Symmetry Induced Heteroclinic Cycles in a CO_2 Laser

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Abstract

The conditions for the existence of heteroclinic connections between the transverse modes of a CO_2 laser whose setup has a perfect cylindrical symmetry are discussed by symmetry arguments for the cases of three, four and five interacting modes. Explicit conditions for the parameters are derived, which can guide observation of such phenomena.

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1 Introduction

Recent evidence of space-time complexity in lasers suggests the possibility of using them as "test-benches" for space time chaos theories [Arecchi *et al.*, 1990,1991,1999] [D'Angelo *et al.*, 1992, Tredicce *et al.*, 1991, Green *et al.*, 1990]. Most lasers used in applications operate with a single Gaussian transverse mode. However, several laboratory configurations dealt with many transverse modes simultaneously active within the optical cavity. The nonlinear coupling between them gives rise to a complicated dynamics, and there is hope that a careful study of the transition from a situation where only few modes are active to a situation with many active modes will shed light on the mechanisms of transition to space time chaotic states. High accuracy and short times involved in laser phenomena give to lasers a relevant role in the study of such kind of processes. Presently, there is a wide spectrum of experimental features observed in transverse laser patterns waiting for theoretical explanation. These range from configurations in which only a few modes are active to ones in which the interaction among many modes leads to a loss of space and time correlation. An example in the first group appears in a laser system with photorefractive materials [Arecchi *et al.*, 1990]. There, the competition between different transverse modes can give rise to periodic or chaotic alternation between them. This phenomenon has been theoretically explained [Arecchi *et al.*, 1992] by symmetry arguments pointing to the existence of heteroclinic connections involving the first three transverse modes, with the assumption that a subcritical bifurcation is responsible for the birth of the fundamental Gaussian mode.

In this paper we consider explicitely a laser system in which the transverse modes are all born supercritically. Our main aim is to discuss the possibility for obtaining heteroclinic cycles when three, four or five modes are active in a CO_2 laser. The paper is organized as follows. In section 2 the existence of heteroclinic connections in dynamical systems with symmetry is reviewed. In section 3, the particular laser system under study is described. The analysis of the mode equations for this system for a perfect symmetry is done in section 4. The conditions that the coefficients of those equations must obey in order to have some kind of heteroclinic connection in a CO_2 laser system are obtained. The main conclusions and a short discussion on possible experiments are gathered in section 5.

2 Symmetry Induced Heteroclinic Cycles

Let P_1, P_2,..., P_n be fixed points of a system of ODEs. A heteroclinic cycle (HC) is a set of trajectories $Q_1(t)$,$Q_2(t)$, ..., $Q_n(t)$ such that

$$\lim_{t\to\infty} Q_k = P_{k+1}, \tag{1}$$

$$\lim_{t\to-\infty} Q_k = P_k, \tag{2}$$

$$P_{n+1} = P_1. \tag{3}$$

Such a connection between invariant sets is non generic for systems of ODEs. But if the system has some symmetries, it has been shown that HCs can be structurally stable [Field, 1980, Melbourne *et al.*, 1989]. The basic idea is that in the presence of symmetries, there are subspaces invariant under the flow. The role of these flow invariant subspaces in building HCs is determinant. Take, for instance, a particular example relevant for the system discussed in this paper. Let

$$\dot{x} = f(x) \tag{4}$$

be a three dimensional dynamical system with $x \in R^3$, Π_i, $i = 1, 2, 3$, three invariant planes and $Q_i(t)$, $i = 1, 2, 3$, three trajectories connecting the invariant planes (see Figure 1). Suppose that the saddle fixed points P_i, $i = 1, 2, 3$, are the asymptotic limits of the trajectories $Q_i(t)$ such as indicated in the relations (1-2). Thus, the unstable set of the fixed point P_i is contained in the invariant plane Π_i and its stable set is contained in the invariant plane Π_{i-1}. Therefore, we have a heteroclinic connection between the points P_i, $i = 1, 2, 3$, forced, in this case, by the symmetries of the system and the boundedness of the trajectories. Let us assume that the planes are induced by subgroups of the symmetry group of the problem. If the system is perturbed with a term equivariant under the same group, it will have the same invariant planes, and, as saddle-sink connections are structurally stable in R^2, the cycle will persist. The situation is quite different if the system is perturbed with a non equivariant term. In that case, if the flow invariant planes are destroyed, one can break the heteroclinic cycle and induce the appearance of periodic orbits, giving rise to the phenomenon of periodic alternation, as it has been recently shown, for instance, in a rotating convection system [Lin & Vivancos, 2002]. We are interested in the possibility of a periodic alternation among modes with different angular momenta for a CO_2 laser in a Fabry-Perot cavity. In a perfect setup, the boundary conditions of this system (cylindrical tube) imposes an $O(2)$ symmetry. It has been shown that slight imperfections in the system (a non perfect coaxial discharge, slight disalignments of the mirrors, etc.) can break the rotational component of $O(2)$, leaving just the Z_2 component (reflections) [D'Angelo *et al.*, 1992][López-Ruiz *et al.*, 1993,1994]. The existence of heteroclinic cycles connecting invariant sets of a $O(2)$ equivariant system was studied by Melbourne *et al.* [Melbourne *et al.*, 1989]. Those cycles were found in the truncated normal forms of Hopf-Hopf mode interactions. In the following section the normal form describing the interactions between modes in the CO_2 laser system is presented and the relationships among the coefficients to have heteroclinic cycles are determined.

