Unifying scheme for generating discrete integrable systems including inhomogeneous and hybrid models

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Abstract

A unifying scheme based on an ancestor model is proposed for generating a wide range of integrable discrete and continuum as well as inhomogeneous and hybrid models. They include in particular discrete versions of sine-Gordon, Landau-Lifshitz, nonlinear Schrödinger (NLS), derivative NLS equations, Liouville model, (non-)relativistic Toda chain, Ablowitz-Ladik model etc. Our scheme introduces the possibility of building a novel class of integrable hybrid systems including multi-component models like massive Thirring, discrete self trapping, two-mode derivative NLS by combining different descendant models. We also construct inhomogeneous systems like Gaudin model including new ones like variable mass sine-Gordon, variable coefficient NLS, Ablowitz-Ladik, Toda chains etc. keeping their flows isospectral, as opposed to the standard approach. All our models are generated from the same ancestor Lax operator (or its q -¿ 1 limit) and satisfy the classical Yang-Baxter equation sharing the same r-matrix. This reveals an inherent universality in these diverse systems, which become explicit at their action-angle level.

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I. INTRODUCTION

Though integrable models represent only a special class of nonlinear systems, their numbers and varieties discovered till today have become amazingly large. Therefore it is particularly important now to have well defined schemes, which will be able to generate them in a systematic way, find out their interrelations, detect the fundamental ones and identify their universal properties. Reduction of Lax operators in AKNS spectral problem [1], classification of soliton bearing equations through self-dual Yang-Mills equation [2], gauge unification of nonlinear Schrödinger (NLS)-type models [3] are few of such successful approaches. However most of these schemes are designed to deal with the continuous models only, whereas the importance and significance of discrete integrable systems have been well emphasized in recent years [4]. Moreover, the algebraic approach in classical integrable models, though has a rich and sophisticated formulation through the classical Yang-Baxter equation and the classical r-matrix [5], as it appears, has not been exploited fully.

Our aim here is therefore to propose an unified algebraic scheme for systematic generation of a large class of integrable discrete models, based on their underlying Poisson bracket (PB) structure. The specialty of this class of models is that , they can be easily quantized to yield the corresponding quantum integrable systems and their classification may be done through the associated classical r-matrix with its known trigonometric and rational solutions. We present an integrable discrete ancestor model linked with the trigonometric r-matrix (and its $q \to 1$ form related naturally to the rational solution of r) and containing a set of arbitrary parameters. Various choices of these external parameters define in turn different underlying algebraic structures and the associated Lax operators. This generates through suitable realizations a wide range of diverse integrable systems sharing the same r-matrix with their ancestor model. They are by construction integrable discrete models with few of them having also well defined field limits.

Our scheme, the basic idea of which is borrowed from the quantum domain [6, 7], appears to be effective not only in classifying an important class of discrete models as well as their field limits, but also their inhomogeneous extensions. Along with the exactly integrable discrete versions of the well known models like sine-Gordon, Landau-Lifshitz equation, NLS, derivative NLS (DNLS), Liouville model, relativistic and nonrelativistic Toda chain, Ablowitz-Ladik model, we also obtain new inhomogeneous models like variable mass sine-Gordon and more general variable coefficient NLSs, as well as Gaudin model, inhomogeneous Ablowitz-Ladik model and Toda chains. As an important application of our scheme we may construct novel families of integrable hybrid models, by combining different descendant models in different domains of the lattice space ¹, or by fusing copies of a single component model to get its multi-component generalization. Moreover, the present method of generating integrable inhomogeneous discrete

¹Very recently such field models attracted attention [8]

and continuum models reveals the intriguing fact that the conventional approach by considering space-time dependent spectral parameter $\lambda(x,t)$ [9]-[14] is rather restricted and even appears to be misleading, since it would lead in general to a dynamical r-matrix spoiling the underlying algebraic structure and forbidding therefore the possible quantization of the models and their usual action-angle formulation. Moreover, for more general inhomogeneous sine-Gordon and NLS models, as we find here, the conventional treatment of nonisospectral flow would likely to fail. In our approach on the other hand the necessary isospectrality is kept intact by taking constant λ as in the original homogeneous case and the inhomogeneity is introduced through arbitrary parameters, which act like Casimir operators in the associated Poisson algebra.

Since all these models, in spite of their manifestly diverse forms and nature, are generated from the same ancestor model sharing the same r-matrix (or its $q \to 1$ form), it reveals an intriguing universality among them which is reflected prominently in their description of complete integrability through action-angle variables.

The paper is arranged as follows. In sec. II we review the theory of integrable systems satisfying classical Yang-Baxter equation associated with classical $r(\lambda - \mu)$ matrix. Sec. III presents the explicit form of the ancestor model and its $q \to 1$ limit together with the underlying PB algebras. We introduce our generating scheme in sec. IV and construct concrete models. Sec. V accounts for the generation of integrable inhomogeneous as well as hybrid models. Sec. VI focuses on the universal property of all descendant models by explicit construction of their action-angle variables. Sec. VII is the concluding section.

II. CLASSICAL YANG-BAXTER EQUATION AND INTEGRABLE SYSTEMS

By integrability of a nonlinear discrete system defined on a lattice with sites j = 1, 2, ..., N, we mean it in the Liouville sense by requiring the existence of its N number of independent conserved quantities $C_n, n = 1, 2, ..., N$ including the Hamiltonian H of the system with the criteria $\{C_n, C_m\} = 0$. Such conserved quantities can be considered as the action variables generated from a spectral parameter λ -dependent transfer matrix as: $\tau(\lambda) = \sum_{n=1}^{N} C_n \lambda^n$ and consequently the integrability criteria may be replaced by the single condition

$$\{\tau(\lambda), \tau(\mu)\} = 0. \tag{1}$$

For deriving this condition therefore along with the conventional linear spectral problem: $T_{k+1}(\lambda) = L_k(\lambda)T_k(\lambda)$ we define also the PB algebra for its Lax operator $L_k(\lambda)$ in a specific form, which is known as the classical Yang-Baxter equation (CYBE) [5]

$$\{L_k(\lambda) \otimes, L_l(\mu)\} = \delta_{kl}[r(\lambda - \mu), L_k(\lambda) \otimes L_k(\mu)]$$
(2)

associated with the classical $r(\lambda - \mu)$ -matrix playing the role of structure constants. For the associativity of algebra (2) ensuring its Jacobi identity, the r-matrix in turn must satisfy another

form of CYBE:

$$[r_{12}(\lambda - \mu), r_{13}(\lambda - \delta)] + [r_{12}(\lambda - \mu), r_{23}(\mu - \delta)] + [r_{13}(\lambda - \delta), r_{23}(\mu - \delta)] = 0.$$
 (3)

