the following retarded Dirichelet and Neumann Green's functions:

$$G_{RD}(\mathbf{x}, t; \mathbf{x}', t') = \frac{2^{n_i}}{L_1 \cdots L_n} H(t - t') \sum_{m_1 = 1}^{\infty} \cdots \sum_{m_n = 1}^{\infty} e^{-ih^2(t - t')} \prod_{l=1}^n \sin(h_l x_l) \sin(h_l x_l'),$$

$$G_{RN}(\mathbf{x}, t; \mathbf{x}', t') = \frac{-i}{L_1 \cdots L_n} H(t - t') \sum_{m_1 \in \mathcal{Z}} \cdots \sum_{m_n \in \mathcal{Z}} e^{-ih^2(t - t')} \prod_{l=1}^n \cos(h_l x_l) \cos(h_l x_l').$$
(35)

In the case of semicompact domains the proper spectral transform generated by the EbR approach has, in general, a continuous support. For instance, in the Dirichelet problem for the Schrödinger equation on the *n*-Quadrant

$$V = \{ \mathbf{x} \in \mathcal{R}^n : \ x_j \ge 0, \ j = 1, .., n \},$$
(36)

the elimination operator  $\mathcal{E} = \Delta_{-}$ , which restricts the definition domain to  $\mathcal{A}' = \mathcal{R}^n \otimes \overline{\mathcal{I}}_q$ , leads to the continuous sine Fourier transform:

$$\tilde{\varphi}_{\mathbf{k},q}(\mathbf{x},t) = (-2i)^n e^{-iqt} \prod_{l=1}^n \sin(k_l x_l), \quad \varphi_{\mathbf{k},q}(\mathbf{x},t) = \frac{e^{i(\mathbf{k}\cdot\mathbf{x}+qt)}}{(2\pi)^{n+1}}, \quad \sum_{\mathbf{k},q} = \int_{\mathcal{R}} dq \int_{\mathcal{R}^n} d\mathbf{k}$$
(37)

and to the continuous multidimensional sine-Fourier representation of the solution:

$$u(\mathbf{x},t) = -\int_{\mathcal{R}^{n+1}} \frac{dq d\mathbf{k}}{(2\pi)^{n+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+qt)} \frac{\Delta_{-}\hat{f}(\mathbf{k},q)}{q+k^{2}-i0} + \int_{\mathcal{R}^{n}} \frac{d\mathbf{k}}{(2\pi)^{n}} e^{i(\mathbf{k}\cdot\mathbf{x}-k^{2}t)} \Delta_{-}\hat{u}_{0}(\mathbf{k}) + \sum_{j=1}^{n} \int_{\mathcal{R}^{n-1}} \frac{d\mathbf{k}_{j}}{(2\pi)^{n-1}} \int_{\gamma} \frac{dk_{j}}{\pi} e^{i(\mathbf{k}\cdot\mathbf{x}-k^{2}t)} k_{j} \Delta_{-}^{(j)} \hat{v}_{0j}(\mathbf{k}_{j},-k^{2}), \quad (\mathbf{x},t) \in \mathcal{D}.$$

$$(38)$$

From the above illustrative examples it appears that the EbR procedure in Fourier space is very effective and, perhaps, simpler than the Green's function approach, which is its counterpart in configuration space. The comparison between these two methods of elimination in examples in which the GF approach fails is postponed to a subsequent paper.

# 3 The Elimination-by-Restriction Approach: The Nonlinear Case

In this section we turn our attention to IBV problems associated with nonlinear evolution PDE's which are integrable by the inverse scattering (spectral) transform method. Because of the limited scope of this paper, we content ourselves with illustrating our method of solution by considering, as a prototype, the nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + c|q|^2 q = 0, \quad q = q(x,t),$$
(39)

where c is an arbitrary real parameter, but this method applies as well to other 1+1 dimensional soliton equations (f.i. to the Korteweg de Vries (KdV) equation). Moreover, we confine our treatment below to solutions of (39) in the first quadrant of the (x,t) plane, namely on the semiline  $0 \le x \le \infty$  for  $t \ge 0$ . Besides the initial value  $q(x,0) = q_0(x)$ , the boundary value which uniquely specifies the solution is

$$f(t) = a_1 v(t) + a_2 w(t), \quad t \ge 0, \tag{40}$$

where we have set

$$v(t) = q(0,t), \quad w(t) = q_x(0,t).$$
 (41)

Here  $a_1$  and  $a_2$  are given real constants and, if  $a_2 = 0$   $(a_1 = 0)$  this is the Dirichelet (Neumann) BV problem. Thus the problem is that of constructing

the solution q(x,t) of (39) when the initial value  $q_0(x)$  and the boundary value f(t) are given functions in an appropriate functional space (we may assume that they are complex valued functions which rapidly decay as  $x \to \infty$  and  $t \to \infty$ ).