3 The System

The system under study is a CO_2 laser in a Fabry-Perot cavity [Solari & Gilmore, 1990]. The cavity has a perfectly reflecting plane mirror at one end (z=0) and a curved mirror with partial reflectivity at the other (z=-1). Physically, the effective curvature of this mirror can be modified inserting a passive optical device. As the pumping strength is changed, different modes will be excited. The resulting dynamics can be studied by reducing the Maxwell-Bloch equations to evolution equations for the amplitudes of the active modes. This reduction procedure is made in two steps. First, the electric field, the polarization and the population inversion are expanded in terms of left and right longitudinal modes ϕ^+, ϕ^-,

$$
\begin{aligned}
E &= (E)e^{iwt}(e^{-\chi z}\phi^+\Phi_+ + e^{\chi z}\phi^-\Phi_-)(s^2+z^2)^{1/2}, && (5)\\
P &= i(P)e^{iwt}\sum[(\phi^+)^n\Pi_n + (\phi^-)^n\Pi_{-n}](s^2+z^2)^{1/2}, && (6)\\
D &= (D)(D_0 + \sum[D_n(\phi^+)^n + D_{-n}(\phi^-)^n]), && (7)
\end{aligned}
$$

where $(E), (P), (D)$ stand for scale factors, χ is the rate of cavity losses, s is a constant that depends on the curvature of the mirrors and z indicates the axial direction. Evolution equations for the slowly varying amplitudes in that expansion are derived. They read

$$
\begin{aligned}
i\frac{\partial}{\partial t}\begin{bmatrix}\Phi_+\\ \Phi_-\end{bmatrix} &= \begin{bmatrix}H^+ + i\chi & 0\\ 0 & H^- - i\chi\end{bmatrix}\begin{bmatrix}\Phi_+\\ \Phi_-\end{bmatrix} - i\begin{bmatrix}\Pi_+\\ \Pi_-\end{bmatrix}, && (8)\\
\begin{bmatrix}\Pi_+\\ \Pi_-\end{bmatrix} &= \begin{bmatrix}D_0 & D_2\\ D_{-2} & D_0\end{bmatrix}\begin{bmatrix}R^+\Phi_+\\ R^-\Phi_-\end{bmatrix}, && (9)\\
I_2 + 1/\gamma\frac{\partial}{\partial t}\begin{bmatrix}D_0 & D_2\\ D_{-2} & D_0\end{bmatrix} &= KI_2 + \left\{\begin{bmatrix}\Phi_+\\ \Phi_-\end{bmatrix}\begin{bmatrix}\Pi_+^* & \Pi_-^*\end{bmatrix} + \right.\\
&\quad + \begin{bmatrix}\Pi_+\\ \Pi_-\end{bmatrix}\begin{bmatrix}\Phi_+^* & \Phi_-^*\end{bmatrix} + && (10)\\
&\quad + \left.\begin{bmatrix}\Phi_-\Pi_-^* + \Phi_-^*\Pi_- & 0\\ 0 & \Phi_+\Pi_+^* + \Phi_+^*\Pi_+\end{bmatrix}\right\}(s^2+z^2)^{-1},\\
R^{+-} &= (\beta - iH^{+-})^{-1}, && (11)
\end{aligned}
$$

where γ is the rate of decay of the population inversion, β is the atomic rate of decay of the atomic polarization, K is the pumping profile, H^{+-} are 3D differential operators and I_2 is the 2D identity matrix. The trivial solution of these equations is

$$
\begin{aligned}
\Phi_{+-} &= 0, && (12)\\
\Pi_{+-} &= 0, && (13)\\
D_{+-2} &= 0, && (14)
\end{aligned}
$$

and

$$D_0 = K. \qquad (15)$$

For certain values of the parameters, different cavity modes are born from a Hopf bifurcation of this trivial solution. The electric field amplitude can be expanded as a combination of those cavity mode amplitudes:

$$\left[\begin{array}{c}\Phi_+\\ \Phi_-\end{array}\right] = \sum z_\mu \left[\begin{array}{c}a_+^\mu\\ a_-^\mu\end{array}\right]. \tag{16}$$