It is crucial to observe that, though there is a variety of Lax operator solutions to (2) with different basic operators and spectral parameter dependence, representing a wide range of integrable systems (for a list see sec. IV), the associated r-matrix solutions satisfying (3) are only of three types: elliptic, trigonometric and rational. Moreover most of the known models are linked to the last two cases only, i.e to the trigonometric r-matrix

$$r_t(\lambda - \mu) = \frac{1}{i\sin(\lambda - \mu)} \left(\frac{1}{2}\cos(\lambda - \mu)\sigma_3 \otimes \sigma_3 + \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ \right)$$
(4)

or to its $q \to 1, \lambda, \mu \to 0$ limit given by the rational solution

$$r_r(\lambda - \mu) = \frac{1}{i(\lambda - \mu)} P$$
, where $P = \frac{1}{2} (I + \vec{\sigma} \cdot \vec{\sigma})$, (5)

P being the permutation operator. The above remarkable observation has motivated us to conjecture that all integrable models satisfying the CYBE (2) must be derivable from an ancestor model with their Lax operators obtained as various reductions of this single ancestor Lax operator and this should make the r-matrix, inherited from their ancestor, to be naturally the same for all these descendant models. In the next section we present such an ancestor model in the explicit form associated with the trigonometric r-matrix (4), from which we will be able to generate a rich collection of integrable discrete and continuum models including inhomogeneous as well as hybrid systems, all satisfying the CYBE and sharing the same r-matrix (4) (or its rational limit (5)). Note that from the CYBE (2) one can go to its global description

$$\{T_N(\lambda)\otimes, T_N(\mu)\} = [r(\lambda - \mu), T_N(\lambda)\otimes T_N(\mu)]$$
(6)

for the monodromy matrix

$$T_N(\lambda) = L_N(\lambda) \cdots L_1(\lambda) == \begin{pmatrix} a_N(\lambda) & b_N(\lambda) \\ c_N(\lambda) & d_N(\lambda) \end{pmatrix}.$$
 (7)

It is important to notice that (6) exhibits exactly the same form as its local relation (2), which reflects a deep underlying Hopf algebra structure, an important characteristic of all such integrable systems [15]. Defining now the transfer matrix as $\tau(\lambda) = trT_N(\lambda) = a_N(\lambda) + d_N(\lambda)$ and taking the trace of (6) one can easily derive (since the rhs being the trace of a commutator is zero) the integrability condition (1) for the system. Therefore going backwards in the logical chain we can conclude that the nonlinear systems with its representative Lax operator and the r-matrix satisfying the CYBE (2) must be an integrable system. We shall see below that the relation (6) also carries important information for deriving action-angle variables and reflects

an universal property for all integrable systems sharing the same r-matrix and hence belonging to the same class.

Note that in this algebraic approach we are not concerned about the usual Lax pair L, M and do not obtain the dynamical equation from the flatness condition involving them. We on the other hand take the Lax operator $L_k(\lambda)$ satisfying the CYBE as the representative of the integrable model and using it construct the monodromy matrix: $T(\lambda) = \prod_k L_k(\lambda)$ and then the transfer matrix from its trace $\tau(\lambda) = trT(\lambda)$. Expanding further the transfer matrix $\tau(\lambda)$ in spectral parameter λ as described above, we derive the conserved quantities including the Hamiltonian H in the explicit form. The dynamical equation can now be obtained as the Hamilton equation $\psi_t = \{\psi, H\}$, using the fundamental PB relations.

At the lattice constant $\Delta \to 0$ one may recover in some cases the corresponding field model: $L_k(\lambda) \to I + \Delta \mathcal{L}(x,\lambda) + O(\Delta^2)$ with $\mathcal{L}(x,\lambda)$ as the field Lax operator. Though the associated r-matrix remains the same, the CYBE gets deformed and the corresponding monodromy matrix $T(\lambda)$ at the infinite interval limit $l = N\Delta \to \infty$ satisfies also a bit different global CYBE [5]. For continuum models one can extract the conserved quantities more conveniently from the Lax operator using the Ricatti equation derived from the linear spectral problem.

III. ANCESTOR MODELS ASSOCIATED WITH TRIGONOMETRIC AND RATIONAL r-MATRIX

As mentioned, our generating scheme for integrable models is based on various reductions of a discrete ancestor Lax operator, which we propose to take in the following form [7]

$$L_k^{trig(anc)}(\xi) = \begin{pmatrix} \xi c_1^+ e^{i\alpha S_k^3} + \xi^{-1} c_1^- e^{-i\alpha S_k^3}, & 2\sin\alpha S_k^- \\ 2\sin\alpha S_k^+, & \xi c_2^+ e^{-i\alpha S_k^3} + \xi^{-1} c_2^- e^{i\alpha S_k^3} \end{pmatrix}, \quad \xi = e^{i\lambda}, \tag{8}$$

and demand it to satisfy the CYBE (2) with the trigonometric r-matrix (4). \bar{S}_k appearing in (8) are the basic dynamical fields PB algebra of which as specified below is dictated by its integrability and c_a^{\pm} , a=1,2 are a set of arbitrary parameters. The structure of the Lax operator (8) becomes clearer if we notice its possible decomposition, after an allowed gauge transformation by $h=e^{i\lambda\sigma_3}$, $L^{t(anc)}(\xi)\to hL^{t(anc)}(\xi)h^{-1}=\xi L_++\xi^{-1}L_-$, where L_{\pm} are spectral parameter ξ -free upper/lower triangular matrices. Note that the r-matrix (4) allows also a similar decomposition (after a similar gauge transformation): $r_t(\frac{\xi}{\eta})\to \frac{\xi}{\eta}r_++(\frac{\xi}{\eta})^{-1}r_-$, $\xi=e^{i\lambda}$, $\eta=e^{i\mu}$ with r_{\pm} being spectral-free upper/lower triangular matrices, which together with L_{\pm} satisfy the FRT-type [16] PB algebra derivable from the CYBE [17]. The demand of integrability on (8) through the CYBE can be shown to be equivalent to the underlying general algebra

$$\{S_k^3, S_l^{\pm}\} = \pm i\delta_{kl}S_k^{\pm}, \quad \{S_k^+, S_l^-\} = i\frac{\delta_{kl}}{\sin\alpha}f(2\alpha S_k^3), \text{ with } f(x) = (M^+\sin(x) + M^-\cos(x)),$$

$$(9)$$

where $M^{\pm} = \pm \frac{1}{2} \sqrt{\pm 1} (c_1^+ c_2^- \pm c_1^- c_2^+)$ are arbitrary parameters acting as central elements with trivial brackets with all others: $\{M^{\pm}, \cdot\} = 0$ and in general may also be site and time dependent. It is important to note that the underlying PB structure (9) is linked with a generalization of the well known quantum group algebra. For generating integrable systems from this ancestor model, we find first a realization of (9) in canonical variables $\{u_k, p_l\} = \delta_{kl}$, in the form

$$S_k^3 = u_k, \quad S_k^+ = e^{-ip_k} g(u_k), \quad S_k^- = g(u_k) e^{ip_k},$$
 (10)

where

$$g(u_k) = [\kappa + \sin \alpha (s - u_k) \{ f(\alpha (u_k + s + 1)) \}]^{\frac{1}{2}} \frac{1}{\sin \alpha},$$
(11)

containing free parameters κ , s and function f(x) as defined in (9). It should be remarked here that realization (10) usually assumes the complex conjugacy $S_k^- = (S_k^+)^*$, which however is not imposed by the integrability condition (9). Note that we have now lots of freedom for generating descendant models from the ancestor Lax operator (8) by using various reductions of (11)

under different choices of the arbitrary parameters c's as well as κ and s or its further realization in bosonic variables: $\{\psi_k, \psi_l^*\} = i\delta_{kl}$ in (10). Moreover we can multiply these Lax operators from left or right by σ_a , a = 1, 2, 3, since such transformations are allowed by the CYBE due to a symmetry of (4) and (5) as $[r, \sigma_a \otimes \sigma_a] = 0$.

