

# Negative Volterra Flows

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November 8, 2018

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## Abstract

Taking the standard zero curvature approach we derive an infinite set of integrable equations, which taken together form the negative Volterra hierarchy. The resulting equations turn out to be nonlocal, which is usual for the negative flows. However, in some cases the nonlocality can be eliminated. Studying the combined action of both positive (classical) and negative Volterra flows, i.e. considering the differential consequences of equations of the extended Volterra hierarchy, we deduce local equations which seem to be promising from the viewpoint of applications. The presented results give answers to some questions related to the classification of integrable differential-difference equations. We also obtain dark solitons of the negative Volterra hierarchy using an elementary approach.

## 1 Introduction.

The present paper is devoted to an integrable hierarchy which can be viewed as an extension of the Volterra hierarchy (VH) by taking into account the so-called 'negative' flows. The idea of negative flows is not new. One of the first examples (and maybe the most striking one) is the relation between the AKNS and sine-Gordon models [1]: the sine-Gordon equation can be derived as the negative flow of the AKNS hierarchy. Some recent results and approaches to this problem can be found in [2, 3, 4, 5]. The negative flows have been constructed for almost all known integrable systems. But there are a few exceptions, and the Volterra model seems to be among them: to our knowledge, the systematic derivation of its 'negative' equations has not been discussed in the literature.

Some of the equations belonging to the extended Volterra hierarchy are already known. First, it contains the famous 2D Toda lattice [6]. We would also like to mention the works of De Lillo and Konotop [7], who discussed a nonlocal modification of the Volterra chain and recent papers by Boiti *et al* [8, 9], who derived and studied new integrable

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discretization of KdV. As will be shown below, these equations belong to the extended VH.

Another indication of the fact that the neagive Volterra flows should be studied more carefully is related to the problem of classification of the integrable discrete equations. In the paper by Yamilov [10] it has been shown that all equations of the form

$$u_{n,t} = f(u_{n+1}, u_n, u_{n-1}) \quad (1)$$

can be divided into three classes. One of them can be presented as

$$u_{n,t} = \frac{p(u_n)u_{n+1}u_{n-1} + q(u_n)(u_{n+1} + u_{n-1}) + r(u_n)}{u_{n+1} - u_{n-1}} \quad (2)$$

where  $p$ ,  $q$  and  $r$  are polynomials in  $n$ . This equation is known to satisfy integrability tests, to posses an infinite set of local conservation laws and generalized symmetries. However, as was stated, for example, in [11], "it is the only example of nonlinear chain of the form (1) which cannot be reduced to the Toda or Volterra equations by Miura transformations". In this paper we will show that there is no need to look for substitutions converting (2) into Volterra chain, equations (2) (at least the simplest of them) are nothing but equations of the extended VH.

The most straightforward approach to the negative flows is to consider them in the framework of the inverse scattering transform. In section 2 we start with the standard zero-curvature representation (ZCR) for the VH and derive equations of the negative Volterra subhierarchy. The resulting equations turn out to be nonlocal. It is usual for the case of negative flows. However, nonlocality does not mean that equations are not interesting from the viewpoint of possible applications. Moreover, sometimes nonlocality can be eliminated. This can be achieved by a redefinition of the variables or by considering some differential consequences of the equations in question. Namely this is the topic of section 3 where we present the local equations which in this or another way are related to the nonlocal Volterra equations. Finally, in section 4 we derive the soliton solutions for the negative VH.

## 2 Zero curvature representation.

The inverse scattering approach for integrable systems is based on the ZCR, when our nonlinear equations are presented as a compatibility condition for some linear system. For example, the Volterra chain,

$$\dot{u}_n = u_n (u_{n+1} - u_{n-1}), \quad n = 0, \pm 1, \pm 2, \dots \quad (3)$$

is the compatibility condition,

$$\dot{U}_n = V_{n+1}U_n - U_nV_n, \quad (4)$$

for the system [12]

$$\begin{cases} \Psi_{n+1} &= U_n \Psi_n \\ \dot{\Psi}_n &= V_n \Psi_n \end{cases} \quad (5)$$

where

$$U_n = U_n(\lambda) = \begin{pmatrix} \lambda & u_n \\ -1 & 0 \end{pmatrix}, \quad V_n = V_n(\lambda) = \begin{pmatrix} u_n & \lambda u_n \\ -\lambda & u_{n-1} - \lambda^2 \end{pmatrix} \quad (6)$$

and  $\lambda$  is an arbitrary constant. Traditionally Volterra and Toda chains are considered more often in the framework of the 'big' Lax representation, when the system is finite ( $n = 1, \dots, N$ ) and  $U$  and  $V$  are  $N \times N$  matrices. In this paper we will use the  $2 \times 2$   $U$ - $V$  pair (6). This approach is equivalent to the  $N \times N$  representation and sometimes even more convenient (that is, it can be more easily modified to the cases of different boundary conditions, including the soliton case when  $N = \infty$ ).