The key-property of the NLS equation (39) is that it is the integrability condition for the following pair of  $2 \times 2$  matrix linear Ordinary Differential Equations (ODEs),

$$\Psi_x = (ik\sigma_3 + Q)\Psi, \qquad \Psi_t = 2ik^2[\sigma_3, \Psi] + M\Psi$$
(42)

where  $\sigma_3 = diag (1, -1)$  and

$$Q(x,t) = \begin{pmatrix} 0 & -c\bar{q}(x,t) \\ q(x,t) & 0 \end{pmatrix}, \qquad M(x,t,k) = 2kQ - i\sigma_3Q_x + iQ^2\sigma_3.$$
(43)

The solution  $\Psi(x, t, k)$  of these equations is defined by the asymptotic condition

$$\Psi(x,t,k)e^{-ikx\sigma_3} \to I, \quad x \to \infty \tag{44}$$

which uniquely defines the scattering matrix S(k, t) in the standard way, namely as the boundary value

$$S(k,t) = \Psi(0,t,k).$$
 (45)

Well-known facts, which will be instrumental in the method below, are the following. The matrix solution  $\Psi$  and, therefore (see (45)), the scattering matrix S have unit determinant,

$$det \Psi(x,t,k) = det S(k,t) = 1.$$
(46)

Moreover, the property

$$Q^{\dagger} = -CQC^{-1}, \quad C := \begin{pmatrix} 1 & 0\\ 0 & c \end{pmatrix}$$
(47)

of the matrix Q, see (43), induces the corresponding property

$$\Psi^{\dagger}(x,t,k) = C\Psi^{-1}(x,t,\bar{k})C^{-1}, \quad S^{\dagger}(k,t) = CS^{-1}(\bar{k},t)C^{-1}, \quad (48)$$

on the Jost solution  $\Psi$  and on the scattering matrix S (the superscript <sup>†</sup> indicates hermitian conjugation). As a consequence, it is convenient to parametrize the matrix S by introducing the two functions  $\alpha(k,t)$  and  $\beta(k,t)$  according to the definition

$$S(k) = \begin{pmatrix} \alpha(k) & -c\bar{\beta}(\bar{k}) \\ \beta(k) & \bar{\alpha}(\bar{k}) \end{pmatrix}.$$
(49)

As for the k-dependence (here the complex spectral variable k plays the same role as the Fourier variable in the linear case), the functions  $\alpha(k,t)$  and  $\beta(k,t)$ turn out to be analytic in the UHP ( $Im \ k > 0$ ) and to have there the asymptotic behaviour

$$\alpha(k,t) = 1 + O(k^{-1}), \qquad \beta(k,t) = O(k^{-1}), \tag{50}$$

for large |k|. Finally, we remind the reader that these analyticity properties of  $\alpha(k,t)$  and  $\beta(k,t)$  provide the way to solve the inverse problem, i.e.  $S(k,t) \rightarrow Q(x,t)$ , for any fixed  $t \geq 0$ ; the basic equations of the inverse problem, which are not reported here, read as either Cauchy-type integral equations in the k variable or, equivalently, as Marchenko-type integral equation in the x-variable.

Let us now look at the time evolution. Here the real crux of the spectral method appears in the evolution equation of the scattering matrix, see (42) and (45),

$$S_t = 2ik^2[\sigma_3, S] + Z(k, t)S,$$
(51)

since the matrix Z(k, t) has a separate dependence on both the boundary data v(t) and w(t) (see (41)) according to the following expressions

$$Z(k,t) = 2kV(t) - i\sigma_3 W(t) + iV^2(t)\sigma_3, V(t) = Q(0,t), \quad W(t) = Q_x(0,t).$$
(52)

As a consequence, the evolution equation (51) cannot be immediately integrated to yield the scattering matrix whose knowledge is essential to reconstruct Q(x,t)via the solution of the inverse problem. Since only the boundary datum (40) is given, in analogy with the elimination strategy of the linear case, we introduce at this point the novel matrix

$$\tilde{S}(k,t) = A^{-1}(k)S^{-1}(-k,t)A(k)S(k,t), \quad A(k) := a_1I + 2ika_2\sigma_3, \tag{53}$$

because of its two important properties. First, its determinant is unit (see (46)) and its asymptotic value as  $|k| \rightarrow \infty$  is the unit matrix:

det 
$$\hat{S}(k,t) = 1$$
,  $\hat{S}(k,t) = I + O(k^{-1});$  (54)

second, it satisfies an evolution equation which contains only the given boundary value (40), namely

$$\tilde{S}_{t} = 2ik^{2}[\sigma_{3}, \tilde{S}] + 4kA^{-1}(k)S^{-1}(-k, t)F(t)S(k, t), 
F(t) = a_{1}V(t) + a_{2}W(t) = \begin{pmatrix} 0 & -c\bar{f}(t) \\ f(t) & 0 \end{pmatrix}.$$
(55)