We will demonstrate in the next section that a class of discrete integrable systems with nontrivial deformation parameter q, which may be interpreted as the *relativistic* parameter can be generated in a systematic way from the ancestor Lax operator (8). The nonrelativistic models on the other hand may be constructed in a similar way from the $q \to 1$ limit of (8) given as

$$L_k^{rat.(anc)}(\lambda) = \begin{pmatrix} c_1^0(\lambda + s_k^3) + c_1^1, & s_k^- \\ s_k^+, & c_2^0(\lambda - s_k^3) - c_2^1 \end{pmatrix},$$
(12)

with $c_a^{0,1}, a=0,1$ being arbitrary parameters. Here due to the corresponding limits of $\vec{S} \to \vec{s}, \{c_a^{\pm}\} \to \{c_a^{0,1}\}, M^+ \to -m^+, M^- \to -\alpha m^-, \xi \to 1+i\lambda$, the PB algebra (9) reduces to

$$\{s_k^+, s_k^-\} = i\delta_{kl}(2m^+s_k^3 + m^-), \qquad \{s_k^3, s_l^\pm\} = \pm i\delta_{kl}s_k^\pm,$$
 (13)

where $m^+ = c_1^0 c_2^0$, $m^- = c_1^1 c_2^0 + c_1^0 c_2^1$ with $\{m^{\pm}, \cdot\} = 0$. Note that a Casimir operator commuting with all other generators of (13) may be constructed as

$$S^{2} = s_{k}^{3}(m^{+}s_{k}^{3} + m^{-}) + s_{k}^{+}s_{k}^{-}$$

$$\tag{14}$$

and a realization of it (13) given by the generalized *Holstein-Primakov* transformation (HPT)

$$s_k^3 = s - N_k, \quad s_k^+ = g_0(N_k)\psi_k, \quad s_k^- = \psi_k^* g_0(N_k), \quad g_0(N_k) = (m^- + m^+ (2s - N_k))^{\frac{1}{2}}, \quad N_k \equiv \psi_k^* \psi_k$$

$$(15)$$

in bosonic variables ψ_k , which in fact is the $\alpha \to 0$ limit of (10) and (11). We stress again that since the conjugacy of s_k^{\pm} is not necessarily imposed by the integrability, ψ, ψ^* in (15) in general may not be complex conjugates. Note that the ancestor model (12) represents the undeformed rational class and satisfies the CYBE with the rational r-matrix (5). (8) and (12) serving as the ancestor Lax operators for the discrete integrable models may also yield for some systems the corresponding field models with the Lax operator $\mathcal{L}(x,\lambda)$. The associated r-matrix however would remain the same at the continuum limit, since it is a global nondynamical object independent of site indices. We shall see in sec. V that parameters c's in general can be spacetime dependent and hence could induce inhomogeneity in the model preserving the constancy of the spectral parameter.

IV. UNIFIED GENERATION OF DISCRETE INTEGRABLE MODELS

From the ancestor models proposed we generate here integrable discrete models belonging to both trigonometric and rational class

A. Relativistic models belonging to trigonometric class

For constructing this class of models we start from the ancestor Lax operator (8) and look into its different realizations by choosing first the arbitrary parameters c's as constants.

1.) Discrete sine-Gordon model: Parameter choice $c_1^{\pm} = -c_2^{\pm} = m\Delta$, with m as the constant mass. This gives $M^- = 0$, $M^+ = -(m\Delta)^2$, and reduces realization (10) correspondingly to yield from (8) (after multiplying it from right by $-i\sigma_1$) the Lax operator

$$L_k(\lambda) = \begin{pmatrix} g(u_k) e^{ip_k \Delta}, & m\Delta \sin(\lambda + \alpha u_k) \\ m\Delta \sin(\lambda - \alpha u_k), & e^{-ip_k \Delta} g(u_k) \end{pmatrix}, \quad g^2(u_k) = 1 - (m\Delta)^2 \cos \alpha (2u_k + 1).$$
 (16)

It is important to note that (16) yields exactly the Lax operator of the integrable discrete sine-Gordon model [18] and at the continuum limit $\Delta \to 0$, when $e^{\pm ip_k\Delta} \to 1 \pm \Delta ip_k$ and $(u_k, p_k) \to (u(x), p(x))$, recovers clearly the field Lax operator

$$L_k(\lambda) = 1 + \Delta \mathcal{L}(x, \lambda), \quad \mathcal{L}(x, \lambda) = ip(x)\sigma_3 + m\sin(\lambda + \alpha u(x))\sigma_+ + m\sin(\lambda - \alpha u(x))\sigma_-, \quad p(x) = \dot{u}(x)$$
(17)

of the well known sine-Gordon model $u_{tt} - u_{xx} + \sin \alpha u = 0$. Remarkably the PB algebra (9) in this case reduces to the classical limit of the celebrated quantum group [15] with its familiar relation $\{S^+, S^-\} = -i[2S^3]_q$. We will see in the next section that a more general choice for the parameters would lead to an inhomogeneous extension of this sine-Gordon model.

2.) Discrete Liouville model: Parameter choice $c_1^+ = c_2^- = \Delta$, $c_1^- = c_2^+ = 0$. This gives $M^{\pm} = \pm \frac{1}{2} \sqrt{\pm 1} \Delta^2$ and correspondingly reduces (10) to derive from the same ancestor (8) (after

multiplying it from right by σ_1) the Lax operator

$$L_k(\xi) = \begin{pmatrix} e^{p_k \Delta} g(u_k) , & \Delta \xi e^{\alpha u_k} \\ \frac{\Delta}{\xi} e^{\alpha u_k} , & g(u_k) e^{-p_k \Delta} \end{pmatrix}, \qquad g^2(u_k) = 1 + \Delta^2 e^{\alpha(2u_k + i)}, \tag{18}$$

which represent the discrete Liouville model [19] and at its field limit ($\Delta \to 0$) the Lax operator $\mathcal{L} = p\sigma_3 + e^{\alpha u}(\xi\sigma_+ + \frac{1}{\xi}\sigma_-)$ of the well known Liouville equation: $u_{tt} - u_{xx} = e^{\alpha u}$. Note that in this case (9) gives a novel PB algebra with exponentially deformed relation like $\{S^+, S^-\} = \frac{1}{2\sin\alpha}e^{2i\alpha S^3}$.