The choice of the  $U$ - $V$  pair for the system (5), even for a given  $U$ -matrix, is not unique. The matrix  $V$  in (6) is a second-order polynomial in  $\lambda$ . By simple algebra one can find other matrices  $V$ , which are polynomials of higher order, such that ZCR (4) will be satisfied for all  $\lambda$  provided  $u_n$  solve some nonlinear evolutionary equations. These equations are called 'higher Volterra equations'. Taken together they constitute the VH.

Now we are going to derive the negative Volterra flows. The idea is simple: let us search for the  $V$ -matrices which are polynomials not in  $\lambda$  but in  $\lambda^{-1}$ . For example, the simplest of such  $V$ -matrices, which is linear in  $\lambda^{-1}$ , is given by

$$V_n = \begin{pmatrix} 0 & -\frac{p_{n-1}}{\lambda} \\ \frac{1}{\lambda p_{n-1}} & \frac{1}{p_{n-1}} \end{pmatrix} \quad (7)$$

It is easy to verify that (5) will be consistent if  $u_n$  and  $p_n$  satisfy the system

$$\begin{cases} \dot{u}_n &= p_{n-1} - p_n \\ u_n &= p_{n-1} p_n \end{cases} \quad (8)$$

This is the first negative Volterra equation.

Now our aim is to obtain an infinite set of the similar  $V$ -matrices which solve ZCR (4). Using the notation

$$V_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad (9)$$

we can rewrite (4) as

$$0 = \lambda(a_{n+1} - a_n) - b_{n+1} - u_n c_n, \quad (10)$$

$$0 = b_n + u_n c_{n+1}, \quad (11)$$

$$0 = \lambda c_{n+1} + a_n - d_{n+1} \quad (12)$$

and

$$\dot{u}_n = u_n (a_{n+1} - d_n) - \lambda b_n \quad (13)$$

or, after eliminating  $b_n$  and  $d_n$ ,

$$0 = \lambda(a_{n+1} - a_n) + u_{n+1}c_{n+2} - u_n c_n, \quad (14)$$

$$\dot{u}_n = u_n [a_{n+1} - a_{n-1} + \lambda(c_{n+1} - c_n)] \quad (15)$$

It is easy to show that (14) and (15) possess solutions where  $a_n$  are polynomials of the  $(2j - 2)$ th order in  $\lambda^{-1}$  while  $c_n$  are polynomials of the  $(2j - 1)$ th order for  $j = 1, 2, \dots$ . In what follows we indicate different polynomials with the upper index,  $a_n^{(j)}, c_n^{(j)}$ , and introduce an infinite set of times,  $\bar{t}_j$ , to distinguish the resulting nonlinear equations.

By simple algebra one can establish the following relations between different polynomials:

$$a_n^{(j+1)} = \lambda^{-2} a_n^{(j)} + \lambda^{-2} \alpha_n^{(j)}, \quad (16)$$

$$c_n^{(j+1)} = \lambda^{-2} c_n^{(j)} + \lambda^{-1} \gamma_n^{(j)} \quad (17)$$

where  $\alpha_n^{(j)}$  and  $\gamma_n^{(j)}$  do not depend on  $\lambda$ . Substituting (16) and (17) into (14) and (15) one can convert our system into

$$\alpha_{n+1}^{(j)} - \alpha_n^{(j)} + u_{n+1} \gamma_{n+2}^{(j)} - u_n \gamma_n^{(j)} = 0, \quad (18)$$

$$\gamma_n^{(j)} + \alpha_n^{(j+1)} + \alpha_{n-1}^{(j+1)} = 0 \quad (19)$$

and

$$\frac{\partial}{\partial \bar{t}_j} \ln u_n = \alpha_{n-1}^{(j)} - \alpha_{n+1}^{(j)} \quad (20)$$

After introducing the tau-functions of the VH,

$$u_n = \frac{\tau_{n+1} \tau_{n-2}}{\tau_n \tau_{n-1}} \quad (21)$$

equations (18) and (19) can be solved,

$$\alpha_n^{(j)} = \frac{\partial}{\partial \bar{t}_j} \ln \frac{\tau_{n-1}}{\tau_n} \quad (22)$$

$$\gamma_n^{(j)} = \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_n}{\tau_{n-2}} \quad (23)$$

and the resulting equations become

$$\tau_{n-1} \tau_{n+1} \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} + \tau_n^2 \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n = 0 \quad (24)$$

together with the first one

$$\tau_{n-1} \tau_{n+1} \frac{\partial}{\partial \bar{t}_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \tau_n^2. \quad (25)$$

