

Parasupersymmetric structure of the Boussinesq-type systems

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Abstract

We study Darboux transformations for a Boussinesq-type equations. The parasupersymmetric structure of link between Boussinesq and modified Boussinesq systems is revealed.

1 Introduction.

In [1] the supersymmetric structure of the KdV and modified KdV (mKdV) systems (including lower KdV equations) is revealed. Well known Miura transformations between KdV and mKdV equations is nothing but the manifestation of this structure. In [2] the Miura-type transformations for the Boussinesq (Bq) and modified Boussinesq (mBq) equations is obtained. Lax pairs for these equations are some third-order linear differential expressions and its can't be elements of supersymmetry (SUSY) algebra. This because SUSY algebra can be realized via even-order linear operators [3].

Some times ago interest has appeared into extensions of SUSY quantum mechanics to systems with three-fold degeneracy of the energy spectrum [4-6]. The related transformations obey the *parasuperalgebra*. By contrast with SUSY which bind one bosonic and one fermionic levels with the same energy, the parasupersymmetry (PSUSY) do the same for the one bosonic and two parafermionic levels [7].

One of our main goals is to show that the algebraic structure of Bq and mBq equations is the PSUSY and that Miura-type transformations between these systems ([2]) can be obtained from this structure.

2 Supersymmetry and parasupersymmetry.

The KdV equation can be obtained from the Lax pair in the form

$$\frac{dL_1}{dt} = [A_1, L_1], \quad (1)$$

where $L_1 \equiv \partial^2 + u_1(x, t) = q^+q$, $\partial = d/dx$, $q = \partial + g$, $q^+ = \partial - g$.

Darboux transformation (DT) for the Schrödinger equation yet [8],

$$L_1 \rightarrow L_2 = \partial^2 + u_2 = qq^+, \quad (2)$$

with $u_2 = u_1 + 2g_x$.

Supersymmetry Hamiltonian H and supergenerators Q, Q^+ are [9],

$$H = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix},$$

so we have superalgebra

$$\{Q, Q^+\} = H, \quad [Q, H] = [Q^+, H] = 0. \quad (3)$$

Note the supergenerators are nilpotent of order two,

$$Q^2 = (Q^+)^2 = 0.$$

To construct PSUSY we use two-times DT

$$L_1 = q^+q \rightarrow L_2 = qq^+ = \tilde{q}^+\tilde{q} + \mu \rightarrow L_3 = \tilde{q}\tilde{q}^+ + \mu,$$

where $\tilde{q} = \partial + \tilde{g}$, $\tilde{q}^+ = \partial - \tilde{g}$. Then the parasuperhamiltonian and parasupergenerators are

$$H = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 - \mu \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ 0 & \tilde{q} & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & q^+ & 0 \\ 0 & 0 & \tilde{q}^+ \\ 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

Parasupergenerators are nilpotent of order three,

$$Q^3 = (Q^+)^3 = 0,$$

and they satisfy parasuperalgebra [6],

$$Q^+QQ^+ = Q^+H, \quad (Q^+)^2Q + Q(Q^+)^2 = Q^+H, \quad [H, Q] = [H, Q^+] = 0, \quad (5)$$

if $\mu = 0$.

Remark. The fact that $\mu = 0$ show that only special kind of DT - *binary DT* - is working to construct parasuperalgebra. In fact, $g = -\psi_x/\psi$, where $L_1\psi = 0$ and $\tilde{g} = -\tilde{\psi}_x/\tilde{\psi}$, where $L_2\tilde{\psi} = \mu\tilde{\psi}$. Therefore for the case $\mu = 0$ there are only two variants for the $\tilde{\psi}$. The first is $\tilde{\psi} = 1/\psi$. In this case $u_3 = L_3 - \partial^2 = u_1$ so $L_1 = L_3$. It is trivial and uninteresting case.

The second variant for the $\tilde{\psi}$ is

$$\tilde{\psi} = \frac{1}{\psi} \int dx \psi^2,$$

so

$$u_3 = u_1 + 2\partial^2 \log \int dx \psi^2. \quad (6)$$

(6) is so called binary DT (it is the DT to square). It is fundamental relationship in the positon theory [10]. In particular, one can show that one-positon (or one-negaton) solution of the KdV equation can be obtained via the formula (6) if $u_1 = 0$. Thus, positon potentials are connected with the PSUSY whereas solitons are the same for the SUSY.

3 PSUSY structure of integrable systems.

Andreev and Burova showed that connection between KdV and mKdV equations has SUSY structure [1]. To show this one need to construct supercharge which is nothing but square root from the SUSY Hamiltonian ¹:

$$\theta = \sqrt{H}.$$

θ is 2×2 matrix operator and *it is the L-operator of the mKdV hierarchy*. This is the crucial point of the work [1].