It is intriguing to observe here that though the underlying PB structure and hence its realization giving the model are fixed by the choice of M^{\pm} , the Lax operator (8) which depends directly on the parameters c's may take different forms for the same model. For example, in the present case with additional choice $c_1^- \neq 0$ would record the same values for M^{\pm} , but a different Liouville Lax operator [20]. This opens up therefore a promizing possibility for systematically obtaining different useful Lax operators for the same integrable model.

3.) Relativistic Toda chain: Different sets of constant choices i) $c_a^+ = 1$, a = 1, 2, or ii) $c_a^- = 1$, a = 1, 2, or iii) $c_1^{\pm} = \pm 1$, or iv) $c_1^+ = 1$, with the rest of c's being zero, lead to $M^{\pm} = 0$, reducing therefore (9) to the simple PB algebra

$$\{S_k^+, S_l^-\} = 0, \ \{S_k^3, S_l^{\pm}\} = \pm i\delta_{kl}S_k^{\pm},$$
 (19)

and the realization (10) (after a canonical interchange of variables: $u \to -ip, p \to -iu$,) to the form

$$S_k^3 = -ip_k, \ S_k^{\pm} = \alpha e^{\mp u_k} \ .$$
 (20)

This generates interestingly from the same ancestor Lax operator (8) different forms of the discrete-time or relativistic Toda chain (RTC). For example, case iii) yields

$$L_k(\xi) = \begin{pmatrix} \frac{1}{\xi} e^{\alpha p_k} - \xi e^{-\alpha p_k}, & \alpha e^{u_k} \\ -\alpha e^{-u_k}, & 0 \end{pmatrix}, \tag{21}$$

recovering the Lax operator found in [21], while iv) generates a different Lax operator [22] for the same model. More famous RTC model of Suris [23] however is obtained in this approach after performing a twisting transformation with twisting parameter taken as $\pm \alpha$ (of these equivalent cases we consider here only $-\alpha$, for definiteness), which deforms the r_t -matrix (4) by adding a constant matrix Ω to it [21]:

$$r_t \to r_\Omega = r_t - \Omega$$
, where $\Omega = i(\sigma_3 \otimes I - I \otimes \sigma_3)$. (22)

As a result the form of the ancestor Lax operator (8) also gets changed with its elements transforming as

$$c_a^{\pm} \to c_a^{\pm} e^{i\alpha S_k^3}, \quad S_k^{\pm} \to \tilde{S}_k^{\pm} = e^{i\frac{1}{2}\alpha S_k^3} S_k^{\pm} e^{i\frac{1}{2}\alpha S_k^3}.$$
 (23)

Implementing the corresponding changes in (21) and using the same realization (20) for the variables (23) we obtain now the explicit form of the Lax operator

$$L_k(\xi) = \begin{pmatrix} \frac{1}{\xi} e^{2\alpha p_k} - \xi, & \alpha e^{u_k} \\ -\alpha e^{2\alpha p_k - u_k}, & 0 \end{pmatrix}, \tag{24}$$

generating that of the Suris RTC [23].

4.) Discrete derivative NLS: Parameter choice as constants:

$$c_1^+ = c_2^+ = 1, \ c_1^- = -iq\Delta, \ c_2^- = \frac{i\Delta}{q}, \ \text{giving } M^+ = 2\Delta\sin\alpha, \ M^- = 2i\Delta\cos\alpha$$
 (25)

gives (10) a q-bosonic realization as $S_k^+ = -Q_k$, $S_k^- = Q_k^*$, $S_k^3 = -N_k$, with a PB algebra induced from (9) as

$$\{Q_k, N_l\} = i\delta_{kl}Q, \ \{Q_k, Q_l^*\} = i\delta_{kl}\frac{\cos(\alpha(2N+1))}{\cos\alpha}, \tag{26}$$

which clearly reduces to the standard bosons ψ_k, ψ_k^* at $\alpha \to 0$. It is worth noting that this new q-bosonic model as realized from the ancestor Lax operator (8) (after introducing Δ) would give

$$L_k(\xi) = \begin{pmatrix} \frac{1}{\xi} q^{-N_k} - i\xi \Delta \ q^{N_k+1}, & Q_k^* \\ Q_k, & \frac{1}{\xi} q^{N_k} + i\xi \Delta \ q^{-(N_k+1)} \end{pmatrix}, \tag{27}$$

which represents an exact lattice version of the DNLS equation [24]. When expressed through bosonic field: $Q = \psi(\Delta \frac{[2N]_q}{2N\cos\alpha})^{\frac{1}{2}}$, $N = \Delta |\psi|^2$, (27) yields at the continuum limit $\Delta \to 0$ the field Lax operator

$$\mathcal{L}(\psi) = -(\frac{1}{4}\xi^2 - |\psi|^2)\sigma_3 + \xi(\psi^*\sigma^+ + \psi\sigma^-)$$
 (28)

of the well known Chen-Lee-Liu DNLS equation [25]: $i\psi_t = \psi_{xx} - 4i|\psi|^2\psi_x$.

5.) Ablowitz-Ladik model: This model involving also another form of q-boson is possible to generate in our scheme, though it needs twisting transformation as in the Suris RTC mentioned above and is associated with the same twisted r_{Ω} -matrix (22) and the twisted ancestor Lax operator with the change (23). Now the the parameter choice $c_1^+ = c_2^- = 0$ with $c_1^- = c_2^+ = 1$ giving $M^{\pm} = \frac{1}{2}\sqrt{\pm 1}$ (compare with the Liouville case) together with the twisting removes dynamical variables from the diagonal elements of the twisted Lax operator as well as modifies the Poisson algebra of the transformed variables \tilde{S}_k^{\pm} as derivable from (9). Therefore naming $b_k = 2\sin\alpha \tilde{S}_k^+$ we get this modified PB relation as $\{b_k, b_l^*\} = i\delta_{kl}(1 - b_k^*b_k)$, confirming the basic variables of the Ablowitz-Ladik model as a type of q-boson with its Lax operator as

$$L_k(\xi) = \begin{pmatrix} \frac{1}{\xi}, & b_k^* \\ b_k, & \xi \end{pmatrix}. \tag{29}$$

related to (22). We will see later how space-time dependent parameters c's give inhomogeneous extensions of this model. Note that another intriguing possibility of generalizing this model arises if we simply consider $c_1^+ \neq 0$ in the above construction. It is not difficult to see that, realizing $S_k^3 = -\ln(1-b_k^*b_k)$ this would generate an extra term $\xi c_1^+(1-b_k^*b_k)^{-2i\alpha}$ in the upper diagonal element of the Ablowitz-Ladik Lax operator (29). Its consequence in the dynamical equation would be an interesting problem to study.