These equations can be rewritten in the Hirota bilinear form as

$$2D_{j+1} \tau_{n+1} \cdot \tau_{n-1} + D_j D_1 \tau_n \cdot \tau_n = 0 \quad (26)$$

and

$$D_1 \tau_{n+1} \cdot \tau_{n-1} = \tau_n^2 \quad (27)$$

where

$$D_j a \cdot b = \frac{\partial}{\partial \xi} a(\dots, \bar{t}_j + \xi, \dots) b(\dots, \bar{t}_j - \xi, \dots) \Big|_{\xi=0}. \quad (28)$$

### 3 Some integrable models.

In the previous section we have obtained an infinite set of integrable equations. These equations are nonlocal and it is difficult to say whether they are of any importance, say, from the viewpoint of applications. However, a hierarchy is more than a set of equations. The key point is that all equations of a hierarchy are compatible (some notes on this question can be found in the appendix). Hence we can consider them simultaneously and study the combined action of different flows. In other words, we can take some finite set of equations of a hierarchy and analyse equations which follow from this system. So, this is the aim of this section. Starting with the equations of the VH (both negative, derived above, and positive, i.e. classical Volterra equations) we will deduce some their consequences, which are local and some of which seem to be more promising for applications.

#### 3.1 2D Toda lattice and the VH.

First let us consider the simplest negative equation (25) together with the Volterra equation (3)

$$\frac{\partial}{\partial \bar{t}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{1}{p_n} \quad (29)$$

$$\frac{\partial}{\partial t} \ln u_n = u_{n+1} - u_{n-1} \quad (30)$$

where  $p_n$  is related to  $\tau_n$  by

$$p_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}. \quad (31)$$

One can derive from the second equation that

$$\frac{\partial}{\partial t} \frac{1}{p_n} = p_{n-1} - p_{n+1} \quad (32)$$

which leads to

$$\frac{\partial^2}{\partial t \partial \bar{t}} \ln \tau_n = -\frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2} \quad (33)$$

where an unessential constant has been omitted. This equation, which can be rewritten in terms of  $p_n$  as

$$\frac{\partial^2}{\partial t \partial \bar{t}} \ln p_n = -p_{n-1} + 2p_n - p_{n+1} \quad (34)$$

is nothing but the 2D Toda lattice [6]. So, we have shown explicitly that tau-function (21) of the VH is at the same time the tau-function of the 2D Toda lattice, or, in other words, that the 2D Toda equation can be embedded into the VH. The reverse statement seems not to be true: not every solution of (33) should solve (29) and (30). In this sense splitting the (2+1)-dimensional equation (33) into two (1+1)-dimensional equations from the VH is a kind of *ansatz*. But the class of solutions of the 2D Toda model which can be obtained by solving Volterra equations (29) and (30) contains a large number of important solutions such as, e.g., N-soliton and finite-gap quasiperiodic solutions. It is obvious that this is a relation between the 2D Toda model and the *extended* VH and could not be revealed if dealing with the classical Volterra equations only.

### 3.2 Singular chain.

Another consequence of (25) is that the quantity

$$x_n = \frac{\partial}{\partial \bar{t}_1} \ln \tau_n \quad (35)$$

satisfies

$$\dot{x}_n = \frac{1}{x_{n-1} - x_{n+1}} \quad (36)$$

where the dot stands for the differentiating with respect to the first 'positive' Volterra time (i.e. time which corresponds to (3)),  $t$ ,  $\dot{x}_n = \partial x_n / \partial t$ . Thus we have come to an equation of the type (2) mentioned in the introduction. So, equations (2) probably can be embedded in the extended VH. Note that the quantity  $x_n$  cannot be expressed locally in terms of the 'positive' Volterra subhierarchy, without invoking the first negative flow.

Equation (36) seems to be non-typical for physical applications. However, it can be easily converted to a more usual form. Indeed, it follows from (36) that the quantities  $Q_n$  and  $P_n$ , given by

$$Q_n = \frac{1}{2} \ln \frac{x_{n+1}}{x_{n-1}} \quad P_n = x_{n+1} x_n \quad (37)$$

satisfy the system

$$2\dot{Q}_n = \frac{1}{P_n - P_{n+1}} + \frac{1}{P_n - P_{n-1}} \quad (38)$$

$$2\dot{P}_n = -\coth(Q_n + Q_{n+1}) - \coth(Q_n + Q_{n-1}) \quad (39)$$

which is a Hamiltonian system

$$\dot{Q}_n = \frac{\partial \mathcal{H}}{\partial P_n} \quad \dot{P}_n = -\frac{\partial \mathcal{H}}{\partial Q_n} \quad (40)$$

with the standard Poisson bracket and the Hamiltonian given by

$$\mathcal{H} = \frac{1}{2} \sum_n \{ \ln |P_n - P_{n-1}| + \ln | \sinh(Q_n + Q_{n-1}) | \}. \quad (41)$$