Though this is an important step, it does not yield the solution of our problem since the unknown scattering matrix S(k,t) still appears in the evolution (55a). Thus one has to find the way to relate S(k,t) and  $\tilde{S}(k,t)$  to each other. The relation  $S(k,t) \to \tilde{S}(k,t)$  is of course trivial as it is given by the definition (53) itself. This relation yields the initial value  $\tilde{S}(k,0)$  for the integration of the evolution equation (55), i.e.  $Q(x,0) \to \Psi(x,0,k) \to S(k,0) = \Psi(0,0,k) \to \tilde{S}(k,0)$ . As for the inverse relation,  $\tilde{S}(k,t) \to S(k,t)$ , one has instead to set up a RH problem, which finally leads to a Cauchy-type integral equation. Starting with rewriting (53) in the form  $A(k)S(k,t) = S(-k,t)A(k)\tilde{S}(k,t)$ , and noting that the first column of S(k,t) in (49) is analytic in the UHP with the asymptotic behaviour (50), one can write down two coupled integral equations for  $\alpha(k,t)$  and  $\beta(k,t)$  in terms of  $\tilde{S}(k,t)$  by going through the standard RH problem technique. By assuming, just for the sake of simplicity, that no poles occur in the UHP, and by substituting  $\tilde{S}(k,t)$  with its expression obtained by formally integrating the evolution equation (55), one finally ends up with the two coupled nonlinear integral equations

$$\begin{aligned} \alpha(k,t) &= 1 + \frac{c}{2\pi i} \int_{-\infty}^{\infty} \frac{dk'}{k' - (k+i0)} e^{-4ik'^2 t} h(k',t) \bar{\beta}(k',t), \\ \beta(k,t) &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk'}{k' - (k+i0)} e^{-4ik'^2 t} h(k',t) \bar{\alpha}(k',t), \end{aligned}$$
(56)

whose nonlinearity is due to the fact that the function h(k, t) depends itself on the unknowns  $\alpha(k, t)$  and  $\beta(k, t)$  through the integral (in t) relation

$$h(k,t) = \{a(k)\alpha_{0}(k)\beta_{0}(-k) - a(-k)\alpha_{0}(-k)\beta_{0}(k) - 4k\int_{0}^{t} dt' e^{4ik^{2}t'} [f(t')\alpha(k,t')\alpha(-k,t') + c\bar{f}(t')\beta(k,t')\beta(-k,t')]\} / \{a(-k)\alpha_{0}(-k)\bar{\alpha}_{0}(k) + ca(k)\beta_{0}(-k)\bar{\beta}_{0}(k) - -4kc\int_{0}^{t} dt' [f(t')\alpha(-k,t')\bar{\beta}(k,t') - \bar{f}(t')\bar{\alpha}(k,t')\beta(-k,t')]\},$$
(57)

where  $a(k) = a_1 + 2ika_2$  and  $\alpha_0$  and  $\beta_0$  are, respectively, the known initial values  $\alpha(k, 0)$  and  $\beta(k, 0)$ .

The discussion of these, admittedly complicate, equations, and of their implications in various directions is beyond the scope of this paper, and will be reported elsewhere. Here we merely note that this formulation naturally single out the so-called linearizable IBV problems, these being those for which the boundary value f(t), see (40), vanishes: f(t) = 0. Indeed, in this case, the kernel function h(k,t) (57) does not depend on the unknowns  $\alpha(k,t)$  and  $\beta(k,t)$ , and the equations (56) become linear. As a side remark, we also note that setting c = 0 eliminates the nonlinearity in all the formulae given above, so that (39) becomes the linear Schrödinger equation and our equations (56), (57) yield  $\alpha(k,t) = 1$  while  $\beta(k,t)$  coincides with the usual explicit expression of the Fourier transform of the solution q(x,t).