B. Nonrelativistic models belonging to rational class

Deformation parameter $q=e^{i\alpha}$, as we have seen in the above models, serves as the relativistic or the deformed bosonic parameter. We consider now the undeformed limit $q\to 1$ or $\alpha\to 0$, when as explained already, the r-matrix reduces to its rational form (5) and the ancestor Lax operator is converted to (12) with the underlying PB algebra (13).

We find that the integrable models belonging to this rational class are mostly nonrelativistic models, which can be generated in a similar way from the rational ancestor model (12) with different constant choices for parameters $c_a^{0,1}$, a = 1, 2 involved in it.

6.) Landau-Lifshitz equation (LLE) Parameter choice $c_a^0 = 1, c_1^1 = -c_2^1 = -l$ compatible with $m^+ = 1, m^- = 0$, reduces (13) to the classical sl_2 spin algebra $\{s_k^{\alpha}, s_l^{\beta}\} = i\delta_{kl}\epsilon^{\alpha\beta\gamma}s_l^{\gamma}$ with spin: $s^2 = (s_k^3)^2 + s_k^+ s_k^- \equiv \vec{s}_k^2$ as the Casimir operator reduced from (14). The ancestor Lax operator (12) simplifies (ignoring an irrelevant multiplicative factor) to

$$L_k(\lambda) = I + \frac{1}{\lambda - l} \vec{s}_k \cdot \vec{\sigma} \tag{30}$$

representing a discrete version of the LLE. At the continuum limit $\Delta \to 0$ putting $\vec{s}_k \to \Delta \vec{s}(x)$ one gets from the Casimir: $\vec{s}^2(x) = 1$ and from the Lax operator $L_k(\lambda) \to I + \Delta \mathcal{L}(\lambda)$, $\mathcal{L}(\lambda) = \frac{1}{\lambda - l} \vec{s}(x) \cdot \vec{\sigma}$, that for the well known LLE [26].

7.) Discrete NLS model: For the same sl_2 spin algebra transformation (15) yields the standard HPT with $g_0(|\psi_k|) = (2s - \Delta |\psi_k|^2)^{\frac{1}{2}}$ (considering ψ_k, ψ_k^* to be complex conjugates and scaling them by $\sqrt{\Delta}$). This realization by considering parameter l = 0 leads (12) to the Lax operator of exactly integrable discrete NLS model [18] given by

$$L_k(\lambda) = \begin{pmatrix} \lambda + s - \Delta |\psi_k|^2 & \sqrt{\Delta} \psi^* g_0(|\psi_k|) \\ \sqrt{\Delta} \psi g_0(|\psi_k|) & \lambda - s + \Delta |\psi_k|^2 \end{pmatrix}.$$
(31)

At the field limit: $\Delta \to 0$, (31) yields (after multiplying it from left by $\sigma_3 \Delta$ and considering $s = 1/\Delta$) the familiar form of the Lax operator

$$\mathcal{L}(\lambda) = \lambda \sigma^3 + \sqrt{2}(\psi^* \sigma^+ - \psi \sigma^-) \tag{32}$$

for the NLS field equation $i\psi_t = \psi_{xx} + 2|\psi|^2\psi$.

8.) Simple lattice NLS: On the other hand a complementary choice $m^+ = 0, m^- = 1$, giving $g_0(N_k) = 1$ converts (15) directly to the realization $s_k^+ = \psi_k$, $s_k^- = \psi_k^*$, $s_k^3 = s - \psi_k^* \psi_k$ in bosonic field: $\{\psi_k, \psi_l^*\} = i\delta_{kl}$. Now a compatible choice of parameters: $c_1^0 = c_2^1 = 1, c_2^0 = c_1^1 = 0$ together with this bosonic realization generates from the ancestor (12) the Lax operator

$$L_k(\lambda) = \begin{pmatrix} \lambda + s - N_k & \psi_k^* \\ \psi_k & -1 \end{pmatrix}, \quad N_k = \psi_k^* \psi_k$$
 (33)

which may be associated with another—simple lattice NLS model proposed in [27] and as noted there ψ , ψ^* may not be complex conjugates at the discrete level. At the continuum limit we recover again the same field Lax operator (32) for the NLS equation and regain also the complex-conjugacy of the fields (see for details [27]).

9.) Nonrelativistic Toda chain: Note that the trivial choice $m^{\pm} = 0$ yields from (13) again the same algebra (19) and therefore we may take the same realization of it as found before. However the rational form of ancestor model (12) generates now simpler Lax operator

$$L_k(\lambda) = \begin{pmatrix} p_k - \lambda & e^{u_k} \\ -e^{-u_k} & 0 \end{pmatrix}. \tag{34}$$

of the nonrelativistic Toda chain associated with the rational r-matrix and described by the Hamiltonian: $H = \sum_k \frac{1}{2} p_k^2 + e^{(u_k - u_{k+1})}$.

Thus we have demonstrated that discrete and continuum integrable models can be obtained in a unified way from the ancestor Lax operator (8) (or its rational limit (12)) by choosing different sets of constant values for the parameters c's involved in the ancestor model and by using different realizations of the underlying PB algebra. In the next section we find how a more general choice of c's can generate further the inhomogeneous extensions of these integrable models.

We find also a convincing answer to an important question raised above asking why different integrable systems with varied Lax operator solutions should have the same r-matrix, by discovering that all these models are basically obtainable from the same ancestor model (8) associated with the trigonometric r-matrix (4). These descendant models, whose explicit Lax operators we derive here satisfy the CYBE (2) inheriting and sharing the same r-matrix (4). We will see in sec. VI that this significant fact induces a universality among these seemingly diverse systems by defining their action-angle variables in the same way.

V. INTEGRABLE INHOMOGENEOUS AND HYBRID MODELS WITH ISOSPECTRAL FLOW

We have seen how by fixing the values of certain parameters we could generate a wide spectrum of integrable models belonging to the trigonometric and rational class. We focus here on some promising possibilities to generalize this procedure for constructing novel integrable families of inhomogeneous and hybrid models.