### 3.3 Integrable dynamics of roots of a 3-polynomial.

The following system is a 'toy' model which can be derived from (36) by imposing the periodic conditions  $x_{n+3} = x_n$ . Noting that the product

$$\Pi = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \quad (42)$$

is a constant and rescaling the time  $t \rightarrow \Pi \cdot t$  one can rewrite (36) as

$$\begin{cases} \dot{x}_1 = (x_1 - x_2)(x_1 - x_3) \\ \dot{x}_2 = (x_2 - x_1)(x_2 - x_3) \\ \dot{x}_3 = (x_3 - x_1)(x_3 - x_2) \end{cases} \quad (43)$$

or

$$\dot{x}_k = \mathcal{P}'(x_k) \quad k = 1, 2, 3 \quad (44)$$

where  $\mathcal{P}$  is an arbitrary monic 3-polynomial, and  $x_k$  are its roots:

$$\mathcal{P}(x) = \prod_{k=1}^3 (x - x_k). \quad (45)$$

Of course, it is a very simple ordinary differential equation, which can be studied without using the inverse scattering transform machinery. In terms of the first symmetric function of the roots,  $\sigma = x_1 + x_2 + x_3$ , it can be rewritten as the stationary KdV,

$$\sigma_{ttt} - 6\sigma_t^2 = 0 \quad (46)$$

(to make the following formulae more readable we indicate differentiations with subscripts) or, after introducing a 'logarithmic potential'  $\psi$ ,

$$\psi_t / \psi = -\sigma, \quad (47)$$

as

$$\psi_{ttt} \psi - 4\psi_{tt} \psi_t + 3\psi_t^2 = 0 \quad (48)$$

which is the expanded form of the simple bilinear equation

$$D_t^4 \psi \cdot \psi = 0. \quad (49)$$

### 3.4 3+1 dimensional example.

The last example we want to discuss differs from those presented above. All previous equations considered in this section were difference-differential systems. Now we derive from the VH some partial derivative equations in (3+1)-dimensional (physical) space-time.

Our starting point is equation (24) which can be presented as

$$p_n \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = -\frac{\partial^2}{\partial \bar{t}_j \partial \bar{t}_1} \ln \tau_n. \quad (50)$$

Acting on both sides of this equation by the  $\partial/\partial \bar{t}_k$  operator one can note that right-hand side is symmetrical in  $j$  and  $k$ , which leads to

$$\left( \frac{\partial}{\partial \bar{t}_j} p_n \frac{\partial}{\partial \bar{t}_{k+1}} - \frac{\partial}{\partial \bar{t}_k} p_n \frac{\partial}{\partial \bar{t}_{j+1}} \right) \ln \frac{\tau_{n+1}}{\tau_{n-1}} = 0. \quad (51)$$

On the other hand, differentiating (50) with respect to the first 'positive' Volterra time  $t$  one gets

$$\frac{\partial}{\partial t} p_n \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = -\frac{\partial}{\partial \bar{t}_j} \left( \frac{\partial^2}{\partial t \partial \bar{t}_1} \ln \tau_n \right) \quad (52)$$

which can be rewritten using (25) as

$$\frac{\partial}{\partial t} p_n \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial}{\partial \bar{t}_j} p_n. \quad (53)$$

It is easy to note that equations (51) and (53) form a closed system

$$\frac{\partial}{\partial \bar{t}_j} p \frac{\partial \Lambda}{\partial \bar{t}_{k+1}} - \frac{\partial}{\partial \bar{t}_k} p \frac{\partial \Lambda}{\partial \bar{t}_{j+1}} = 0 \quad (54)$$

$$\frac{\partial}{\partial t} p \frac{\partial \Lambda}{\partial \bar{t}_{j+1}} = \frac{\partial p}{\partial \bar{t}_j} \quad (55)$$

for two functions

$$p = p_n \quad \text{and} \quad \Lambda = \ln \frac{\tau_{n+1}}{\tau_{n-1}} \quad n = \text{constant} \quad (56)$$

with  $n$  being fixed.

From this multidimensional system one can derive a rather interesting consequence in 3+1 dimensions. To this end we denote

$$(x, y, z) = (\bar{t}_1, \bar{t}_2, \bar{t}_3) \quad (57)$$

and introduce vector  $\vec{u}$  by

$$\vec{u} = \left( \frac{\partial \Lambda}{\partial \bar{t}_2}, \frac{\partial \Lambda}{\partial \bar{t}_3}, \frac{\partial \Lambda}{\partial \bar{t}_4} \right)^T. \quad (58)$$



In these terms equation (55) can be rewritten as

$$\frac{\partial}{\partial t} p \vec{u} = \nabla p \quad (59)$$

where  $\nabla$  is the three-dimensional gradient,  $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ , while equation (54) gives

$$\text{curl } p \vec{u} = 0. \quad (60)$$

Eliminating  $p$  from (59) and (60) one obtains that vector  $\vec{u}$  satisfies the following equation:

$$\text{curl } \vec{u} = \left[ \vec{u} \times \frac{\partial \vec{u}}{\partial t} \right] \quad (61)$$

where  $[\dots \times \dots]$  is the usual three-dimensional wedge (vector) product.

