Exact soliton solutions of the one-dimensional complex Swift-Hohenberg equation

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Abstract

Using Painlevé analysis, the Hirota multi-linear method and a direct ansatz technique, we study analytic solutions of the $(1+1)$ -dimensional complex cubic and quintic Swift-Hohenberg equations. We consider both standard and generalized versions of these equations. We have found that a number of exact solutions exist to each of these equations, provided that the coefficients are constrained by certain relations. The set of solutions include particular types of solitary wave solutions, hole (dark soliton) solutions and periodic solutions in terms of elliptic Jacobi functions and the Weierstrass \wp function. Although these solutions represent only a small subset of the large variety of possible solutions admitted by the complex cubic and quintic Swift-Hohenberg equations, those presented here are the first examples of exact analytic solutions found thus far.

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1 Introduction

Complicated pattern-forming dissipative systems can be described by the Swift-Hohenberg (S-H) equation [1,2]. Classic examples are the Rayleigh-Bénard problem of convection in a horizontal fluid layer in the gravitational

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field [3,4], Taylor-Couette flow [5], some chemical reactions [6] and large-scale flows and spiral core instabilities [7]. Examples in optics include synchronouslypumped optical parametric oscillators [8], three-level broad-area cascade lasers [9] and large aspect-ratio lasers [10,11,12,13]. In the context of large-aperture lasers with small detuning between the atomic and cavity frequencies, the complex cubic Swift-Hohenberg (CCSH) equation was derived asymptotically [14,15,16]. This equation is also believed to be relevant for oscillatory convection in binary fluids. The appearance of spatial patterns is the most remarkable feature of the solutions.

Despite their visual complexity, spatial patterns are actually formed from a certain number of ordered or chaotic combinations in space of some simpler localized structures. It is known that the S-H equation admits the existence of transverse localized structures and phase domains [14]. These coherent structures can be considered as bright or dark solitons. Consequently, these simple localized structures and their stability are of great interest in the study of any pattern-forming system. If we keep these visual properties of patterns in mind, we can split the problem into several steps. Firstly, we should study the simplest stationary objects (localized structures). Secondly, we have to investigate their stability. As a last step, we should study their interaction and the combination structures. The distinction between stationary and moving patterns is defined by the nature of the interaction forces between the localized structures. The above program cannot be carried out in one move. Moreover, in studying stationary localized structures, we have to start with the simplest cases, *viz.* $(1+1)$ -dimensional structures. This sequential step-bystep approach allows us to avoid any possible confusion inherent in trying to explain a complicated structure in one step.

Whichever part of the above program we consider, it is clear that it can mainly be done only using computer simulations. This has been done in the majority of papers published so far. We will analyse only $(1+1)$ -dimensional bright and dark solitons, and leave aside, for the moment, the question of their stability and the interaction between them. Moreover, our task here is to obtain some analytic results. The latter point will further restrict the class of solutions which are of interest in our investigation. Nonetheless, this is a step of paramount importance because, until now, no analytic solutions have been known.

In many respects, localized structures of the S-H equation are similar to those observed in systems described by the previously-studied complex Ginzburg-Landau equation(CGLE). We recall that some analytic solutions for the cubic and quintic CGLE are known [17,18,19,20]. For the cubic $(1+1)$ -D equation, analytic solutions describe all possible bright and dark soliton solutions. In contrast, in the case of the quintic equation, the analytic solutions of the CGLE represent only a small subclass of its soliton solutions. Moreover, the stable soliton solutions are located outside of this subclass [20]. Therefore, practically useful results can only be obtained numerically.

We note that, apart from some exceptions, the CGLE generally has only isolated solutions [20,21] i.e. they are fixed for any particular set of equation parameters. This property is fundamental for the whole set of localized solutions of the CGLE. The existence of isolated solutions is one of the basic features of dissipative systems in general. The qualitative physical foundations of this property are given in [22]. This property is one of the reasons why the above program is possible at all. Like the CGLE, the S-H equation models dissipative systems, and we expect that it will have this property. Indeed, preliminary numerical simulations support this conjecture.

The main difference between the S-H equation [23] and the CGLE lies in its more involved diffraction term. The latter is important in describing more detailed features of an actual physical problem. However, these complications prevent us from easily analyzing the solutions. In fact, it was not clear that such solutions could exist at all [8,11]. In this work, we study the quintic complex S-H equation in 1D and report various new exact solutions.

Before going into further details, we should distinguish between the real Swift-Hohenberg(RSH) equation and the CCSH equation. The former can be considered as a particular case of the latter. Moreover, the RSH equation is a phenomenological model which cannot be rigorously derived from the original equations [3]. In contrast, the CCSH equation is derived asymptotically and is rigorous in the long-wavelength limit. In standard notations, the CCSH equation can be written as

$$
\psi_t = r\psi - (1 + ic)|\psi|^2 \psi + ia\Delta\psi - d(\Lambda + \Delta)^2 \psi , \qquad (1)
$$

where r is the control parameter and α characterizes the diffraction properties of the active medium. In the limit of $\Lambda \to 0$, i.e., in the long-wavelength limit, the differential nonlinearities formally have higher orders and therefore can be dropped. In this case the wave-number-selecting term, $(\Lambda + \Delta)^2 \psi$, is just a small correction to the diffraction term $ia\Delta\psi$. Thus the CCSH equation can be treated as a perturbed cubic CGLE. For us the main interest here is $(1+1)$ dimensional case when Δ has only x-derivatives.

In physical problems, the quintic nonlinearity can be of equal or even higher importance to the cubic one [24] as it is responsible for stability of localized solutions. Sakaguchi and Brand made a numerical investigation of the complex quintic Swift-Hohenberg (CQSH) equation

$$
\psi_t = a\psi + b|\psi|^2\psi - c|\psi|^4\psi - d(1+\partial_{xx})^2\psi + if\partial_{xx}\psi\,,\tag{2}
$$

where all coefficients $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, and the order parameter ψ are complex, but d and f are real [23]. For a, b and d real and $c \equiv 0$ and $f \equiv 0$, the original Swift-Hohenberg equation, which was derived as an order parameter equation for the onset of Rayleigh-Bénard convection in a simple fluid, is recovered. Then eq.(2) can be rewritten as

$$
i\psi_t + (f + 2id)\psi_{xx} + id\psi_{xxxx} + (b_2 - ib_1)|\psi|^2 \psi + (-c_2 + ic_1)|\psi|^4 \psi
$$

= $(-a_2 + i(a_1 - d))\psi$. (3)

This equation can be generalized to

$$
i\psi_t + \beta \psi_{xx} + \gamma \psi_{xxxx} + \mu |\psi|^2 \psi + \nu |\psi|^4 \psi = i\delta \psi , \qquad (4)
$$

where all coefficients $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\mu = \mu_1 + i\mu_2$ and $\nu = \nu_1 + i\nu_2$ are complex and δ is real. Eq.(4) is written in the form which we will refer to as the generalized CQSH (or GCQSH) equation in the rest of this study. Then the CCHS, real quintic Swift-Hohenberg (RQSH) and real cubic Swift-Hohenberg (RCSH) equations are particular cases of eq.(4) with some of the coefficients being equal to zero.

The interpretation of the variables depends on the particular problem. In optics, t is the propagation distance or the cavity round-trip number(treated as a continuous variable), x is the transverse variable, β_1 is the 2nd order diffraction, γ_1 is the 4th order diffraction, μ_2 is a nonlinear gain (or 2-photon absorption if negative) and δ represents a difference between linear gain and loss. The angular spectral gain is represented by the coefficients β_2 and γ_2 .

Sakaguchi and Brand observed some soliton-like structures in their numerical work [23]. They also showed that the CQSH equation admits a stable hole solution. Their results indicate that exact solutions may exist, but till now none have been found. In this paper, we study the CQSH and CCSH equations by using Painlevé analysis and the Hirota *multi-linear* method. Our method is an extension of the modified Hirota method used earlier by Nozaki and Bekki [25] and the modification of the Berloff-Howard method [26]. We note that Nozaki and Bekki solved the CGLE by using the Hirota bilinear method, while Berloff and Howard solved some real-coefficient non-integrable equations using the Weiss-Tabor-Carnevale (WTC) method and the Hirota multi-linear method. In addition, we confirm that our solutions are valid by using a direct ansatz approach. The solutions we obtain here can be considered as a basis for further development.

The rest of the paper is organized as follows. The methodology and analytical procedures are described in Sec. 2. Painlev´e analysis of the CQSH and CCSH equations is performed in Sec. 3. Exact solutions of the generalized CQSH and CCSH equations and particular cases of them are obtained in Sec. 4. Finally, we summarize with our conclusions in Sec. 5.