A. Inhomogeneous models

Returning again to the ancestor model (8) we may notice that the parameters c_a^{\pm} , a=1,2 entering in it (similarly, $c_a^{0,1}$, a=1,2 in its rational limit (12)) act like external parameters having trivial PB with all basic variables in their local algebra and therefore, apart from constants as earlier they may be considered in general as site (time) dependent arbitrary functions. As a result M^{\pm} in (10), (11) in turn also become functions $M_k^{\pm}(t), k = 1, 2, \dots, N$ (and similarly $m_k^{\pm}(t)$ in (15)) and lead to new integrable descendant models, which are inhomogeneous extensions of the discrete and continuum models constructed above. However this integrable family of inhomogeneous models is obtained in our scheme by keeping the usual isospectral flow. Moreover, such constancy of spectral parameters (except some trivial transformations like shifting etc.) is essential in this algebraic formalism for satisfying the CYBE (6) with spectral dependent global and nondynamical $r(\lambda - \mu)$ -matrix. It is important also to notice that the inhomogeneity is introduced here through a set of different independent parameters: c_{ak}^{\pm} (or $c_{ak}^{0,1}$) with a=1,2and therefore it may not be always possible to absorb them in the single spectral parameter, even by declaring it to be nonisospectral. Therefore we see that, contrary to the standard approach the inhomogeneous models can not be described in general as nonisospectral flow, at least those that belong to the present family. Moreover isospectrality is a necessary criterion for the CYBE solution, as explained already.

10.) Variable mass sine-Gordon model:

The construction is parallel to that of the constant mass sine-Gordon model obtained above, where in place of constants we choose now the parameters as four different variable mass: $c_1^{\pm} = m_{1k}^{\pm}(t)\Delta$, $c_2^{\pm} = m_{2k}^{\pm}(t)\Delta$. This would generate from the ancestor Lax operator (8) and realization (10) a general form of a new inhomogeneous sine-Gordon model, which is integrable and satisfies the CYBE associated to the trigonometric r-matrix (4). Particular choices of the inhomogeneities would yield naturally different forms of the variable mass sine-Gordon model, discrete as well as continuum, which seem to have been never considered before.

For a demonstration we take up the simplest case when all mass parameters coincide: $m_{ak}^{\pm}(t) = m_k(t)$. This variable mass discrete sine-Gordon model can be described again by the same form (16) by replacing constant m by a variable $m_k(t)$. At the continuum limit this would correspond to a sine-Gordon field model with variable mass m(x,t). If the mass parameter is assumed to be factorized: $m(x,t) \equiv m_0(t)m_1(x)$, by introducing a new coordinate system through nonlinear transformation $(t,x) \to (T,X)$, $T = \int^t m_0(t')dt'$, $X = \int^x m_1(x)dx'$, the Hamiltonian of the model can be written formally again as the standard sine-Gordon model with unit mass. Nevertheless we notice that even in a further simplified case with $m_0(t) = 1$, the soliton solutions in the original system might have quite interesting character depending on the form of the variable mass $m_1(x)$ (see Fig. 1).

11.) Inhomogeneous NLS model: Since this model belongs to the rational class, in accordance

with our strategy we start with the ancestor Lax operator (12) and consider the parameters involved in it and in realization (15) to be site and time dependent functions: $c_{ak}^{0,1}(t), a = 1, 2$. With all of them different we naturally get the general inhomogeneous discrete NLS model, which retains its integrability and contrary to the standard approach also its isospectrality, as explained already. For constructing the corresponding field model we take the parameters in the form $c_{1k}^0 = g_{1k}, c_{2k}^0 = -g_{2k}, c_{1k}^1 = \frac{1}{\Delta} + f_{1k}, c_{2k}^1 = -\frac{1}{\Delta} + f_{2k}, s = \Delta$ and at the limit $\Delta \to 0$ obtain the Lax operator

$$\mathcal{L}(\lambda) = \begin{pmatrix} \Lambda_1(x,t) & Q^* \\ -Q & -\Lambda_2(x,t) \end{pmatrix}, \text{ where } \Lambda_a(x,t) = \lambda g_a(x,t) + f_a(x,t), \ a = 1,2$$
 (35)

and $Q = \psi g_0(x,t)$ with $g_0(x,t) = (g_1(x,t) + g_2(x,t))^{\frac{1}{2}}$, representing an inhomogeneous NLS field model with inhomogeneities introduced by the independent functions $g_a(x,t)$, $f_a(x,t)$, a=1,2. It may be stressed again that here the spectral parameter λ is strictly constant and when $\Lambda_1 \neq \Lambda_2$, all inhomogeneous parameters apparently can not be absorbed in this single parameter. It is challenging to derive the explicit form of this integrable variable coefficient general NLS equation, associated with the rational r-matrix (5). For showing that the Lax operator of many known inhomogeneous NLS equations can actually be derived from (35), we consider the particular situation $g_1 = g_2 \equiv g(x,t)$, $f_1 = f_2 \equiv f(x,t)$ and rewrite (35) as

$$\mathcal{L}(\lambda) = \Lambda(x,t)\sigma_3 + Q^*\sigma_+ - Q\sigma_-, \quad \Lambda(x,t) = \lambda g(x,t) + f(x,t). \tag{36}$$

It is remarkable that from this single operator we recover at $g=1, f=\alpha t$ the Lax operator of [9], at $g=\frac{1}{t}, f=\frac{4x}{t}$ that of [10] and similarly at $g=T(t), f=\frac{\alpha}{2}xT(t)$ that of [11]. Note however that the actual form of the equations depend also on the time evolution operator M, which is likely to be different in our approach from the known ones, since in our construction the fundamental canonical PB structure is always preserved. Therefore it would be a challenging problem to derive these new integrable inhomogeneous NLS equations explicitly from their Hamiltonian using the canonical PB.

- 12.) Gaudin model: It is intriguing that by just by considering the parameter l in the Lax operator (30) for the discrete LLE to be site dependent: $l \to l_k$, one recovers the Lax operator $L_k(\lambda) = I + \frac{1}{\lambda l_k} \vec{s}_k \cdot \vec{\sigma}$ for the celebrated Gaudin model, given by the integrable Hamiltonians $H_k = \sum_{l \neq k}^{N} \frac{1}{l_k l_l} (\vec{s}_k \cdot \vec{s}_l), \ k = 1, 2, \dots, N$ [28]. This model is associated also with the rational r_r matrix (5).
- 13.) Inhomogeneous relativistic and nonrelativistic Toda chains: It is not difficult to see that by repeating the construction of the Toda chains but taking the parameters to be nonconstants we get integrable inhomogeneous extensions of such models. For example, considering in ancestor Lax operator (12) the parameters to be $c_1^{\pm} = f_k^{\pm}(t)$, $c_2^{\pm} = 0$, but using the same realization as for the original relativistic Toda chain, we get an extension of its Lax operator

(21) to include inhomogeneity through arbitrary functions $f_k^{\pm}(t)$:

$$L_k(\xi) = \frac{1}{2} \left(\frac{f_k^-}{\xi} e^{\alpha p_k} - f_k^+ \xi e^{-\alpha p_k} \right) (I + \sigma_3) + \alpha (e^{u_k} \sigma_+ - e^{-u_k} \sigma_-),$$

which therefore would represent a new integrable family of inhomogeneous relativistic Toda chain. We will not present here its explicit form.