Equation (61), which can be termed as 'dual to' the Euler equation,  $\partial \vec{u} / \partial t = [\vec{u} \times \text{curl } \vec{u}]$ , has already been discussed in the literature. For example, it has been proposed as a relativistic generalization of the  $\text{curl } \vec{v}_s = 0$  condition for the velocity of a superfluid condensate (see, e.g., [13]). However, its relation with the integrable Volterra system (which also implies integrability of this model) seems to be new, and we are going to present more elaborated studies of (61) in future papers.

## 4 Soliton solutions.

Here we would like to recall that equations are not only to be derived or classified, but should also be solved. In this section we present some class of solutions of the negative VH, namely the dark soliton ones.

There are many ways to derive pure-soliton solutions for an integrable model: the inverse problem, the dressing method, Backlund-Darboux transforms, Hirota's ansatz. The key idea of the last approach is that soliton solutions of all the integrable models possess the same structure, and the only thing one has to do while solving a particular equation is to determine some constants. In this paper we also exploit this fact. However, we do not use Hirota's technique but solve our equations using the elementary matrix calculus. Our starting point is that usually  $N$ -soliton solutions can be constructed of determinants of some combinations of the  $N \times N$  matrices having the following structure:

$$(A_n)_{jk} = \frac{\ell_j a_{nk}(t)}{L_j - R_k} \quad (62)$$

i.e. matrices which satisfy the equation

$$LA_n - A_n R = |\ell\rangle \langle a_n| \quad (63)$$

Here we use the 'bra-ket' notation,

$$\langle a_n| = (a_{n,1}, \dots, a_{n,N}) \quad |\ell\rangle = (\ell_1, \dots, \ell_N)^T \quad (64)$$

and  $L$  and  $R$  are some diagonal matrices

$$L = \text{diag}(L_1, \dots, L_N) \quad R = \text{diag}(R_1, \dots, R_N) \quad (65)$$

#### 4.1 Algebra of matrices (62).

Before proceeding further we present some formulae related to matrices (62) which we need to construct soliton solutions of the negative Volterra equations under the so-called 'finite-density' boundary conditions. We do not consider the most general case of (63), but restrict ourselves to

$$L = R^{-1} \quad (66)$$

and specify from the beginning the  $n$ -dependence by

$$A_{n+1} = A_n R \quad (67)$$

(as will be shown below, namely these matrices we need for our purposes). From (63), which we rewrite now as

$$R^{-1}A_n - A_{n+1} = |\ell\rangle\langle a_n| \quad (68)$$

or

$$A_n Z - Z A_{n+1} = Z |\ell\rangle\langle a_n| Z \quad (69)$$

where

$$Z = (I + R^{-1})^{-1} \quad (70)$$

and  $I$  is  $N \times N$  unit matrix, one can derive that matrices  $F_n$  inverse to  $I + A_n$ ,

$$F_n = (I + A_n)^{-1} \quad (71)$$

satisfy the identities

$$F_{n+1} R^{-1} - R^{-1} F_{n-1} = F_{n+1} |\ell\rangle\langle a_{n-1}| F_{n-1} \quad (72)$$

and

$$Z F_{n+1} - F_n Z = Z F_{n+1} |\ell\rangle\langle a_n| F_n Z. \quad (73)$$

These relations, together with the following formulae

$$1 - \langle a_n | F_n Z | \ell \rangle = \frac{\omega_{n+1}}{\omega_n} \quad (74)$$

$$1 + \langle a_{n-1} | Z F_n | \ell \rangle = \frac{\omega_{n-1}}{\omega_n} \quad (75)$$

where

$$\omega_n = \det(I + A_n) \quad (76)$$

lead to

$$\omega_n F_n Z |\ell\rangle = \omega_{n+1} Z F_{n+1} |\ell\rangle \quad (77)$$

$$\omega_n \langle a_n | F_n Z = \omega_{n+1} \langle a_n | Z F_{n+1}. \quad (78)$$

Formulae (74) and (75) follow from (67) and the fact that for any bra-vector (i.e.  $N$ -row)  $\langle u |$  and any ket-vector (i.e.  $N$ -column)  $|v\rangle$ ,  $\det(I + |v\rangle\langle u|) = 1 + \langle u|v\rangle$ , while (77) and (78) can be obtained from (73) by multiplying it by  $|\ell\rangle$  and  $\langle a_n|$ . Formulae (72)–(78) are all we need in the following consideration.