Finally, we deem of interest to report here also an approach to the IBV problem for the NLS equation (39) which is different from the one given above, and yet equivalent as it eventually leads to the same equations (56) and (57). The main feature of this approach is that the IBV problem is reformulated on the whole line, i.e. for  $x \in (-\infty, \infty)$ , and that the matrix  $\tilde{S}(k, t)$ , as defined by (53), acquires now a spectral meaning within the standard direct and inverse problem associated with the Lax equation, say the first ODE in (42). The price one pays to arrive at this more familiar formulation is that the nonlinear PDE one has to solve now is the NLS equation with a inhomogeneous source term rather than the NLS equation (39). This approach is briefly sketched here with

two limitations which are merely dictated by the sake of simplicity; namely we confine our treatment to the Dirichelet and Neumann BV problems and, second, we assume that the spectral data at any time  $t \ge 0$  have no discrete spectrum component.

The starting observation is that, if q(x,t) is a solution of the NLS equation (39) for  $x \in (0, \infty)$  and  $t \ge 0$ , then the function

$$\tilde{q}(x,t) = q(x,t)H(x) - \eta q(-x,t)H(-x), \quad \eta = \pm 1,$$
(58)

as defined for any real value of x, satisfies the PDE

$$i\tilde{q}_t + \tilde{q}_{xx} + 2c|\tilde{q}|^2\tilde{q} = (1+\eta)v(t)\delta'(x) + (1-\eta)w(t)\delta(x).$$
(59)

where  $\delta(x)$  is the Dirac delta distribution,  $\delta'(x)$  is its derivative and v, w are defined in (41). Obviously,  $\eta = 1$  ( $\eta = -1$ ) is the appropriate choice when one deals with the Dirichelet (Neumann) IBV problem.

As implied by the spectral method based on the Lax equations, it is convenient to rewrite (58) and (59) in matrix form by introducing the  $2 \times 2$  off-diagonal matrix (see (43))

$$\tilde{Q}(x,t) = Q(x,t)H(x) - \eta Q(-x,t)H(-x), \quad \eta = \pm 1,$$
 (60)

and the PDE

$$i\tilde{Q}_t - \sigma_3(\tilde{Q}_{xx} - 2\tilde{Q}^3) = \Sigma(x, t)$$
(61)

which is, of course, equivalent to (59) if the source term is (see (52b))

$$\Sigma(x,t) = -\sigma_3[(1+\eta)V(t)\delta'(x) + (1-\eta)W(t)\delta(x)].$$
(62)

The spectral approach to the equation (61) is based on the spectral equation

$$\tilde{\Psi}_x = (ik\sigma_3 + \tilde{Q}(x,t))\tilde{\Psi}, \qquad \tilde{\Psi} = \tilde{\Psi}(x,t,k),$$
(63)

and it is standard. The Jost solution  $\tilde{\Psi}$  is defined by the asymptotic condition (44),  $\tilde{\Psi}exp(-ikx\sigma_3) \to I$ ,  $x \to \infty$ , which readly provides its expression in terms of the solution  $\Psi(x, t, k)$  introduced above on the semiline,

$$\tilde{\Psi}(x,t,k) = \Psi(x,t,k)H(x) + E\Psi(-x,t,-k)E\tilde{S}(k,t)H(-x), \qquad (64)$$

where  $E = diag(1, \eta)$  and

$$\hat{S}(k,t) = ES^{-1}(-k,t)ES(k,t),$$
(65)

is precisely the scattering matrix which is defined in the usual way, namely

$$\tilde{\Psi}(x,t,k) \to e^{ikx\sigma_3}\tilde{S}(k,t), \quad x \to -\infty.$$
(66)

At this point we note that this scattering matrix  $\tilde{S}(k,t)$  coincides with the matrix (53) with  $a_2 = 0$  in the Dirichelet case  $(\eta = 1)$  and with  $a_1 = 0$  in the Neumann case  $(\eta = -1)$ .

It is common expedient now to introduce also the other Jost solution of (63),

$$\tilde{\Phi}(x,t,k) = \tilde{\Psi}(x,t,k)\tilde{S}^{-1}(k,t),$$
(67)

and to take into account the identity

$$\tilde{S}_t + 2ik^2[\tilde{S}, \sigma_3] = i \int_{-\infty}^{\infty} dx \tilde{\Phi}^{-1}(x, t, k) [i\tilde{Q}_t - \sigma_3(\tilde{Q}_{xx} - 2\tilde{Q}^3)] \tilde{\Psi}(x, t, k), \quad (68)$$

which, together with the inhomogeneous PDE (61), entails the evolution equation for the scattering matrix,

$$\tilde{S}_t = 2ik^2[\tilde{S}, \sigma_3] + i \int_{-\infty}^{\infty} dx \tilde{\Phi}^{-1}(x, t, k) \Sigma(x, t) \tilde{\Psi}(x, t, k).$$
(69)