2 Methodology

2.1 Painlevé analysis and Hirota multi-linear method

Any solution of an equation must be in accord with the singularity structure of that equation. Painlevé analysis $[27]$ is a tool for investigating that structure. This analysis can be applied both for ordinary differential equations(ODEs) and partial differential equations (PDEs). The power of the Painlevé test lies in its easy algorithmic implementability. The main requirement is the representation of any possible solution in the form of a Laurent expansion in the neighborhood of a movable singularity:

$$
u = F^{-r} \sum_{j=0}^{\infty} u_j F^j , \qquad (5)
$$

where r is the leading-order exponent, $F(= 0)$ is a singularity manifold given by $F(z)$, and u_j is a set of analytic functions of z.

There are two necessary conditions for an ODE to pass the Painlevé test:

- the leading-order r must be an integer, and
- it must be possible to solve the recursion relation for the coefficients $u_i(z)$ consistently to any order.

The general expansion of a non-integrable equation will fail the Painlevé test at one of these two steps. Leading-order analysis can be done by balancing the highest-order derivative in x with the strongest nonlinearity.

Weiss *et al.* developed the singular manifold method to introduce the Painlevé property into the theory of PDEs [28]. A PDE is said to possess the Painlevé property if its solutions are single-valued about the movable singularity manifold. To be more specific, if the singularity manifold is given by $F(z_1, z_2, \dots, z_n) =$ 0, then a solution of the PDE must have an expansion of the form of eqn.(5). Substitution of this expansion into the PDE determines the positive value of r (from leading order analysis) and defines the recursion relation for the u_j .

Weiss *et al.* truncated the expansion at the "constant term " level [28], i.e.,

$$
u = u_0 F^{-r} + u_1 F^{-r+1} + \dots + u_{m-1} F^{-1} + u_m.
$$
 (6)

Substituting back into the PDE, one obtains an over-determined system of equations for F and u_j . The benefit of the singular manifold method is that this expansion for a nonlinear PDE contains a lot of information about the PDE.

Most nonlinear non-integrable equations do not possess the Painlevé property, i.e., they are not free of movable critical singularities [29,30,31,32]. For some equations it is still possible to obtain single-valued expansions by putting a constraint on the arbitrary function in the Painlevé expansion. Such equations are said to be 'partially integrable' systems, as presented by Hietarinta as a generalization of the Hirota bilinear formalism for non-integrable systems [33]. He conjectured that all completely integrable PDEs can be put into a bilinear form. There are also non-integrable equations that can be put into the bilinear form and then the partial integrability is associated with the levels of integrability defined by the number of solitons that can be combined in an N-soliton solution. Partial integrability means that the equation allows a restricted number of multisoliton solutions. Berloff and Howard suggested combining this treatment of the partial integrability with the use of the Painlevé expansion, truncated before the constant term level, as a transform to reduce a non-integrable PDE to a multi-linear equation [26]. The Berloff-Howard method is a powerful tool for solving non-integrable equations. In this section, we give an example to show how to obtain solutions using this method.

Firstly, we consider the generalized RCSH equation

$$
u_t + \alpha u_{xx} + \beta u_{xxxx} + \gamma u - \delta u^3 = 0, \qquad (7)
$$

and show how to find exact solutions of non-integrable dissipative partial differential equations by using the Painlevé test and the Hirota multi-linear method.

We take the transform truncated at the term before the constant term:

$$
u = u_0 F^{-r} + u_1 F^{-r+1} + \dots + u_{m-1} F^{-1}.
$$
\n(8)

Analysis of the leading order terms gives $r = 2$. By substituting this expansion (8) into the RCSH equation (7) and equating the coefficients of the highest powers of F to zero, we obtain expressions for u_0, u_1 in terms of F, and these lead to the transform

$$
u = \frac{\sqrt{\frac{120\beta}{\delta}}F_x^2}{F^2} - \frac{\sqrt{\frac{120\beta}{\delta}}F_{xx}}{F} = -\sqrt{\frac{120\beta}{\delta}}\frac{d^2}{dx^2}\log F. \tag{9}
$$

This transform leads to an equation which is quadrilinear in F , meaning that each term contains a product of four functions involving F and its derivatives.

$$
\gamma F^{2}F_{x}^{2} - 2F F_{t}F_{x}^{2} + 6\alpha F_{x}^{4} + 2F^{2}F_{t}F_{xt} - \gamma F^{3}F_{xx} + F^{2}F_{t}F_{xx}
$$

\n
$$
-12\alpha F F_{x}^{2}F_{xx} + 3\alpha F^{2}F_{xx}^{2} - 90\beta F_{x}^{2}F_{xx}^{2} + 90\beta F F_{xx}^{3} - F^{3}F_{xxt}
$$

\n
$$
+4\alpha F^{2}F_{x}F_{xxx} + 120\beta F_{x}^{3}F_{xxx} - 120\beta F F_{x}F_{xx}F_{xxx} + 10\beta F^{2}F_{xxx}^{2}
$$

\n
$$
-\alpha F^{3}F_{xxxx} - 30\beta F F_{x}^{2}F_{xxxx} + 15\beta F^{2}F_{xx}F_{xxxx} + 6\beta F^{2}F_{x}F_{xxxxx}
$$

\n
$$
-\beta F^{3}F_{xxxxxx} = 0.
$$

\n(10)

Substituting $F = 1 + e^{2kx + 2\omega t}$ into eq.(10) and equating the coefficients of different powers of e to zero, we arrive at the one-soliton solution:

$$
u(x) = \frac{\pm \sqrt{\frac{30\gamma}{\delta}}e^{2kx}}{(1 + e^{2kx})^2} = \pm \frac{1}{4} \sqrt{\frac{30\gamma}{\delta}} \text{sech}^2(kx) , \qquad (11)
$$

where

$$
k = \pm \frac{1}{2\sqrt{2}} \left(\frac{\gamma}{\beta}\right)^{\frac{1}{4}},\tag{12}
$$

and we require $\alpha = -\frac{5\sqrt{\beta\gamma}}{2}$ $\frac{2p}{2}$ as a constraint on the equation parameters. This solution is reminiscent of the sech²-solution, representing an ultrashort soliton at the minimum-dispersion wavelength, taking into account the effects of fourth order dispersion [20,34,35] of the equation

$$
i\frac{\partial\psi}{\partial t} + \alpha \frac{\partial^2\psi}{\partial x^2} + \beta \frac{\partial^4\psi}{\partial x^4} - \delta|\psi|^2 \psi = 0.
$$
\n(13)

Actually, the simple ansatz $\psi = u(x) \exp(-i\gamma t)$ reduces this equation to the stationary real cubic Swift-Hohenberg equation (7).

Now we consider the generalized RQSH equation in the form [36]

$$
u_t + \alpha u_{xx} + \beta u_{xxxx} + \gamma u - \delta u^3 - \eta u^5 = 0.
$$
\n⁽¹⁴⁾

Analysis of the leading-order terms gives us $r = 1$. By substituting eq.(8) into the RQSH equation (14) and equating the coefficients of the highest powers of F to zero, one obtains expressions for u_0 in terms of F that lead to the transform

$$
u = \left(\frac{24\beta}{\eta}\right)^{\frac{1}{4}} \frac{F_x}{F} \,. \tag{15}
$$

This transform again leads to a quadrilinear equation in F :

$$
F_{tx}F^3 - F_t F_x F^2 + \beta F_{xxxxx} F^3 - 5\beta F_{xxxx} F_x F^2 - 10\beta F_{xxx} F_{xx} F^2 + 20\beta F_{xxx} F_x^2 F + \alpha F_{xxx} F^3 + 30\beta F_{xx}^2 F_x F - 60\beta F_{xx} F_x^3 - 3\alpha F_{xx} F_x F^2 - 2\sqrt{\frac{6\beta}{\eta}} \delta F_x^3 F + 2\alpha F_x^3 F + \gamma F_x F^3 = 0.
$$
 (16)

On substituting $F = 1 + e^{2kx + 2\omega t}$ into the quadrilinear equation (16) and equating the coefficients of different powers of e to zero, we obtain four algebraic equations, but these algebraic equations don't have solutions.