## 4.2 Auxiliary sytem.

Now we are ready to solve auxiliary equations

$$\bar{\partial}_1 \ln \frac{\omega_{n+1}}{\omega_{n-1}} = \frac{\omega_n^2}{\omega_{n+1}\omega_{n-1}} - 1 \quad (79)$$

and

$$\bar{\partial}_{j+1} \ln \frac{\omega_{n+1}}{\omega_{n-1}} = -\frac{\omega_n^2}{\omega_{n+1}\omega_{n-1}} \bar{\partial}_{1j} \ln \omega_n. \quad (80)$$

Solutions of these equations then can be easily modified to become solutions of the negative VH under the non-vanishing boundary conditions  $u_n \rightarrow u_\infty$  as  $n \rightarrow \pm\infty$

The crucial point of the ansatz (62) is that dependence on times  $\bar{t}_1, \bar{t}_2, \dots$  is incorporated in the  $N$ -rows  $\langle a_n | = \langle a_n(\bar{t}_1, \bar{t}_2, \dots) |$ . Moreover, we assume (which is usual for pure soliton solutions) that  $\ln a_{nk}$  is a *linear* function of times. Hence, differentiating the  $A$ -matrices leads to multiplication from the right by some constant matrices,  $C_j$ ,

$$\frac{\partial}{\partial \bar{t}_j} A_n = A_n C_j \quad (81)$$

which should be determined. As follows from (81),

$$\frac{\partial}{\partial \bar{t}_j} \ln \omega_n = \text{tr } F_n A_n C_j = \text{tr } (I - F_n) C_j \quad (82)$$

which gives

$$\frac{\partial}{\partial \bar{t}_j} \ln \frac{\omega_{n+1}}{\omega_{n-1}} = \text{tr } (F_{n-1} - F_{n+1}) C_j. \quad (83)$$

Using (72) and the fact that  $R$  and  $C_j$  commute, the last equation can be rewritten as

$$\frac{\partial}{\partial \bar{t}_j} \ln \frac{\omega_{n+1}}{\omega_{n-1}} = \text{tr } (R^{-1} F_{n-1} - F_{n+1} R^{-1}) C_j R \quad (84)$$

$$= -\langle a_{n-1} | F_{n-1} C_j R F_{n+1} | \ell \rangle \quad (85)$$

On the other hand, by multiplying (74) and (75) one can derive the identity

$$\frac{\omega_n^2}{\omega_{n-1}\omega_{n+1}} = 1 - \langle a_{n-1} | F_{n-1} X_n F_{n+1} | \ell \rangle \quad (86)$$

where

$$X_n = Z(I + A_{n+1}) - (I + A_{n-1})RZ + Z|\ell\rangle\langle a_n|Z \quad (87)$$

Applying (72) one gets

$$X_n = (I - R)Z = \text{constant} \quad (88)$$

i.e.

$$\frac{\omega_n^2}{\omega_{n-1}\omega_{n+1}} - 1 = \langle a_{n-1} | F_{n-1}(R - I)ZF_{n+1} | \ell \rangle \quad (89)$$

Comparing (85) and the last formula one can conclude that  $\omega_n$  solve (79) if

$$C_1 = (R^{-1} - I)Z \quad (90)$$

or

$$C_1 = (I - R)(I + R)^{-1}. \quad (91)$$

Now our aim is to solve (80). Differentiating (82) with respect to  $\bar{t}_1$  we get

$$\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \omega_n = \text{tr } A_n F_n C_1 F_n C_j. \quad (92)$$

From definition (91) of  $C_1$  and (69) one can derive

$$A_n C_1 = ZA_n - A_n Z + Z|\ell\rangle\langle a_{n-1}|Z \quad (93)$$

which leads to

$$A_n F_n C_1 F_n = F_n Z - ZF_n + F_n Z|\ell\rangle\langle a_{n-1}|ZF_n. \quad (94)$$

Noting that  $Z$  and  $C_j$  commute and that the trace of a commutator is zero, the right-hand side of (92) can be rewritten as

$$\text{tr } A_n F_n C_1 F_n C_j = \langle a_{n-1} | ZF_n C_j F_n Z | \ell \rangle \quad (95)$$

which gives, together with (77) and (78),

$$\frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \omega_n = \frac{\omega_{n-1}\omega_{n+1}}{\omega_n^2} \langle a_{n-1} | F_{n-1} C_j Z^2 F_{n+1} | \ell \rangle. \quad (96)$$

Comparing this expression and equation (85) for the derivative with respect to the  $(j+1)$ th time,  $\bar{t}_{j+1}$ , one can conclude that  $\omega_n$  is a solution of (80), if matrices  $C_{j+1}$  and  $C_j$  are related by

$$C_{j+1} = C_j Z^2 R^{-1}. \quad (97)$$

This recurrence together with the 'initial condition' (91) can be easily solved:

$$C_j = C_1 (R + R^{-1} + 2I)^{1-j} = C_1 (R^{1/2} + R^{-1/2})^{2-2j}. \quad (98)$$

In such a way we have shown that if the time-dependence of the  $A$ -matrices is given by (81) and (98), then the quantities  $\omega_n$  given by (76) solve (80).