At $\alpha \to 0$ this family of relativistic models would go to its nonrelativistic limit represented by the Lax operator

$$L_k(\lambda) = \begin{pmatrix} (p_k - \lambda) + g_{2k} & (c_{1k}^0)^{-1} e^{u_k} \\ -(c_{1k}^0)^{-1} e^{-u_k} & 0 \end{pmatrix}, \tag{37}$$

which is an obvious extension of (34) (by introducing the inhomogeneous parameter $g_{2k} \equiv \frac{c_{1k}^1}{c_{1k}^0}$ and normalizing it by c_{1k}^0). Without defining any time evolution operator M, we can directly construct from (37) the explicit form of the Hamiltonian through the conserved quantity as $H = C_{N-1}$ and derive the Hamilton equations using the canonical PB between u_k, p_l , yielding $\dot{u}_k = p_k + g_{2k}$ and hence the inhomogeneous Toda chain equation as

$$\frac{d^2}{dt^2}u_k = g_1(k)e^{u(k-1)-u(k)} - g_1(k+1)e^{u(k)-u(k+1)} + \dot{g}_2(t) + boundary \ terms \tag{38}$$

with arbitrary parameters $g_1(k) = (c_{1k}^0 c_{1k+1}^0)^{-1}$ and $g_2(k)$. Different choices of these parameters would generate from (38) different inhomogeneous Toda chains. For example the particular choices: $g_1(k) = k$, $g_2(k) = \alpha_0 t$ and $g_1(k) = 4k^2 + 1$, $g_2(k) = kt$ derives the Toda chains found in [12], though in contrast we recover this result in a completely isospectral way.

14.) Inhomogeneous Ablowitz-Ladik model: It is easy to notice again that if instead of constants as in the original model, we choose the parameters through arbitrary function $\Gamma(t)$ as $c_1^- = (c_2^+)^{-1} = e^{\Gamma(t)}$, keeping the same trivial choice for $c_1^+ = c_2^- = 0$, we generate from (8) the Lax operator

$$L_k(\xi) = \begin{pmatrix} \frac{1}{\xi} e^{\Gamma(t)}, & b_k^* \\ b_k, & \xi e^{-\Gamma(t)} \end{pmatrix}. \tag{39}$$

Remarkably, in spite of our isospectral approach, (39) recovers exactly the Lax operator of [13] for arbitrary $\Gamma(t)$ and that of [14] for $\Gamma(t) = \alpha t$, representing known inhomogeneous Ablowitz-Ladik models.

In a similar way by generalizing the constant parameters to inhomogeneous functions one can generate systematically inhomogeneous extensions of other integrable models constructed here. Note again that all such extensions retain the integrability of the system as well as the isospectrality and the same r-matrix solution.

B. Integrable hybrid models

Our scheme for generating different integrable models from an ancestor model sharing the same r-matrix opens up a possibility of constructing new families of integrable models by hybridizing these descended models.

Such constructions can be of two types. The first type of hybrid models may be constructed by using different descendant Lax operators obtained directly from (8) (or alternatively from (12)) as its different but consistent reductions and realizations at different lattice sites. Since all representative Lax operators of these constituent models: $L_k^{d(k)}(\lambda)$, with d(k) denoting different members of the same descendant class inserted at sites k, should share the same r-matrix, the monodromy matrix of this hybrid model: $T^{\{d\}}(\lambda) = \prod_{d(k),k} L_k^{d(k)}(\lambda)$ must satisfy the global CYBE (6) and represent therefore an integrable system with the set of conserved quantities including the Hamiltonian obtainable as usual through expansion of $\tau^{hyb}(\lambda) = tr(T^{\{d\}}(\lambda))$ in the spectral parameter. One can generate in this way some exotic hybrid models by combining for example, sine-Gordon and Liouville models, different types of relativistic Toda chain or discrete NLS model etc. constructed above. These hybrid models presumably would show different dynamics at different domains in the coordinate space. It is encouraging to note that very recently such models have received well deserved attention, though only at the continuum level [8]. We hope that the present idea, based on discrete approach and r-matrix formalism would prove to be promizing and fruitful for analyzing such hybrid integrable models.

A second type of hybrid models may be constructed by considering different representation of the Lax operator for different components of the field and inserting their direct product at the same lattice site. As a result one can build new multi-component generalization of a scalar model through the fused Lax operator: $L_k^{\{m\}} = \prod_m L_k^{(m)}$, where each entries in the product would represent individual components. Note that unlike the vector generalization, which needs also enlarged matrix realization for the Lax operator, our multi-component hybrid models would yield only 2×2 matrix Lax operators. For elaborating this idea we present the detail construction of an integrable hierarchy of two-component DNLS model.

15.) Integrable hierarchy of two-component DNLS: Note that in constructing the discrete DNLS model in sec. IV, the values of M^{\pm} fixed by (25) actually determined the underlying algebra as well as the required realization. It is however crucial to notice now that interchanging the parameters $c_1^{\pm} \leftrightarrow c_2^{\mp}$ would not change the values of M^{\pm} and therefore would lead to the same algebra and its realization, but result to a complementary form for the Lax operator, though representing the same DNLS model. In our construction of the two-component model with fields $\psi_k^{(\beta)}$, $\beta=1,2$ having PB relations $\{\psi_k^{(\beta)},\psi_l^{*(\gamma)}\}=i\delta_{\beta\gamma}\delta_{kl}$, we take c_a^{\pm} , a=1,2 as in (25) for building the Lax operator $L_k^{(1)}(\psi^{(1)})$ as (27) for the first component. However, for the corresponding construction of $\tilde{L}_k^{(2)}(\psi^{(2)})$ related to the second component we take the complementary choice by considering $c_1^{\pm} \leftrightarrow c_2^{\mp}$. The fused Lax operator taking the form

 $L_k^{(1,2)}(\psi^{(\beta)}) = L_k^{(1)}(\psi^{(1)})\tilde{L}_k^{(2)}(\psi^{(2)})$ represents now a new discrete multi-component DNLS satisfying the CYBE (2) with the same r-matrix (4). At the continuum limit $\Delta \to 0$, repeating the construction for the scalar DNLS model (28), it is easy to see that the field Lax operator of this two-component model is given simply as a linear superposition $\mathcal{L}(\psi^{(1)}, \psi^{(2)}) = \mathcal{L}(\psi^{(1)}) + \tilde{\mathcal{L}}(\psi^{(2)})$ with the explicit form

$$\mathcal{L}(\psi^{(1)}, \psi^{(2)}) = (\frac{1}{4}(\frac{1}{\xi^2} - \xi^2) + |\psi|_1^2 - |\psi|_2^2)\sigma_3 + (\xi\psi_1^* + \frac{1}{\xi}\psi_2^*)\sigma^+ + (\xi\psi_1 + \frac{1}{\xi}\psi_2)\sigma^-, \ \xi = e^{i\lambda}$$
(40)

It is interesting to show by direct construction that this Lax operator generates an integrable hierarchy of multi-component DNLS model through the expansion $\ln \tau(\lambda) = \sum_{n=0}^{\infty} C_{\pm n} \lambda^{\mp 2n}$ with $H_2 = C_2 + C_{-2}$ introducing a new two-component generalization of the Chen-Lee-Liu DNLS equation. We are not giving here its explicit form, which can be worked out with little patience. The higher conserved quantities will yield higher order equations. Some similar class of discrete matrix and multi-component DNLS models was proposed recently [30].