We suppose a different transform:

$$
u = \frac{G}{F} \,. \tag{17}
$$

This leads to an equation which is pentalinear in F and G :

$$
F^{4}G - F_{x}F^{3}G - \beta F_{xxxx}F^{3}G + 8\beta F_{xxx}F_{x}F^{2}G - 4\beta F_{xxx}F^{3}G_{x}
$$

+6\beta F_{xx}^{2}F^{2}G - 36\beta F_{xx}F_{x}^{2}FG + 24\beta F_{xx}F_{x}F^{2}G_{x} - 6\beta F_{xx}F^{3}G_{xx}
-\alpha F_{xx}F^{3}G + 24\beta F_{x}^{4}G - 24\beta F_{x}^{3}FG_{x} + 12\beta F_{x}^{2}F^{2}G_{xx}
+2\alpha F_{x}^{2}F^{2}G - 4\beta F_{x}F^{3}G_{xxx} - 2\alpha F_{x}F^{3}G_{x} + F^{4}G_{t} + \beta F^{4}G_{xxxx}
+\alpha F^{4}G_{xx} - \delta F^{2}G^{3} - \gamma\eta G^{5} = 0. (18)

Substituting $F = 1 + e^{2kx + 2\omega t}$ and $G = 2u_0 e^{kx + \omega t}$ into equation (18) and equating the coefficients of different powers of e to zero, we get the one-soliton solution:

$$
u = u_0 \operatorname{sech}(kx),\tag{19}
$$

where

$$
k = \pm u_0 \left(\frac{\eta}{24\beta}\right)^{\frac{1}{4}}, u_0^2 = \frac{2}{3} \left(-\frac{\delta}{\eta} \pm \frac{\sqrt{\delta^2 + 6\gamma\eta}}{\eta}\right),
$$

$$
-24\sqrt{6\beta}\gamma - \sqrt{6\beta}nu^4
$$
 (20)

$$
\alpha = \frac{-24\sqrt{6\beta\gamma} - \sqrt{6\beta\eta u_0^4}}{12\sqrt{\eta u_0^2}}.\tag{21}
$$

Substituting $F = 1 + e^{2kx + 2\omega t}$ and $G = u_0(-1 + e^{2kx + 2\omega t})$ into equation (18) and equating the coefficients of different powers of e to zero, we obtain the kink solution:

$$
u = u_0 \tanh(kx), \tag{22}
$$

where

$$
k = \pm u_0 \left(\frac{\eta}{24\beta}\right)^{\frac{1}{4}}, u_0^2 = -\frac{\delta}{2\eta} \pm \frac{\delta^2 + 4\gamma\eta}{2\eta}, \alpha = \frac{3\sqrt{6\beta\gamma} + 2\sqrt{6\beta\eta}u_0^4}{3\sqrt{\eta}u_0^2}.
$$
 (23)

Generally speaking, the application of this technique to complex variable equations is difficult because of the complex leading order. However, we will show that this method works in some cases such as the complex Swift-Hohenberg equation.

2.2 Direct ansatz method

It is well-known that solutions of many nonlinear differential equations can be expressed in terms of hyperbolic functions like tanh or sech. This fact motivates the direct method of solution in which the starting point is a suitable ansatz so that the p.d.e. is expressible as a polynomial in terms of tanh or sech functions. Clearly this method is not as general as the Hirota method, since it will not work if there is no solution of the assumed form. However, in principle, the method is more straightforward than the Hirota method, and it is more useful in obtaining periodic solutions (in terms of elliptic functions).

In this section, we give an outline of the direct ansatz method and illustrate facets of its application by considering some straightforward examples.

First, we consider the RCSH equation (7) and show how to get exact solutions of non-integrable dissipative partial differential equations by using the direct ansatz method. The analysis of the leading-order terms gives us $r = 2$, and so we suppose the solution $u = u_0 \text{sech}^2(kx)$. We obtain the soliton solution by substituting this ansatz into the RCSH equation. This yields an algebraic equation in terms of tanh-functions by using formulae for hyperbolic functions (see Appendix), and we then equate the coefficients of different powers of the tanh function to zero. The solution is identical to that found by using the Hirota method. We find that the constraint needed on the equation parameters is $\gamma = \frac{4\alpha^2}{25.6}$ $rac{4\alpha^2}{25\beta}$. Assuming this is satisfied, the solution is:

$$
u(x) = \pm \alpha \sqrt{\frac{3}{10\beta\delta}} \operatorname{sech}^2(kx), \qquad (24)
$$

where

$$
k^2 = -\frac{\alpha}{20\beta} \,. \tag{25}
$$

Now we consider the RQSH equation (14). Analysis of the leading-order terms gives us $r = 1$, and so we suppose the solution $u = u_0 \operatorname{sech}(kx)$. Substitution of this ansatz into the RQSH equation yields an algebraic equation in terms of the tanh function. Again, by equating the coefficients of different powers of the tanh function to zero, we obtain the soliton solution, and note that it is identical with the Hirota method solution.

We find that the constraint needed on the equation parameters is now

$$
-\frac{9\alpha^2}{100\beta} + \gamma + \frac{3\delta^2}{50\eta} \pm \frac{2\sqrt{6}\alpha\delta}{25\sqrt{\beta\eta}} = 0,
$$
\n(26)

With this satisfied, we have:

$$
u = u_0 \operatorname{sech}(kx),\tag{27}
$$

where

$$
k = \pm \frac{u_0 \sqrt[4]{\eta}}{\sqrt[4]{24\beta}}, \ u_0^2 = -\frac{6\sqrt{\beta}\delta \pm \sqrt{6\eta}\alpha}{5\sqrt{\beta}\eta}, \tag{28}
$$

In the same way, we obtain a kink solution by the ansatz $u_0 \tanh(kx)$

$$
u = u_0 \tanh(kx), \tag{29}
$$

where

$$
k = \pm \frac{u_0 \sqrt[4]{\eta}}{\sqrt[4]{24\beta}}, \ u_0^2 = -\frac{\delta}{2\eta} \pm \frac{\delta^2 + 4\gamma\eta}{2\eta}, \tag{30}
$$

and the constraint needed on the equation parameters is

$$
\alpha = \frac{3\sqrt{6\beta}\gamma + 2\sqrt{6\beta}\eta u_0^4}{3\sqrt{\eta}u_0^2} \,. \tag{31}
$$

It is easy to develop this method to get solutions in terms of elliptic functions. If we substitute a Jacobi elliptic function, as an ansatz, into the RQSH equation (14), we obtain an algebraic equation in terms of some Jacobi elliptic function. By using the formulae in the Appendix, and equating the coefficients of different powers of the elliptic function to zero, we then get the elliptic solutions.

The elliptic function solutions of the RQSH equation (14) are as follows:

Jacobi cn function

$$
u = u_0 \operatorname{cn}(kx, q), \tag{32}
$$

where

$$
k = \pm \sqrt{\frac{\alpha - 2\alpha q \pm \sqrt{(2q - 1)^2 \alpha^2 - 4\beta \gamma A}}{2\beta A}},
$$
\n(33)

$$
A = 16q^2 - 16q + 1,
$$
\n(34)

$$
u_0^2 = -\frac{2k^2q(\alpha + 10\beta(2q - 1)k^2)}{\delta},\tag{35}
$$

and the constraint needed on the equation parameters is

$$
\eta = \frac{24\beta q^2 k^4}{u_0^4} \,. \tag{36}
$$

Jacobi sn function

$$
u = u_0 \operatorname{sn}(kx, q), \tag{37}
$$

where

$$
k = \pm \sqrt{\frac{\alpha + \alpha q \pm \sqrt{(q+1)^2 \alpha^2 - 4\beta \gamma A}}{2\beta A}},
$$
\n(38)

$$
A = q2 + 14q + 1,
$$
\n
$$
u_02 = \frac{2k^2q(\alpha - 10\beta(q+1)k^2)}{\delta},
$$
\n(39)

and the constraint needed on the equation parameters is

$$
\eta = \frac{24\beta q^2 k^4}{u_0^4} \,. \tag{41}
$$

Jacobi dn function

$$
u = u_0 \operatorname{dn}(kx, q) \,, \tag{42}
$$

where

$$
k = \pm \sqrt{\frac{(q-2)\alpha \pm \sqrt{(q-2)^2 \alpha^2 - 4\beta \gamma A}}{2\beta A}},
$$
\n(43)

$$
A = q^2 - 16q + 16,
$$
\n(44)

$$
u_0^2 = \frac{2k^2(-\alpha + 10(q - 2)\beta k^2)}{\delta},\tag{45}
$$

and the constraint needed on the equation parameters is

$$
\eta = \frac{24\beta k^4}{u_0^4} \,. \tag{46}
$$

In principle, these Jacobi elliptic function solutions can be also obtained from multi-linear forms, because each Jacobi elliptic function can be expressed by theta functions which are Hirota τ -functions in multi-linear form [37,38]. However, calculation by this method is tedious, so we do not provide details of it here.

Next we consider an elliptic function solution of the RCSH equation. We know from Painlevé analysis that the RCSH equation has a double pole. Thus we look for elliptic functions having double poles. We suppose

$$
u = u_0 + \wp(kx) \tag{47}
$$

Substituting this ansatz into the RCSH equation (7) yields an algebraic equation in terms of a Weierstrass \wp function. By using formulae from the Appendix, and equating the coefficients of different powers of the Weierstrass \wp function to zero, we obtain the following elliptic function solution:

$$
u = u_0 + \wp(kx),\tag{48}
$$

where

$$
k = \pm \left(\frac{\delta}{120\beta}\right)^{\frac{1}{4}}, u_0 = \pm \frac{\alpha}{\sqrt{30\beta\delta}},
$$
\n(49)

$$
g_2 = \frac{2(10\beta\gamma - \alpha^2)}{3\beta\delta}, \ g_3 = \pm \frac{2\sqrt{2}\alpha(\alpha^2 - 5\beta\gamma)}{3\sqrt{15\beta^3\delta^3}}.
$$
\n
$$
(50)
$$

From its relation to the Weierstrass σ function (see Appendix), we know that the Weierstrass σ function is a τ -function of the Hirota multi-linear form. Thus, we can construct this solution by the Hirota multi-linear method. However, we do not give the detail of this approach here.