### 4.3 Dark-soliton solutions of the VH.

It is easy to see that equations (79) and (80) differ only in a constant term on the right-hand side of (79), which can be taken into account by  $\exp(n\bar{t}_1/2)$  multiplier: functions

$$\tau_n = \exp\left(\frac{n\bar{t}_1}{2}\right) \omega_n \quad (99)$$

solve (25). The limiting values of the corresponding  $u$ -functions are unity and to meet general 'finite-density' boundary conditions,

$$\lim_{n \rightarrow \pm\infty} u_n = u_\infty \quad (100)$$

one has to add the  $u_\infty^{n^2/4}$  factor and to rescale the times,  $\bar{t}_j \rightarrow \bar{t}_j/u_\infty^{j-1/2}$ . The final formulae for the  $N$ -soliton solutions of the VH can be written as

$$\tau_n = u_\infty^{n^2/4} \exp\left(\frac{n\bar{t}_1}{2\sqrt{u_\infty}}\right) \det \left| 1 + \frac{R_k^n a_k(\bar{t})}{R_j^{-1} - R_k} \right|_{j,k=1,\dots,N}. \quad (101)$$

The time dependence of  $a_k$  on negative times is given by

$$a_k(\bar{t}) = a_k(\bar{t}_1, \bar{t}_2, \dots) = a_k^{(0)} \exp\left(\sum_{j=1}^{\infty} \nu_k^{(j)} \bar{t}_j\right) \quad (102)$$

with

$$\nu_k^{(j)} = \frac{1}{\sqrt{u_\infty}} \frac{1 - R_k}{1 + R_k} \left[ u_\infty (2 + R_k + R_k^{-1}) \right]^{1-j} \quad (103)$$

where  $R_k, a_k^{(0)}, k = 1, \dots, N$  are arbitrary constants. Note that constants  $\ell_j$  appearing in (62) have been incorporated in  $a_k^{(0)}$  by the transform  $A_n \rightarrow M^{-1}A_nM$  with  $M = \text{diag}(\ell_1, \dots, \ell_N)$  which does not change determinants.

## 5 Conclusion.

In the present paper we studied the negative Volterra flows. The main result of this work is that even in the theory of, so to say, classical systems there are some questions which have not been discussed in the literature yet. We applied the widely used zero curvature approach to the well-known scattering problem and obtained some results which seem to be new. Deriving equations (24) and (25) or (26) and (27) is not a very difficult problem, though it fills some gaps in the theory of one of the oldest integrable systems. However, a hierarchy is more than a set of equations, and one can hardly enumerate all equations which are contained in this or that hierarchy, i.e. to study all possible differential consequences of equations of a given hierarchy. In section 3 we demonstrated that from the extended VH, which is a set of differential-difference equations, some of which are nonlocal, one can extract rather different systems. For example, in our opinion it is very difficult to say *a priori* that equation (61) (a vector partial differential equation in 3+1 dimensions) has any relation to the Volterra chain. In future papers we are going to present some other integrable systems which can be 'embedded' into (or extracted from) the extended VH, and we hope that the results presented above demonstrate that the theory of integrable systems is far from finished.

## Acknowledgements

VEV is supported by Ministerio de Educación, Cultura y Deporte under grant SAB2000-0256.

## Appendix.

The question of compatibility of the equations of a hierarchy (or, in other words, of commutativity of the corresponding flows) is of vital importance and usually not trivial. The most straightforward approach to this problem is to derive the Hamiltonians generating these flows and to show that they are in involution. However, the Hamiltonian representation of the negative Volterra hierarchy is still unknown and is a subject of future studies. Thus here we do not discuss this rather serious problem in the most general way and restrict ourselves to proving the commutativity of the first (classical) Volterra flow (3) and all the negative flows derived in this paper, a fact which is crucial for what has been presented in section 3.

This claim can be verified directly in the simplest cases. Let us first consider the Volterra equation (3), rewriting it now as

$$\frac{\partial}{\partial t_1} \ln \frac{\tau_n}{\tau_{n-1}} = u_n, \quad (104)$$

which in its turn leads to

$$\frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = u_{n+1} + u_n, \quad (105)$$

and show that it is compatible with (25),

$$\frac{\partial}{\partial \bar{t}_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{1}{p_n}. \quad (106)$$

To do this we need auxiliary formulae for the derivatives of  $u_n$  and  $p_n$ . The first one (which follows from (106)) is

$$\frac{\partial}{\partial \bar{t}_1} \ln u_n = \frac{\partial}{\partial \bar{t}_1} \left( \ln \frac{\tau_{n+1}}{\tau_{n-1}} - \ln \frac{\tau_n}{\tau_{n-2}} \right) = \frac{1}{p_n} - \frac{1}{p_{n-1}} \quad (107)$$

while the second one is a consequence of (104):

$$\frac{\partial}{\partial t_1} \ln p_n = \frac{\partial}{\partial t_1} \left( \ln \frac{\tau_{n+1}}{\tau_n} - \ln \frac{\tau_n}{\tau_{n-1}} \right) = u_{n+1} - u_n. \quad (108)$$

Now it is easy to derive

$$\frac{\partial}{\partial \bar{t}_1} \left( \frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \right) = u_{n+1} \left( \frac{1}{p_{n+1}} - \frac{1}{p_n} \right) + u_n \left( \frac{1}{p_n} - \frac{1}{p_{n-1}} \right) = -p_{n+1} + p_{n-1} \quad (109)$$

(we have used (107) and the fact that  $u_n = p_n p_{n-1}$ ).