Now let consider Bq system

$$a_{1t} = (2b_1 - a_{1x})_x, \quad b_{1t} = \left(b_{1x} - \frac{2}{3}a_{1xx} - \frac{1}{3}a_1^2 \right)_x, \quad (7)$$

and a certain "modified" version of it given by ([2])

$$\begin{aligned} f_{1t} &= f_{1xx} - 2f_1f_{1x} - \frac{2}{3}(2f_1 + f_2)_{xx} - \frac{2}{3}\left(f_1f_2 - (f_1 + f_2)^2\right)_x, \\ f_{2t} &= f_{2xx} - 2f_2f_{2x} - \frac{2}{3}(f_2 - f_1)_{xx} - \frac{2}{3}\left(f_1f_2 - (f_1 + f_2)^2\right)_x. \end{aligned} \quad (8)$$

One of the goal of the work [2] was to relate solutions a_1, b_1 of (7) and f_1, f_2 of (8) by Miura-type transformation. Authors did it but what is the algebraic structure which allow one to obtain this Miura-type transformation? Our aim here is to show that it is possible because (7) and (8) are connected via PSUSY. To show this we are starting with Lax representation (1) for the (7) where

$$L_1 = \partial^3 + a_1\partial + b_1, \quad A_1 = \partial^2 + \frac{2}{3}a_1.$$

It is well known that ([2]),

$$L_1 = (\partial + f_3)(\partial + f_2)(\partial + f_1) = q_3q_2q_1,$$

where

$$\begin{aligned} f_1 + f_2 + f_3 &= 0, \\ a_1 &= (f_2 + 2f_1)_x - f_1^2 - f_2^2 - f_1f_2, \\ b_1 &= f_{1xx} + f_1(f_2 - f_1)_x - f_1f_2(f_1 + f_2). \end{aligned} \quad (9)$$

By the analogy with (2) we get two DT

$$L_1 \rightarrow L_2 \rightarrow L_3,$$

or

$$q_3q_2q_1 \rightarrow q_1q_3q_2 \rightarrow q_2q_1q_3,$$

where

$$L_2 = \partial^3 + a_2\partial + b_2, \quad L_3 = \partial^3 + a_3\partial + b_3,$$

with

$$a_2 = a_1 - 3f_{1x}, \quad a_3 = a_2 - 3f_{2x}. \quad (10)$$

We don't need b_2 and b_3 .

¹There are two such operators: θ and $\theta' = i\sigma_3\theta$, where σ_3 is Pauli matrix.

As for the usual SUSY one can construct the first parasuperhamiltonian,

$$H_I = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix}. \quad (11)$$

To contrast with (4), L_i ($i = 1, 2, 3$) are third-order linear differential operators.

To construct parasupercharge M we must calculate the cube root from the (11): $M = H^{1/3}$. It easy to verify that

$$M = \begin{pmatrix} 0 & 0 & q_3 \\ q_1 & 0 & 0 \\ 0 & q_2 & 0 \end{pmatrix}. \quad (12)$$

The rest roots can be obtained by the multiplication of M to the matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & (\lambda_1 \lambda_2)^{-1} \end{pmatrix},$$

where $\lambda_{1,2}$ are arbitrary (nonvanishing) complex numbers.

Operator (12) is contained in [2]. Namely, the Lax equation for the (8) is

$$\frac{dM}{dt} = [H_{II}, M],$$

thus parasupercharge M is the L -operator for the (8). Therefore it is clear why we can call (8) as modified Bq system. The A -operator H_{II} has the form

$$H_{II} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}. \quad (13)$$

where $A_i = \partial^2 + \frac{2}{3}a_i$ (see (10)). So (13) look as the parasuperhamiltonian (4) exactly. To show that (13)=(4) (with $\mu = 0$) we need to find two functions g and \tilde{g} such that

$$A_1 = (\partial - g)(\partial + g), \quad A_2 = (\partial + g)(\partial - g) = (\partial - \tilde{g})(\partial + \tilde{g}), \quad A_3 = (\partial + \tilde{g})(\partial - \tilde{g}).$$