We can find another elliptic function solution of the RCSH equation,

$$
u = u_0 + \text{cn}^2(kx, q), \tag{51}
$$

where

$$
k = \pm \left(\frac{\delta}{120\beta q^2}\right)^{\frac{1}{4}},\tag{52}
$$

$$
q = \frac{6 + 19u_0 + 15u_0^2 \pm \sqrt{A}}{8 + 38u_0 + 60u_0^2 + 30u_0^3},
$$
\n(53)

$$
A = -15u_0^4 - 30u_0^3 - 3u_0^2 + 12u_0 + 4,
$$
\n(54)

and the constraints needed on the equation parameters are

$$
\alpha = 20\beta (1 - (2 + 3u_0)q), \tag{55}
$$

$$
\gamma = 8\beta k^4 (8 - (23 + 30u_0)q + (23 + 60u_0 + 45u_0^2)q^2).
$$
 (56)

We note that all above hyperbolic-function solutions can be derived in the limit of $q \rightarrow 1$ of above elliptic function solutions.

3 Painlevé analysis of the complex Swift-Hohenberg equation

It is difficult to use a full expansion because the CQSH equation possesses both a complex leading order and non-integer resonances, and it generally has no consistency conditions. To see this, the CQSH equation is first rewritten as the following system of equations:

$$
i\psi_t + (\beta_1 + i\beta_2)\psi_{xx} + (\gamma_1 + i\gamma_2)\psi_{xxxx} + (\mu_1 + i\mu_2)|\psi|^2\psi + (\nu_1 + i\nu_2)|\psi|^4\psi = i\delta\psi,
$$
 (57)

$$
-i\psi_t^* + (\beta_1 - i\beta_2)\psi_{xx}^* + (\gamma_1 - i\gamma_2)\psi_{xxxx}^* + (\mu_1 - i\mu_2)|\psi|^2\psi^*
$$

$$
+(\nu_1 - i\nu_2)|\psi|^4\psi^* = -i\delta\psi^*.
$$
 (58)

To leading order, we set ψ and ψ^* as

$$
\psi \sim \psi_0 F^{-\xi_1}, \psi^* \sim \psi_0^* F^{-\xi_2}, \qquad (59)
$$

where ξ_1 and ξ_2 are the leading-order exponents. Upon equating the exponents of the dominant balance terms in CQSH

$$
(\gamma_1 + i\gamma_2)\psi_{xxxx} + (\nu_1 + i\nu_2)|\psi|^4\psi \sim 0, \qquad (60)
$$

we find

$$
\xi_1 + \xi_2 = 2. \tag{61}
$$

To find ξ_1 and ξ_2 we must also equate the coefficients in front of these terms. From eq.(57) we get

$$
|\psi_0|^4 = -\frac{\gamma_1 + i\gamma_2}{\nu_1 + i\nu_2} \xi_1(\xi_1 + 1)(\xi_1 + 2)(\xi_1 + 3) \left(\frac{\partial F}{\partial x}\right)^4,\tag{62}
$$

and from eq.(58)

$$
|\psi_0|^4 = -\frac{\gamma_1 - i\gamma_2}{\nu_1 - i\nu_2} \xi_2 (\xi_2 + 1)(\xi_2 + 2)(\xi_2 + 3) \left(\frac{\partial F}{\partial x}\right)^4.
$$
 (63)

Eqs.(62) and (63) are combined to give

$$
\frac{\gamma_1 + i\gamma_2}{\nu_1 + i\nu_2} \xi_1(\xi_1 + 1)(\xi_1 + 2)(\xi_1 + 3) = \frac{\gamma_1 - i\gamma_2}{\nu_1 - i\nu_2} \xi_2(\xi_2 + 1)(\xi_2 + 2)(\xi_2 + 3)
$$
 (64)

Eqs.(61) and (64) are now the two equations we need to solve for ξ_1 and ξ_2 . The result is

$$
\xi_1 = 1 + i\alpha \,, \, \xi_2 = 1 - i\alpha \,, \tag{65}
$$

where α satisfies

$$
(\alpha^4 - 35\alpha^2 + 24)(\gamma_1\nu_2 - \gamma_2\nu_1) + 10\alpha (5 - \alpha^2)(\gamma_1\nu_1 + \gamma_2\nu_2) = 0. \quad (66)
$$

(To have $\alpha = 0$, we need $\gamma_1 \nu_2 = \gamma_2 \nu_1$ as the condition for a chirp-less solution.) The leading-order exponents $\xi_1 = 1+i\alpha$ and $\xi_2 = 1-i\alpha$ are not integers unless $\alpha = 0$. This means that the CQSH equation already fails the Painlevé test at the first step if $\alpha \neq 0$.

When $\gamma_1 \nu_2 - \gamma_2 \nu_1 = 0$ we can have $\alpha = 0$ (or $\alpha = \sqrt{5}$). In fact this is the condition of the next section, giving the chirp-less solution. This plain solution is not possible for the quintic CGL equation.

The expansions for ψ and ψ^* therefore take the form

$$
\psi = (\psi_0 F^{-1-i\alpha} + \psi_1 F^{-i\alpha} + \cdots) \exp(iKx + i\Omega t), \qquad (67)
$$

$$
\psi^* = (\psi_0^* F^{-1+i\alpha} + \psi_1 F^{i\alpha} + \cdots) \exp(-iKx - i\Omega t). \tag{68}
$$

The resonances for the CQSH equation can be calculated from the recursion relations for the ψ_j 's and ψ_j^* j 's, but we do not include them here because they are not important for our purpose.

We can expect the following dependent variable transformation by the above analysis,

$$
\psi = \frac{G}{F^{1+i\alpha}} \exp(i\Omega t), \quad \psi^* = \frac{G^*}{F^{1-i\alpha}} \exp(-i\Omega t), \tag{69}
$$

(In general, the transform should be $\psi = \frac{G}{F^{1+i\alpha}} \exp(iKx + i\Omega t)$, however, in our case, this transformation also gives solutions in the next section.)

In the same way, we can obtain the dependent variable transformation of the CCSH equation,

$$
\psi = \frac{G^2}{F^{2+i\alpha}} \exp(i\Omega t), \quad \psi^* = \frac{G^{*2}}{F^{2-i\alpha}} \exp(-i\Omega t). \tag{70}
$$

In the next section, we show the existence of exact solutions by using these transformations.

4 Exact solutions of the complex Swift-Hohenberg equation

4.1 The complex quintic Swift-Hohenberg equation

We consider the CQSH equation in [23],

$$
i\psi_t + (f + 2id)\psi_{xx} + id\psi_{xxxx} + (b_2 - ib_1)|\psi|^2 \psi + (-c_2 + ic_1)|\psi|^4 \psi
$$

= $(-a_2 + i(a_1 - d))\psi$, (71)

(In the numerical work of Sakaguchi and Brand, a_2 was 0.)

Substituting the transformation (69) into the CQSH equation (71), we obtain a pentalinear equation

$$
(id - \Omega a_1 + a_2 - i)F^4 G + (b_2 - ib_1)F^2 G^2 G^* - (c_2 - ic_1)G^3 G^{*2}
$$

\n
$$
-iF^3 F_t G + iF^4 G_t + (2f + 4id)F^2 F_x^2 G + 24idF_x^4 G
$$

\n
$$
-(2f + 4id)F^3 F_x G_x - 24idF F_x^3 G_x - (2id + f)F^3 F_{xx} G
$$

\n
$$
-36idF F_x^2 F_{xx} G + 24idF^2 F_x F_{xx} G_x + 6idF^2 F_{xx}^2 G + (f + 2id)F^4 G_{xx}
$$

\n
$$
+12idF^2 F_x^2 G_{xx} - 6idF^3 F_{xx} G_{xx} + 8idF^2 F_x F_{xxx} G - 4idF^3 F_{xxx} G_x
$$

\n
$$
-4idF^3 F_x G_{xxx} - idF^3 F_{xxxx} G + idF^4 G_{xxxx} = 0.
$$
 (72)

Putting F, G and G^{*} as polynomials in terms of $\exp(kx+\omega t)$ and substituting these functions into eq.(72) and equating the coefficients of different powers of e to zero, we obtain the following exact solutions. We can also obtain same solutions by using direct ansatz method.