On the other hand, it follows from (106) and (108)

$$\frac{\partial}{\partial t_1} \left( \frac{\partial}{\partial \bar{t}_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \right) = \frac{1}{p_n} (u_n - u_{n+1}) = p_{n-1} - p_{n+1}. \quad (110)$$

Since the right-hand sides of (109) and (110) are equal, one can conclude that

$$\frac{\partial}{\partial \bar{t}_1} \frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial}{\partial t_1} \frac{\partial}{\partial \bar{t}_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \quad (111)$$

which proves the compatibility of (104) and (106).

Further we proceed by induction. Suppose that Volterra flow (104) commutes with all the negative flows  $\partial/\partial \bar{t}_1, \partial/\partial \bar{t}_2, \dots, \partial/\partial \bar{t}_j$ . Now our aim is to show that it commutes with the  $(j+1)$ th negative flow as well. Indeed, differentiating equation (24), which can be rewritten as

$$\frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = -\frac{1}{p_n} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n \quad (112)$$

with respect to  $t_1$ , one can obtain

$$\begin{aligned} \frac{\partial}{\partial t_1} \left( \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \right) &= -\frac{\partial}{\partial t_1} \left( \frac{1}{p_n} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n \right) \\ &= -\left( \frac{\partial}{\partial t_1} \frac{1}{p_n} \right) \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n - \frac{1}{p_n} \frac{\partial}{\partial \bar{t}_j} \left( \frac{\partial^2}{\partial t_1 \partial \bar{t}_1} \ln \tau_n \right) \end{aligned}$$

$$= (p_{n+1} - p_{n-1}) \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n + \frac{1}{p_n} \frac{\partial}{\partial \bar{t}_j} p_n \quad (113)$$

Here we have used the commutativity of  $\partial/\partial t_1$  and  $\partial/\partial t_j$  as well as the identity

$$\frac{\partial^2}{\partial t_1 \partial \bar{t}_1} \ln \tau_n = -p_n \quad (114)$$

which can be derived by differentiating (106) with respect to  $t_1$  (see section 3.1). Noting that

$$\frac{\partial}{\partial \bar{t}_j} p_n = -p_n^2 \frac{\partial}{\partial \bar{t}_j} \frac{1}{p_n} = -p_n^2 \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \quad (115)$$

one can rewrite (113) as

$$\frac{\partial}{\partial t_1} \left( \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \right) = (p_{n+1} - p_{n-1}) \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n + p_n \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \frac{\tau_{n-1}}{\tau_{n+1}}. \quad (116)$$

On the other hand,

$$\frac{\partial}{\partial \bar{t}_{j+1}} \left( \frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \right) = \frac{\partial}{\partial \bar{t}_{j+1}} (u_{n+1} + u_n). \quad (117)$$

One can obtain from (112) the following formula for the derivative of  $u_n$ :

$$\begin{aligned} \frac{\partial}{\partial \bar{t}_{j+1}} u_n &= p_n p_{n-1} \frac{\partial}{\partial \bar{t}_{j+1}} \left( \ln \frac{\tau_{n+1}}{\tau_{n-1}} - \ln \frac{\tau_n}{\tau_{n-2}} \right) \\ &= p_n \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_{n-1} - p_{n-1} \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n \end{aligned} \quad (118)$$

Substituting this expression in (117) leads to

$$\frac{\partial}{\partial \bar{t}_{j+1}} \left( \frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \right) = (p_{n+1} - p_{n-1}) \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \tau_n + p_n \frac{\partial^2}{\partial \bar{t}_1 \partial \bar{t}_j} \ln \frac{\tau_{n-1}}{\tau_{n+1}} \quad (119)$$

Comparing the right-hand sides of (116) and (119) one can easily see that they coincide, which means that

$$\frac{\partial}{\partial t_1} \frac{\partial}{\partial \bar{t}_{j+1}} \ln \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\partial}{\partial \bar{t}_{j+1}} \frac{\partial}{\partial t_1} \ln \frac{\tau_{n+1}}{\tau_{n-1}} \quad (120)$$

This leads by induction to the fact that all the negative Volterra flows  $\partial/\partial \bar{t}_j$  commute with the classical one,  $\partial/\partial t_1$ .



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