Using (9) one get

$$g = f_1 + c_1, \quad \tilde{g} = f_2 + c_2,$$

with some constants c_1 and c_2 . In this case functions f_1 and f_2 are not arbitrary. After calculation we get the nonlinear equation for the f_1 ,

$$2(2c_2 - f_1)(f_{1x} + 2(c_2 - c_1)f_1)_x + f_{1x}^2 + \left((f_1 + 2c_1 - c_2)^2 - 3(c_1^2 + c_2^2) \right) (3(f_1 - c_2)^2 - (c_1 + c_2)^2 - 2c_1c_2) = 0, \quad (14)$$

and

$$f_2 = \frac{f_{1x} + f_1^2 - 2c_1f_1 - c_1^2 - 2c_2^2}{2(2c_2 - f_1)}, \quad f_3 = \frac{f_{1x} - f_1^2 + 2(2c_2 - c_1)f_1 - c_1^2 - 2c_2^2}{2(f_1 - 2c_2)}. \quad (15)$$

The equation (14) can be written in more compact form ,

$$2FF_{xx} - F_x^2 + 4\alpha FF_x - ((F - 3c_2 + 2\alpha)^2 - 3(\alpha^2 + 2c_2^2 - 2c_2\alpha)) \times (3(F - c_2)^2 - \alpha^2 - 6c_2^2 + 6\alpha c_2) = 0, \quad (16)$$

where $F = 2c_2 - f_1$, $\alpha = c_2 - c_1$.

Substituting (14-15) into the (8) one get

$$\begin{aligned} f_{1t} &= -2c_1 (f_1^2 - 2c_1 f_1 + 2f_1 f_2 - 4c_2 f_2 - c_1^2 - 2c_2^2) \\ f_{2t} &= 2c_2 (f_2^2 - 2c_2 f_2 + 2f_1 f_2 - 4c_1 f_1 - c_2^2 - 2c_1^2). \end{aligned}$$

Thus, if $c_1 = c_2 = 0$ then we get stationary solutions of the mBq equation.

The equations for the f_{1x} (f_{2x}) is compatible with the equation for the f_{1t} (f_{2t}) if $c_1 = c_2$ or if

$$F_t = 2c_1 F_x.$$

Therefore, if $c_1 \neq c_2$ then $F = F(\xi)$ with $\xi = x + 2c_1 t$ and $F(\xi)$ should be solution of the (16) with substitution $F_x \rightarrow F_\xi$.

Thus H_{II} (13) is parasuperhamiltonian if

$$\begin{aligned} \frac{2}{3}a_1 &= f_{1x} - (f_1 + c_1)^2, & \frac{2}{3}a_2 &= -f_{1x} - (f_1 + c_1)^2 \\ \frac{2}{3}a_3 &= 4c_1 f_1 - 2f_1 f_2 - 2f_2^2 + 2c_1^2, \end{aligned}$$

where f_1 and f_2 are defined by (14), (15).

4 Complete PSUSY algebra.

As we have seen, the usual PSUSY (5) is valid for the special kind of potentials only. On the other hand, the complete PSUSY algebra must be connected with parasuperhamiltonian H_I (11) rather than H_{II} (13). This because H_{II} is connected with the auxiliary dynamical problem whereas all information about mBq equation is contained in H_I . Using this operator one can obtain the complete PSUSY algebra. In contrast to SUSY algebra (3) the complete PSUSY algebra is defined by superhamiltonian H_I and three parasupergenerators,

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ q_1 & 0 & 0 \\ 0 & q_2 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & q_3 \\ 0 & 0 & 0 \\ 0 & q_2 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & q_3 \\ q_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whereas for the SUSY (3) it is enough to have superhamiltonian and two supergenerators Q and Q^+ .

There are useful relations,

$$\begin{aligned} M^3 &= H_I, & M^2 &= Q_1^2 + Q_2^2 + Q_3^2, & \{Q_i, Q_k\} &= M^2, & i &\neq k \\ Q_1 Q_2 Q_3 &= Q_2 Q_3 Q_1 = Q_3 Q_1 Q_2 = Q_1^2 = Q_2^2 = Q_3^2 = 0 \\ Q_1 Q_3 Q_2 + Q_2 Q_1 Q_3 + Q_3 Q_2 Q_1 &= 2H_I, & [Q_k, H_I] &= 0, \end{aligned} \tag{17}$$

with $i, k = 1, 2, 3$.

(17) is para-generalization of the (3). To proof three last equations one need to use the intertwining relations,

$$q_1 L_1 = L_2 q_1, \quad q_2 L_2 = L_3 q_2, \quad q_3 L_3 = L_1 q_3.$$

5 Conclusion.

Thus PSUSY algebra is underlie of algebraic structure of link between Bq and modified Bq equations. Now it is easy to find Miura transformation using the method from the [1]. The results is well known (see [2]) so we omit them here.

We conclude that parasupersymmetry is useful not only in the quantum theory but in the theory of integrable systems.

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