Bright Soliton

$$
\psi = g \operatorname{sech}(kx) e^{i\Omega t},\tag{73}
$$

where

$$
k = \pm \sqrt{\frac{\sqrt{a_1} - \sqrt{d}}{\sqrt{d}}}, \ \Omega = -f + \frac{f\sqrt{a_1}}{\sqrt{d}} + a_2, \ |g|^2 = \frac{2f(\sqrt{a_1} - \sqrt{d})}{\sqrt{d}b_2}.
$$
 (74)

The coefficients must satisfy the following relations:

$$
b_1 = b_2 \frac{2(4d - 5\sqrt{da_1})}{f}, \ c_1 = -\frac{6b_2^2 d}{f^2}, \ c_2 = 0 \,. \tag{75}
$$

Dark Soliton

$$
\psi = g \tanh(kx) e^{i\Omega t},\tag{76}
$$

where

$$
k = \pm \sqrt{\frac{fb_1 + 2db_2}{20db_2}}, \ \Omega = -2fk^2 + a_2 \,, \ |g|^2 = -\frac{2fk^2}{b_2} \,. \tag{77}
$$

The coefficients must satisfy the following relations:

$$
a_1 = d(16k^4 - 4k^2 + 1), c_1 = -\frac{6db_2^2}{f^2}, c_2 = 0.
$$
 (78)

Chirped Bright Soliton

$$
\psi = g \operatorname{sech}(kx) e^{-i\alpha \log(\operatorname{sech}(kx))} e^{i\Omega t},\tag{79}
$$

where

$$
k = \pm \sqrt{\frac{f\alpha + d(\alpha^2 - 1) \pm \sqrt{d(d - a_1)A + (f\alpha + d(\alpha^2 - 1))^2}}{dA}},
$$
 (80)

$$
A = \alpha^{4} - 6\alpha^{2} + 1,
$$

\n
$$
\Omega = k^{2}(f - f\alpha^{2} + 4d\alpha + 4d\alpha(1 - \alpha^{2})k^{2})) + a_{2},
$$
\n(81)

$$
|g|^2 = \frac{k^2(2d\alpha(23 - 7\alpha^2)k^2 + 6d\alpha - f(\alpha^2 - 2))}{b_2}.
$$
\n(83)

The coefficients must satisfy the following relations:

$$
b_1 = \frac{k^2(3f\alpha - 2d(2 - \alpha^2) - 2d(10 - 19\alpha^2 + \alpha^4)k^2)}{|g|^2},
$$
\n(84)

$$
c_1 = -b_2^2 \frac{d(\alpha^4 - 35\alpha^2 + 24)}{(2d\alpha(23 - 7\alpha^2)k^2 + 6d\alpha - f(\alpha^2 - 2))^2},
$$
\n(85)

$$
c_2 = -b_2^2 \frac{10d\alpha(\alpha^2 - 5)}{(2d\alpha(23 - 7\alpha^2)k^2 + 6d\alpha - f(\alpha^2 - 2))^2}.
$$
 (86)

Chirped Dark Soliton

$$
\psi = g \tanh(kx) e^{-i\alpha \log(\text{sech}(kx))} e^{i\Omega t},\tag{87}
$$

where

$$
k = \pm \sqrt{\frac{2}{A} - \frac{3f\alpha}{2dA} \pm \frac{\sqrt{(4d - 3f\alpha)^2 - 4d(d - a_1)A}}{2dA}},
$$
\n(88)

$$
A = 16 - 15\alpha^2,
$$
\n
$$
A = 16 - 15\alpha^2.
$$

$$
\Omega = k^2 ((-2f - 6d\alpha) + 30d\alpha k^2) + a_2,
$$

\n
$$
|g|^2 = -\frac{k^2 (2f + 6d\alpha - f\alpha^2 + 10d\alpha(\alpha^2 - 8)k^2)}{b_2}.
$$
\n(90)

$$
\left(\begin{array}{c} \cdot & \cdot \\ \cdot & \cdot \end{array} \right)
$$

$$
(92)
$$

The coefficients must satisfy the following relations:

$$
b_1 = -\frac{k^2(-4d + 3f\alpha + 2d\alpha^2 + 10d(4 - 5\alpha^2)k^2)}{|g|^2},
$$
\n(93)

$$
c_1 = -\frac{dk^4(\alpha^4 - 35\alpha^2 + 24)}{|g|^4}, \ c_2 = -\frac{10d\alpha k^4(\alpha^2 - 5)}{|g|^4}.
$$
\n(94)

Elliptic function solutions of the CQSH equation are the following:

Jacobi cn function solution

$$
\psi = g \operatorname{cn}(kx, q)e^{i\Omega t},\tag{95}
$$

where

$$
k = \pm \sqrt{\frac{d(1 - 2q) \pm \sqrt{d(-12dq(q - 1) + a_1A)}}{dA}},
$$
\n(96)

$$
A = 16q^{2} - 16q + 1,
$$
\n
$$
Q = \frac{f^{2}(2 - 1)}{1 + 1} + \frac{1}{2} + \frac{2fqk^{2}}{1 + 2fqk^{2}}
$$
\n(97)

$$
\Omega = f k^2 (2q - 1) + a_2 \,, \, |g|^2 = \frac{2f q k^2}{b_2} \,. \tag{98}
$$

The coefficients must satisfy the following relations:

$$
b_1 = -\frac{4dqk^2(1+5(2q-1)k^2)}{|g|^2}, \ c_1 = -\frac{24dq^2k^4}{|g|^4}, \ c_2 = 0. \tag{99}
$$

Jacobi sn function solution

$$
\psi = g \operatorname{sn}(kx, q)e^{i\Omega t},\tag{100}
$$

where

$$
k = \pm \sqrt{\frac{d(q+1) \pm \sqrt{d(-12dq + a_1A)}}{dA}},
$$
\n(101)

$$
A = q^2 + 14q + 1,\tag{102}
$$

$$
\Omega = -fk^{2}(q+1) + a_{2}, \, |g|^{2} = -\frac{2fqk^{2}}{b_{2}}.
$$
\n(103)

The coefficients must satisfy the following relations:

$$
b_1 = -\frac{4dqk^2(-1+5(q+1)k^2)}{|g|^2}, \ c_1 = -\frac{24dq^2k^4}{|g|^4}, \ c_2 = 0. \tag{104}
$$

Jacobi dn function solution

$$
\psi = g \operatorname{dn}(kx, q) e^{i\Omega t},\tag{105}
$$

where

$$
k = \pm \sqrt{\frac{d(q-2) \pm \sqrt{d(12d(q-1) + a_1 A)}}{dA}},
$$
\n(106)

$$
A = q^2 - 16q + 16 \,,\tag{107}
$$

$$
\Omega = -fk^2(q-2) + a_2, \, |g|^2 = \frac{2fk^2}{b_2} \,. \tag{108}
$$

The coefficients must satisfy the following relations:

$$
b_1 = \frac{4dk^2(-1+5(q-2)k^2)}{|g|^2}, \ c_1 = -\frac{24dk^4}{|g|^4}, \ c_2 = 0. \tag{109}
$$

4.2 the generalized complex quintic Swift-Hohenberg equation

Now we consider the generalized complex quintic Swift-Hohenberg(GCQSH) equation

$$
i\psi_t + \beta \psi_{xx} + \gamma \psi_{xxxx} + \mu |\psi|^2 \psi + \nu |\psi|^4 \psi = i\delta \psi , \qquad (110)
$$

where all coefficients $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\mu = \mu_1 + i\mu_2$ and $\nu = \nu_1 + i\nu_2$ are complex and δ is real.

This equation can be easily normalized by rescaling $t' = \mu_1 t$, $x' = \sqrt{\frac{mu_1}{\beta_1}} x$, so that β_1 and μ_1 can be 1 if β_1 and μ_1 are non zero.

Substituting the transformation (69) into the GCQSH equation (110), we obtain a pentalinear equation

$$
\nu G^{3} G^{*2} + (24 + 50i\alpha - 35\alpha^{2} - 10i\alpha^{3} + \alpha^{4})\gamma F_{x}^{4} G
$$

+2(-6 - 11i\alpha + 6\alpha^{2} + i\alpha^{3})\gamma(2F F_{x}^{3} G_{x} + 3F F_{x}^{2} F_{xx} G)
+ \mu F^{2} G^{2} G^{*} - 6(-2 - 3i\alpha + \alpha^{2})\gamma(2F^{2} F_{x} F_{xx} G_{x} + F^{2} F_{x}^{2} G_{xx})
-(-2 - 3i\alpha + \alpha^{2})(\beta F^{2} F_{x}^{2} G + 3\gamma F^{2} F_{xx}^{2} G + 4\gamma F^{2} F_{x} F_{xxx} G)
-i(-i + \alpha)(2\beta F^{3} F_{x} G_{x} + 6\gamma F^{3} F_{xx} G_{xx} + 4\gamma F^{3} F_{xxx} G_{x} +
4\gamma F^{3} F_{x} G_{xxx} + iF^{3} F_{t} G + \beta F^{3} F_{xx} G + \gamma F^{3} F_{xxxx} G)
+(-i\delta - \Omega) F^{4} G + iF^{4} G_{t} + \beta F^{4} G_{xx} + \gamma F^{4} G_{xxxx}. (111)

Putting F, G and G^{*} as polynomials in terms of $\exp(kx+\omega t)$ and substituting these functions into pentalinear equation (111) and equating the coefficients of different powers of e to zero, we get the following solutions. We can also obtain same solutions by using direct ansatz method.