- 16.) Massive Thirring model: It is remarkable that (40) constructed through a novel hybridization from our ancestor model coincides also with the Lax operator of the bosonic massive Thirring model [29]. Hamiltonian of this relativistic model may be given through the same conserved quantities constructed for the above model in the form $H_1 = C_1 + C_{-1}$.
- 17.) Integrable discrete self trapping model: A discrete self trapping model with two-bosonic modes $\psi^{(a)}, \psi^{*(a)}, a = 1, 2$ given by the Hamiltonian

$$H = -\left[\frac{1}{2}\sum_{a}^{2}(s_a - N^{(a)})^2 + (\psi^{*(1)}\psi^{(2)} + \psi^{*(2)}\psi^{(1)})\right]$$

was studied in [31]. This integrable model is associated with a Lax operator, which may be constructed by fusing two operators as $L(\lambda) = L^{(1)}(\lambda)L^{(2)}(\lambda)$, where $L^{(a)}(\lambda)$ are given by the same Lax operator (33) for each of the modes a = 1, 2.

An interesting line of investigation would be to apply this hybridization method for constructing possible multi-component extensions of other models like relativistic Toda chain, Abowitz-Ladik, Liouville model, LLE, Gaudin model etc. The linear superposition of Lax operators for building new nonlinear integrable systems, as revealed here, seems to be a promizing idea worth pursuing.

VI. UNIVERSAL PROPERTIES OF INTEGRABLE DESCENDANT MODELS

We have seen that diverse forms of integrable models: discrete and continuum, homogeneous and inhomogeneous, multi-component and hybrid models can be generated in a systematic way in our ancestor model scheme. Among this diversity however we find also an unexpected universality. Indeed, as we have found, a wide range of models, namely sine-Gordon, Liouville, DNLS, relativistic Toda chain etc. including their discrete, inhomogeneous and hybrid variants

belong to the trigonometric class, while models like NLS, Toda chain etc. and their related discrete and inhomogeneous extensions are in the same rational class, which being in fact the undeformed $q \to 1$ or the nonrelativistic limit of the former class.

The crucial observation is that the diversity of all descendant models belonging to the same class seems to disappear at the global level allowing their description through a universal action-angle variable. The reason for this is very simple. Though these models differ widely at their local level having different forms of the Lax operator, their monodromy matrix $T_N(\lambda)$ (7) satisfies the same global relation (6) with the same r-matrix, which is inherited from their ancestor model and shared by all of them.

As a result, for all models belonging for example to the trigonometric class, the PB relations should be given by the same structure constants expressed through the elements of the r_t -matrix (4). For the twisted models, e.g. Suris RTC and Ablowitz-Ladik model the structure constants should similarly be given by the twisted r_{Ω} -matrix (22). In the same way all models from the rational class should have analogous property expressed through the elements of rational r_r -matrix (5). However, while the action variables are constants in time, the time evolution of angle variables depends on the definition of the Hamiltonians through conserved quantities, which usually differs for different models.

Such differences also bear some additional imprint at the continuum limit, when the monodromy matrix is defined as $T(\lambda) = \lim_{N\to\infty} L_{\infty}^{-N}(\lambda) T_N(\lambda) L_{\infty}^{-N}(\lambda)$ and the corresponding CYBE is modified as [5] $\{T(\lambda)\otimes, T(\mu)\} = r_+(\lambda-\mu)T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu)r_-(\lambda-\mu)$, where $r_{\pm}(\lambda-\mu) = \lim_{N\to\infty} L_{\infty}^{\mp N}(\lambda)\otimes L_{\infty}^{\mp N}(\mu)r(\lambda-\mu)L_{\infty}^{\pm N}(\lambda)\otimes L_{\infty}^{\pm N}(\mu)$. Therefore though the action-angle description for such models of the same class are again basically the same, the influence of the individual models also enters now due to the appearance of r_{\pm} modified by the asymptotic forms $L_{\infty}(\lambda)$ of the individual Lax operators. Nevertheless it is startling to check that the canonical action-angle variables for widely different field models like DNLS [24] and the sine-Gordon [1] are defined exactly in the same way: $p(\xi) = \frac{1}{2\pi c\xi} \ln |a(\xi)|$, $q(\xi) = \arg b(\xi)$ for the continuum modes with PB $\{q(\xi), p(\eta)\} = \delta(\xi-\eta)$, $\xi > 0$ and $p_k = \frac{1}{2c} \ln \xi_k$, $q_k = \ln b_k$ for the discrete set with $\{q_k, p_l\} = \delta_{kl}$ and similarly for their conjugates \bar{q}_k, \bar{p}_l .

Therefore we may conclude that all integrable models presented here may be described universally through the ancestor Lax operator (8) and the r_t -matrix (4) (or twisted r_{Ω}) or similarly by the $q \to 1$ limit as the ancestor model (12) and the r_r -matrix (5), where the global relations like action-angle variables are determined by the r-matrix elements alone. The individuality of the models may be reflected only in the definition of their Hamiltonians through conserved quantities and for continuum models, additionally in the limiting forms of their Lax operators.

VII. CONCLUDING REMARKS

We have presented here a unifying scheme based on PB algebra for systematically generating a large class of integrable discrete and continuum models from a single ancestor model. Such models include well known and new integrable systems as well as inhomogeneous models. Based on our construction we conclude that more general and logical approach for inhomogeneous integrable models, at least for models with nondynamical r-matrix, would be to describe them as isospectral flow in inhomogeneous external fields. As another fruitful application of the present scheme we have proposed a simple method for constructing new families of integrable hybrid models by fusing different types of descendant models. In spite of the vastly diverse form of these models their common ancestor and common r-matrix reveal an inherent universality in their description through action-angle variables.

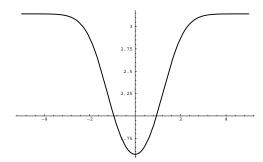
We strongly hope that the algebraic approach linked with the quantum group structure formulated here for generating classical integrable discrete as well as field models, though a bit uncommon in the community working in classical integrability, would prove to be much powerful due to its systematic and algorithmic nature. Similarly the novel ideas of construction introduced here, like generating integrable hybrid and multi-component models and creating integrable inhomogeneity in isospectral flow, are expected to be equally promizing.

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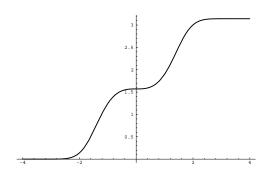


Fig 1: How the kink solution (for $m_1(x) = 1$) deforms in variable mass sine-Gordon model depending on the mass parameter $m_1(x) = x$ and $m_1(x) = x^2$, respectively.