It is now easy to show that inserting in the integral in the RHS of this equation the expressions (67), (64) and (62) yields precisely the evolution equation (55) for the Dirichelet and Neumann IBV problems, say

$$\tilde{S}_t = 2ik^2[\sigma_3, \tilde{S}] + 2k(1+\eta)S^{-1}(-k, t)V(t)S(k, t) - i(1-\eta)\sigma_3S^{-1}(-k, t)W(t)S(k, t)$$
(70)

We end this paper remarking that a good side of the present approach, which we will refer to as the "source-method", is that one may take advantage of the more traditional inverse scattering (spectral) technique on the whole line. In particular, to investigate the large time behaviour of the solution q(x,t) of the IBV problem since the asymptotic expression, if the boundary value rapidly vanishes as  $t \to \infty$ , are readly at hand in the usual spectral theory on the whole line.

## 4 Literature

The classical idea of eliminating unknown boundary values from the representation of the solution is the essence of the Green's function approach [1], which makes essential use of the image method [2] to construct the proper Green's function which eliminates the unknown boundary values. The Elimination-by-Restriction approach introduced here is a natural and effective implementation of the elimination strategy in Fourier space. To the best of our knowledge this method for linear PDEs has never been presented before.

An alternative method, which we term the *Analyticity* approach, is also possible, and it is motivated by Fokas discovery of the global relation and of

its use to solve IBV problems [3], [4], [5], [6], [7]. Our contribution to this method consists in using systematically the analyticity properties of all the Fourier transforms involved in (8), to derive a set of analyticity constraints which allow one to express unknown boundary values in terms of known ones and, in general, to study the unique solvability of IBV problems (see [8]).

Different approaches to deal with the problem of unknown boundary data in the study of IBV problems for soliton equations have been developed during the last few decades. In [9], Fokas introduced a nonlinear analogue of the sine transform. In [10], Sabatier constructed an "elbow scattering" in the (x, t)plane to deal with the semiline problem for KdV, leading to a Gel'fand - Levitan - Marchenko formulation. In [3, 4] a different approach, based on a simultaneous x-t spectral transform, has been introduced by Fokas and rigorously developed in [11, 12], to solve IBV problems for soliton equations on the semiline. It allows one for a rigorous asymptotics [13] and captures in a natural way the known cases [14] of linearizable boundary value problems. In [15] we have introduced two alternative approaches to the study of IBV problems for soliton equations on the segment and on the semiline. In the first method we expressed the unknown boundary values in terms of elements of the scattering matrix S(k, t), thus obtaining a nonlinear integro-differential evolution equation for S. In the second method, which can be viewed as the nonlinear analogue of the EbR approach developed in  $\S2$ , we constructed the nonlinear evolution equation (55) for S, which does not contain unknown boundary values and captures in a natural way the case of linearizable IBV problems.

In some nongeneric cases of soliton equations corresponding to singular dispersion relations, like the stimulated Raman scattering (SRS) equations and the sine Gordon (SG) equation in light-cone coordinates, the evolution equation of the scattering matrix does not contain unknown boundary data. The SG equation on the semiline has been treated using the x-t spectral transform [3]; the SRS and the SG equations on the semiline have also been treated using a more traditional x- transform method respectively in [16] by Leon and Mikhailov and in [17] by Leon and Spire; the x- spectral data used in this last approach satisfy a nonlinear evolution equation of Riccati type.

Apart from the simultaneous x - t transform, all the above approaches are based on the traditional IST [18]. The spectral formalism for studying forced soliton equations has been developed by several authors, expecially in connection to the theory of perturbations (see, for instance, [19]).

The results presented here in the nonlinear context are: i) the construction, via the RH technique, of the closed form integral equation (56), (57) satisfied by the elements of the scattering matrix. ii) The formulation of the Dirichelet and Neumann IBV problems on the semiline for soliton equations as forced initial value problems on the whole line. The equivalence between the semiline and whole-line problems has been already used in [9], although the relevant equations for the spectral data given there differ from those presented here.

#### Acknowledgments

The present work has been carried out during several visits and meetings. We gratefully acknowledge the financial contributions provided by the RFBR Grant 01-01-00929, the INTAS Grant 99-1782 and by the following Institutions: the University of Rome "La Sapienza" (Italy), the Istituto Nazionale di Fisica Nucleare (Sezione di Roma), the Landau Institute for Theoretical Physics, Moscow (Russia) and the Isaac Newton Institute, Cambridge (UK), within the programme "Integrable Systems".

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