First, we consider chirp-less($\alpha = 0$) solutions. From eq.(66) we need $\gamma_1 \nu_2 =$ $\gamma_2 \nu_1$.

Bright Soliton

$$
\psi = g \operatorname{sech}(kx) e^{i\Omega t},\tag{112}
$$

where

$$
k = \pm \sqrt{\frac{-\beta_2 \pm \sqrt{\beta_2^2 + 4\delta \gamma_2}}{2\gamma_2}},\tag{113}
$$

$$
\Omega = k^2 (1 + \gamma_1 k^2), \ |g|^2 = 2k^2 (1 + 10\gamma_1 k^2). \tag{114}
$$

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{2k^2(\beta_2 + 10\gamma_2 k^2)}{|g|^2}, \ \nu_1 = -\frac{24\gamma_1 k^4}{|g|^4}, \ \nu_2 = -\frac{24\gamma_2 k^4}{|g|^4}, \tag{115}
$$

and $\beta_1 = \mu_1 = 1$ (normalized coefficients).

Dark Soliton

$$
\psi = g \tanh(kx) e^{i\Omega t},\tag{116}
$$

where

$$
k = \pm \frac{1}{4} \sqrt{\frac{\beta_2 \pm \sqrt{\beta_2^2 + 16\delta \gamma_2}}{\gamma_2}},\tag{117}
$$

$$
\Omega = 2k^2(8\gamma_1 k^2 - 1), \ |g|^2 = 2k^2(20\gamma_1 k^2 - 1). \tag{118}
$$

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{2k^2(20\gamma_2 k^2 - \beta_2)}{|g|^2}, \ \nu_1 = -\frac{24\gamma_1 k^4}{|g|^4}, \ \nu_2 = -\frac{24\gamma_2 k^4}{|g|^4}, \tag{119}
$$

and $\beta_1 = \mu_1 = 1$ (normalized coefficients).

Chirped Bright Soliton

$$
\psi = g \operatorname{sech}(kx) e^{-i\alpha \log(\operatorname{sech}(kx))} e^{i\Omega t},\tag{120}
$$

where

$$
k = \pm \sqrt{\frac{2\alpha + (\alpha^2 - 1)\beta_2 \pm \sqrt{(2\alpha + (\alpha^2 - 1)\beta_2)^2 + 4\delta A}}{2A}},
$$
\n(121)

$$
A = 4\alpha(\alpha^2 - 1)\gamma_1 + (\alpha^4 - 6\alpha^2 + 1)\gamma_2, \n\Omega = k^2((1 - \alpha^2) + 2\alpha\beta_2)
$$
\n(122)

$$
+k^4((1-6\alpha^2+\alpha^4)\gamma_1+(4\alpha-4\alpha^3)\gamma_2),
$$
\n(123)

$$
|g|^{2} = k^{2}((2 - \alpha^{2}) + 3\alpha\beta_{2})
$$

+2k^{4}((10 - 19\alpha^{2} + \alpha^{4})\gamma_{1} + (23\alpha - 7\alpha^{3})\gamma_{2}). (124)

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{k^2((2-\alpha^2)\beta_2 - 3\alpha)}{|g|^2} + \frac{2k^4((10-19\alpha^2 + \alpha^4)\gamma_2 - (23\alpha + 7\alpha^3)\gamma_1)}{|g|^2},
$$
\n(125)

$$
|g|^{2},
$$

\n
$$
\nu_{1} = -\frac{k^{4}((24 - 35\alpha^{2} + \alpha^{4})\gamma_{1} - 10\alpha(\alpha^{2} - 5)\gamma_{2})}{|g|^{4}},
$$
\n(126)

$$
\nu_2 = -\frac{k^4((24 - 35\alpha^2 + \alpha^4)\gamma_2 + 10\alpha(\alpha^2 - 5)\gamma_1)}{|g|^4},\tag{127}
$$

and $\beta_1 = \mu_1 = 1$ (normalized coefficients).

Chirped Dark Soliton

$$
\psi = g \tanh(kx) e^{-i\alpha \log(\text{sech}(kx))} e^{i\Omega t}, \qquad (128)
$$

where

$$
k = \pm \sqrt{\frac{3\alpha - 2\beta_2 \pm \sqrt{(3\alpha - 2\beta_2)^2 - 4\delta A}}{2A}},
$$
\n(129)

$$
A = 30\alpha\gamma_1 + (15\alpha^2 - 16)\gamma_2, \tag{130}
$$

$$
\Omega = k^2(-2 - 3\alpha\beta_2 + (16\gamma_1 - 15\alpha^2\gamma_1 + 30\alpha\gamma_2)k^2),
$$
\n(131)
\n
$$
|g|^2
$$

$$
= k2((\alpha2 - 2) - 3\alpha\beta_2 + 10(4\gamma_1 - 5\alpha2\gamma_1 + 8\alpha\gamma_2 - \alpha3\gamma_2)k2). \qquad (132)
$$

The coefficients must satisfy the following relations:

 μ_2

$$
=\frac{k^2((\alpha^2-2)\beta_2+3\alpha+10(4\gamma_2-5\alpha^2\gamma_2-8\alpha\gamma_1+\alpha^3\gamma_1)k^2)}{|g|^2},\qquad(133)
$$

$$
\nu_1 = -\frac{k^4((24 - 35\alpha^2 + \alpha^4)\gamma_1 - 10\alpha(\alpha^2 - 5)\gamma_2)}{|g|^4},\tag{134}
$$

$$
\nu_2 = -\frac{k^4((24 - 35\alpha^2 + \alpha^4)\gamma_2 + 10\alpha(\alpha^2 - 5)\gamma_1)}{|g|^4},\tag{135}
$$

and $\beta_1=\mu_1=1$ (normalized coefficients).

Elliptic function solutions of the GCQSH equation are the following:

Jacobi cn function solution

$$
\psi = g \operatorname{cn}(kx, q)e^{i\Omega t},\tag{136}
$$

where

$$
k = \pm \sqrt{\frac{(1 - 2q)\beta_2 \pm \sqrt{(1 - 2q)^2 \beta_2^2 + 4\delta \gamma_2 A}}{2\gamma_2 A}},
$$
\n(137)

$$
\Omega = k^2((2q - 1) + \gamma_1 Ak^2),\tag{138}
$$

$$
A = 16q2 - 16q + 1,
$$
\n(139)

$$
|g|^2 = 2qk^2(1+10\gamma_1(2q-1)k^2). \tag{140}
$$

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{2qk^2(\beta_2 + 10\gamma_2(2q - 1)k^2)}{|g|^2},\tag{141}
$$

$$
\nu_1 = -\frac{24\gamma_1 q^2 k^4}{|g|^4}, \quad \nu_2 = -\frac{24\gamma_2 q^2 k^4}{|g|^4},\tag{142}
$$

and $\beta_1=\mu_1=1$ (normalized coefficients).

Jacobi sn function solution

$$
\psi = g \operatorname{sn}(kx, q)e^{i\Omega t},\tag{143}
$$

where

$$
k = \pm \sqrt{\frac{(1+q)\beta_2 \pm \sqrt{(q+1)^2 \beta_2^2 + 4\delta \gamma_2 A}}{2\gamma_2 A}},
$$
\n(144)

$$
\Omega = k^2(-(q+1) + \gamma_1 Ak^2),\tag{145}
$$

$$
A = q^2 + 14q + 1,\tag{146}
$$

$$
|g|^2 = 2qk^2(-1+10\gamma_1(q+1)k^2).
$$
 (147)

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{2qk^2(-\beta_2 + 10\gamma_2(q+1)k^2)}{|g|^2},\tag{148}
$$

$$
\nu_1 = -\frac{24\gamma_1 q^2 k^4}{|g|^4}, \quad \nu_2 = -\frac{24\gamma_2 q^2 k^4}{|g|^4}, \tag{149}
$$

and $\beta_1=\mu_1=1$ (normalized coefficients).

Jacobi dn function solution

$$
\psi = g \operatorname{dn}(kx, q)e^{i\Omega t},\tag{150}
$$

where

$$
k = \pm \sqrt{\frac{(q-2)\beta_2 \pm \sqrt{(q-2)^2 \beta_2^2 + 4\delta \gamma_2 A}}{2\gamma_2 A}},
$$
\n(151)

$$
\Omega = k^2(-(q-2) + \gamma_1 Ak^2),\tag{152}
$$

$$
A = q^2 - 16q + 16\,,\tag{153}
$$

$$
|g|^2 = 2k^2(1 - 10\gamma_1(q - 2)k^2). \tag{154}
$$

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{2k^2(\beta_2 - 10\gamma_2(q - 2)k^2)}{|g|^2},\tag{155}
$$

$$
\nu_1 = -\frac{24\gamma_1 k^4}{|g|^4}, \quad \nu_2 = -\frac{24\gamma_2 k^4}{|g|^4}, \tag{156}
$$

and $\beta_1 = \mu_1 = 1$ (normalized coefficients).

4.3 The complex cubic Swift-Hohenberg equation

Now we consider the CCSH equation

$$
\psi_t = (\mu + i\nu)\psi + if \psi_{xx} - d(\psi + 2\psi_{xx} + \psi_{xxxx})
$$

-($\delta + i\gamma$)|\psi|² ψ . (157)

Substituting the transformation (70) into the CCSH equation (157), we obtain a hexalinear equation

$$
(-i\gamma - \delta)G^{4}G^{*2} - d(120 + 154i\alpha - 71\alpha^{2} - 14i\alpha^{3} + \alpha^{4})F_{x}^{4}G^{2}
$$

+2d(24 + 26i\alpha - 9\alpha^{2} - i\alpha^{3})(4FF_{x}^{3}GG_{x} + 3FF_{x}^{2}F_{xx}G^{2})
+ (\alpha^{2} - 5i\alpha - 6)(12dF^{2}F_{x}^{2}G_{x}^{2} + 12d(2F^{2}F_{x}F_{xx}GG_{x} + F^{2}F_{x}^{2}GG_{xx})
+ (2dF^{2}F_{x}^{2}G^{2} - ifF^{2}F_{x}^{2}G^{2} + 3dF^{2}F_{xx}^{2}G^{2} + 4dF^{2}F_{x}F_{xxx}G^{2}))
+ (2 + i\alpha)(12dF^{3}F_{xx}G_{x}^{2} + 24dF^{3}F_{x}G_{xx}G_{xx} + 8dF^{3}F_{x}GG_{x}
-4ifF^{3}F_{x}GG_{x} + 12dF^{3}F_{xx}GG_{xx} + 8dF^{3}F_{xxx}GG_{x} + 8dF^{3}F_{x}GG_{xxx}
+ F^{3}F_{t}G^{2} + 2dF^{3}F_{xx}G^{2} - ifF^{3}F_{xx}G^{2} + dF^{3}F_{xxxx}G^{2}) + 2ifF^{4}G_{x}^{2}
-(d - \mu - i\nu + i\Omega)F^{4}G^{2} - 4dF^{4}G_{x}^{2} - 6dF^{4}G_{xx}^{2} - 8dF^{4}G_{x}G_{xxx}
- 2F^{4}GG_{t} - 4dF^{4}GG_{xx} + 2ifF^{4}GG_{xx} - 2dF^{4}GG_{xxxx}(158)

Putting F, G and G^* as polynomials in terms of $\exp(kx+\omega t)$ and substituting these functions into hexalinear equation (158) and equating the coefficients of different powers of e to zero, we get the following solutions. We can also obtain same solutions by using direct ansatz method.

Chirped Bright Soliton

$$
\psi = g \operatorname{sech}^2(kx) e^{-\alpha \log(\operatorname{sech}(kx))} e^{i\Omega t},\tag{159}
$$

$$
k = \pm \frac{1}{\sqrt{\alpha^2 - 10}}, \ |g|^2 = -(120 - 71\alpha^2 + \alpha^4)dk^4,
$$
\n(160)

$$
\mu = 4(9 + 8\alpha^2)dk^4, \ f = -12\alpha dk^2, \tag{161}
$$

$$
\Omega = \nu - \alpha dk^4 (96 - 12\alpha^2) , \gamma = \frac{(154\alpha - 14\alpha^3)dk^4}{|g|^2} . \tag{162}
$$

Now we look for solutions of elliptic function. We suppose

$$
\psi = (g + \wp(kx))e^{i\Omega t}.
$$
\n(163)

Substitution this ansatz into the CCSH equation (157) yields an algebraic equation in terms of a Weierstrass \wp function by using formulas in Appendix, and equating the coefficients of different powers of a Weierstrass \wp function to zero, we obtain the following elliptic function solution:

$$
\psi = (g + \wp(kx))e^{i\Omega t},\tag{164}
$$

where

$$
k = \pm \left(\frac{-\delta}{120d}\right)^{\frac{1}{4}}, \ g = \pm \sqrt{-\frac{2d}{15\delta}}, \ \Omega = \nu \,, \tag{165}
$$

$$
g_2 = \frac{4d(5\mu - 3d)}{3\delta}, \ g_3 = \pm \frac{4(d - 5\mu)}{3} \sqrt{\frac{-2d}{15\delta^3}}, \tag{166}
$$

$$
f = 0, \gamma = 0. \tag{167}
$$

4.4 The generalized complex cubic Swift-Hohenberg equation

We consider the generalized complex cubic Swift-Hohenberg(GCCSH) equation

$$
i\psi_t + \beta \psi_{xx} + \gamma \psi_{xxxx} + \mu |\psi|^2 \psi = \delta \psi , \qquad (168)
$$

where all coefficients $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\mu = \mu_1 + i\mu_2$ and $\delta = \delta_1 + i\delta_2$ are complex.

This equation can be easily normalized by rescaling $t' = \mu_1 t$, $x' = \sqrt{\frac{mu_1}{\beta_1}} x$, so that β_1 and μ_1 can be 1 if β_1 and μ_1 are non zero.

Substituting the transformation (70) into the GCCSH equation (168), we obtain a hexalinear equation

$$
\mu G^{*2} G^4 + (120 + 154i\alpha - 71\alpha^2 - 14i\alpha^3 + \alpha^4)\gamma F_x^4 G^2
$$

+2(-24 - 26i\alpha + 9\alpha^2 + i\alpha^3)\gamma(4FGF_x^2F_xG_x + 3FGF_x^2GF_{xx})
-(-6 - 5i\alpha + \alpha^2)(12\gamma F^2F_x^2G_x^2 + 24\gamma F^2F_xGG_xF_{xx}
+12\gamma F^2F_x^2GG_{xx} + \beta F^2F_x^2G^2 + 3\gamma F^2F_{xx}^2G^2 + 4\gamma F^2F_xF_{xxx}G^2)
-i(-2i + \alpha)F^3(12\gamma(G_x^2F_{xx} + 2F_xG_xG_{xx})
+4(\beta F_xGG_x + 3\gamma F_{xx}GG_{xx} + 2\gamma F_{xxx}GG_x + 2\gamma F_xGG_{xxx})
+iF_tG^2 + \beta F_{xx}G^2 + \gamma F_{xxxx}G^2) + (-i\delta - \Omega)F^4G^2
+2(\beta F^4G_x^2 + 3\gamma F^4G_{xx}^2 + 4\gamma F^4G_xG_{xxx})
+2(iF^4GG_t + \beta F^4GG_{xx} + \gamma F^4GG_{xxxx}). \qquad (169)

Putting F, G and G^* as polynomials in terms of $\exp(kx + \omega t)$ and substituting these functions into hexalinear equation (169) and equating the coefficients of different powers of e to zero, we get the following solutions. We can also obtain same solutions by using direct ansatz method.

Bright Soliton

$$
\psi = g \operatorname{sech}^2(kx) e^{i\Omega t},\tag{170}
$$

where

$$
k = \pm \frac{1}{4} \sqrt{\frac{5\delta_2}{\beta_2}}, \ \Omega = \frac{16k^2}{5} - \delta_1 \,, \tag{171}
$$

$$
|g|^2 = 6k^2, \ \mu_2 = \beta_2, \ \gamma_1 = -\frac{4\beta_2}{25\delta_2}, \ \gamma_2 = -\frac{4\beta_2^2}{25\delta_2} \,. \tag{172}
$$

and $\beta_1=\mu_1=1$ (normalized coefficients).

Chirped Bright Soliton

$$
\psi = g \operatorname{sech}^2(kx) e^{-i\alpha \log(\operatorname{sech}(kx))} e^{i\Omega t}, \qquad (173)
$$

where

$$
k = \pm \sqrt{\frac{-2(\alpha^2 + 2\alpha + 10)(\alpha^2 - 2\alpha + 10)\delta_2}{6\alpha(\alpha^4 + 16\alpha^2 + 96)\beta_1 + \beta_2 A}},
$$
\n(174)

$$
A = \alpha^{6} + 10\alpha^{4} + 8\alpha^{2} - 640,
$$
\n(175)

$$
\Omega = \frac{k^2(-(\alpha^6 + 10\alpha^4 + 8\alpha^2 - 640) + 6\alpha\beta_2(\alpha^4 + 16\alpha^2 + 96))}{2(\alpha^2 + 2\alpha + 10)(\alpha^2 - 2\alpha + 10)} - \delta_1, (176)
$$

$$
|g|^2 = \frac{k^2((-\alpha^6 - 3\alpha^4 + 94\alpha^2 + 1200) + 4\alpha(2\alpha^4 + 33\alpha^2 + 205)\beta_2)}{2(\alpha^2 + 2\alpha + 10)(\alpha^2 - 2\alpha + 10)}.
$$
 (177)

The coefficients must satisfy the following relations:

$$
\mu_2 = \frac{k^2((-\alpha^6 - 3\alpha^4 + 94\alpha^2 + 1200)\beta_2 - 4\alpha(2\alpha^4 + 33\alpha^2 + 205))}{2|g|^2(\alpha^2 + 2\alpha + 10)(\alpha^2 - 2\alpha + 10)},
$$
(178)

$$
\gamma_1 = \frac{(\alpha^2 - 10) + 6\alpha\beta_2}{2k^2(\alpha^2 + 2\alpha + 10)(\alpha^2 - 2\alpha + 10)},
$$
\n(179)

$$
\gamma_2 = \frac{(\alpha^2 - 10)\beta_2 + 6\alpha}{2k^2(\alpha^2 + 2\alpha + 10)(\alpha^2 - 2\alpha + 10)},
$$
\n(180)

and $\beta_1=\mu_1=1$ (normalized coefficients).

Now we look for solutions of elliptic function. We suppose

$$
\psi = (g + \wp(kx))e^{i\Omega t}.
$$
\n(181)

Substitution this ansatz into the GCCSH equation (168) yields an algebraic equation in terms of a Weierstrass \wp function by using formulas in Appendix,

and equating the coefficients of different powers of a Weierstrass \wp function to zero, we obtain the following elliptic function solution:

$$
\psi = (g + \wp(kx))e^{i\Omega t},\tag{182}
$$

where

$$
k = \pm \left(\frac{-1}{120\gamma_1}\right)^{\frac{1}{4}}, \ g = \pm \frac{1}{\sqrt{-30\gamma_1}}, \ \Omega = \frac{-2 + \gamma_1(3g_2 - 20\delta_1)}{20\gamma_1}, \tag{183}
$$

$$
g_2 = \frac{2\beta_2^2 + 20\gamma_2\delta_2}{3\gamma_2\delta_2}, \ g_3 = \pm \frac{2\sqrt{30}(1 + 5\gamma_1(\Omega + \delta_1))}{45\sqrt{-\gamma_1^3}}.
$$
 (184)

The coefficients must satisfy the following relations:

$$
\mu_2 = -120\gamma_2 k^4, \ \beta_2^2 = -30\gamma_2 \mu_2 g^2 \,, \tag{185}
$$

$$
135\gamma_2^3\mu_2^3g_3^2 = -8\beta_2^2(\beta_2^2 + 5\gamma_2\delta_2)^2\,,\tag{186}
$$

and $\beta_1 = \mu_1 = 1$ (normalized coefficients).

We find another elliptic function solution,

$$
\psi = (g + \text{cn}^2(kx, q))e^{i\Omega t},\tag{187}
$$

where

$$
k = \pm \left(\frac{-1}{120\gamma_1 q^2}\right)^{\frac{1}{4}},\tag{188}
$$

$$
\Omega = -\delta_1 - 8\gamma_1 k^4 (8 - (23 + 30g)q + (23 + 60g + 45g^2)q^2),\tag{189}
$$

$$
q = \frac{6 + 19g + 15g^2 \pm \sqrt{A}}{2(g+1)(15g^2 + 15g + 4)},
$$
\n(190)

$$
A = -15g^4 - 30g^3 - 3g^2 + 12g + 4,
$$
\n(191)

$$
g = \frac{1 - 6q}{3q} - \frac{1}{60\gamma_1 q}.
$$
\n(192)

The coefficients must satisfy the following relations:

$$
\mu_2 = -120\gamma_2 q^2 k^4 \,,\tag{193}
$$

$$
\beta_2 = 20\gamma_2(1 - (2 + 3g)q),\tag{194}
$$

$$
\delta_2 = -8\gamma_2 k^4 (8 - (23 + 30g)q + (23 + 60g + 45g^2)q^2), \qquad (195)
$$

and $\beta_1 = \mu_1 = 1$ (normalized coefficients).

5 Conclusion

A number of pattern formation phenomena are descrived by the $(2+1)$ dimensional complex Swift-Hohenberg equation. Among them is an approximation of a Maxwell-Bloch system for the transverse dynamics of a two-level class B laser, filamentation in wide aperture semiconducter lasers etc. $(1+1)$ dimensional version of this equation is one of the basic equations for modelling temporal behaviour of laser systems. Among others, it describes, short pulse generaion by passively mode-locked lasers with complicated spectral filtering elements. The knowledge of its solutions presented in any form will help to understand better the processes in such systems. In this paper, using Painlevé analysis, Hirota multi-linear method and direct ansatz technique, we studied analytic solutions of the $(1+1)$ -dimensional complex cubic and quintic Swift-Hohenberg equations. We considered both, standard and generalized versions of these equations. We have found that a number of exact solutions exist to each of these equations provided that coefficients are bound by special relations. The set of solutions include particular types of solitary wave solutions, hole (dark soliton) solutions and periodic solutions in terms of elliptic Jacobi functions and Weierstrass \wp function. Clearly these solutions represent only a small subset of large variety of possible solutions admitted by the complex cubic and quintic Swift-Hohenberg equations. Nevertheless, the solutions presented here are found for the first time and they might serve as seeding solutions for a wider class of localised structures which, no doubt, exist in these systems. We also hope that they will be useful in further numerical analysis of various solutions to the complex Swift-Hohenberg equation.

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A Formulas of hyperbolic functions and elliptic functions

Let $Dn = dn(kx, q)$, $Sn = sn(kx, q)$, $Cn = cn(kx, q)$.

Jacobi elliptic functions satisfy the following two relations:

$$
Sn2 + Cn2 = 1,
$$
\n(A.1)\n
$$
Dr2 = 1 - Cn2
$$
\n(A.2)

 $Dn^2 = 1 - qSn^2$. . $(A.2)$

We have the following differential formulas:

$$
\frac{1}{k^2} \frac{\text{Dn}''(x)}{\text{Dn}(x)} = q(-1 + 2\text{Sn}^2),\tag{A.3}
$$

and

$$
\frac{1}{qk^4} \frac{\text{Dn}''''(x)}{\text{Dn}(x)} = 4 + q - 4(2 + 5q)\text{Sn}^2 + 24q\text{Sn}^4. \tag{A.4}
$$

Now the limit $q = 1$ reduces this to $dn(kx, 1) = sech(kx) = S(x)$, $sn(kx, 1) =$ $tanh(kx) = T(x)$. Then

$$
\frac{1}{k^2} \frac{S''(x)}{S(x)} = -1 + 2T^2,
$$
\n(A.5)

and

$$
\frac{1}{k^4} \frac{S''''(x)}{S(x)} = 5 - 28T^2 + 24T^4.
$$
\n(A.6)

The other main function we need for the periodic solutions is Jacobi sn function. We have the following differential formulas:

$$
\frac{1}{k^2} \frac{\text{Sn}''(x)}{\text{Sn}(x)} = 2q \text{Sn}^2(x) - q - 1,\tag{A.7}
$$

and

$$
\frac{1}{k^4} \frac{\text{Sn}''''(x)}{\text{Sn}(x)} = 1 + 14q + q^2 - 20q(q+1)\text{Sn}^2(x) \n+ 24q^2\text{Sn}^4(x).
$$
\n(A.8)

Now the limit $q = 1$ reduces this to $\text{sn}(kx, 1) = \tanh(kx) = T(x)$. Then

$$
\frac{1}{k^2} \frac{T''(x)}{T(x)} = 2(-1+T^2),\tag{A.9}
$$

and

$$
\frac{1}{k^4} \frac{T''''(x)}{T(x)} = 8(2 - 5T^2 + 3T^4). \tag{A.10}
$$

We give the following formula of Jacobi cn function:

$$
\frac{1}{k^2} \frac{\text{Cn}''(x)}{\text{Cn}(x)} = -1 + 2q \text{Sn}^2(x),\tag{A.11}
$$

and

$$
\frac{1}{k^4} \frac{\text{Cn}''''(x)}{\text{Cn}(x)} = 1 + 4q - 4q(2q + 5)\text{Sn}^2(x) \n+ 24q^2 \text{Sn}^4(x).
$$
\n(A.12)

Finally, we give the following differential formulas of Weierstrass \wp function:

$$
[\wp'(z)]^2 = 4[\wp(z)]^2 - g_2\wp(z) - g_3,
$$
\n(A.13)

$$
\wp''(z) = 6[\wp(z)]^2 - \frac{1}{2}g_2, \qquad (A.14)
$$

$$
\wp'''(z) = 12 \left(10[\wp(z)]^3 - \frac{3}{2}g_2 \wp(z) - g_3 \right) , \qquad (A.15)
$$

where g_2 and g_3 are constants. Weierstrass \wp function is connected to Weierstrass σ function by

$$
\wp(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \sigma(x) \,. \tag{A.16}
$$

Weierstrass \wp function can be expressed by Jacobi elliptic functions:

$$
\wp(x) = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(kx, q)}, \ k^2 = e_1 - e_3, \ q = \frac{e_2 - e_3}{e_1 - e_3}, \tag{A.17}
$$

where

$$
e_1 + e_2 + e_3 = 0
$$
, $e_1e_2 + e_1e_3 + e_2e_3 = -\frac{g_2}{4}$, $e_1e_2e_3 = \frac{g_3}{4}$. (A.18)

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