Introducing this expansion into the Maxwell-Bloch equations (8-11) one arrives to a set of evolution equations for the cavity mode amplitudes:

$$\dot{z}_\alpha = L_{\alpha\beta} z_\beta + M_{\alpha\mu\nu\beta} z_\mu z_\nu^* z_\beta + h.o.t. \tag{17}$$

The coefficients in these equations can be explicitly computed:

$$\begin{aligned}
L_{\alpha\beta} &= \delta_{\alpha\beta}((-\chi + \frac{K\beta}{\beta^2+\Omega_\alpha^2}) - i\Omega_\alpha(1 - \frac{K}{\beta^2+\Omega_\alpha^2})), && (18)\\
M_{\alpha\mu\nu\beta} &= -G_{\alpha\mu\nu\beta} D_{\alpha\mu\nu\beta}, && (19)\\
D_{\alpha\mu\nu\beta} &= \frac{1}{1+i(\Omega_\mu - \Omega_\nu)/\gamma}(\frac{1}{\beta - i\Omega_\mu} + (\frac{1}{\beta - i\Omega_\nu})^*) \times && (20)\\
&\quad \times \frac{1}{\beta - i\Omega_\beta},\\
G_{\alpha\mu\nu\beta} &= \int((a_+^{\alpha*}a_+^\mu + a_-^{\alpha*}a_-^\mu)(a_+^{\nu*}a_+^\beta + a_-^{\nu*}a_-^\beta) + && (21)\\
&\quad +(a_+^{\alpha*}a_+^\mu a_+^{\nu*}a_+^\beta + a_-^{\alpha*}a_-^\mu a_-^{\nu*}a_-^\beta))\frac{K}{s^2+z^2}dv,
\end{aligned}$$

with Ω_α the slow temporal frequency of the empty cavity modes.
In the following section the relationships between the coefficients of these equations to have a HC are derived.

4 Solutions of the Amplitude Equations

Now the conditions for the existence of heteroclinic solutions in the CO_2 laser system are discussed. If the Gaussian mode interacts with the modes of angular momenta ±1, HCs are not possible. This result is generic for $O(2)$ equivariant systems when all modes are born supercritically from the origin. Then one must go one step further considering four modes (of angular momentum ±1, ±2) interacting in the cavity. It is interesting to analyze this case because this is the simplest situation in which heteroclinic cycles can be built generically for $O(2)$ equivariant systems. However, due to the specific form of the physical equations we prove that HC are not possible in a CO_2 laser in this case. The next step is to consider five modes with angular momenta 0, ±1, ±2. The restrictions among the coefficients of the normal form to have a HC are determined. We separate the discussion of each case in separated subsections.

4.1 Three Mode Interaction

The equations describing the interaction between the modes with angular momenta 0, ±1 near the Hopf-Hopf bifurcation (the parameter value for which the primary branches in which the modes are born cross each other) are:

$$\dot{z}_1 = \lambda_1 z_1 - (A(z_1 z_1^* + 2z_2 z_2^*) + D z_0 z_0^*) z_1, \tag{22}$$

$$\dot{z}_2 = \lambda_1 z_2 - (A(2z_1 z_1^* + z_2 z_2^*) + D z_0 z_0^*) z_2, \tag{23}$$

$$\dot{z}_0 = \lambda_0 z_0 - (E(z_1 z_1^* + z_2 z_2^*) + B z_0 z_0^*) z_0. \tag{24}$$

The coefficients A, B, E, D have been explicitly computed in the previous section. Setting $z_i = \rho_i e^{i\phi_i}$, the amplitude equations read

$$\dot{\rho}_1 = \lambda_1^r \rho_1 - (A^r(\rho_1^2 + 2\rho_2^2) + D^r \rho_0^2) \rho_1, \tag{25}$$

$$\dot{\rho}_2 = \lambda_1^r \rho_2 - (A^r(2\rho_1^2 + \rho_2^2) + D^r \rho_0^2) \rho_2, \tag{26}$$

$$\dot{\rho}_0 = \lambda_0^r \rho_0 - (E^r(\rho_1^2 + \rho_2^2) + B^r \rho_0^2) \rho_0. \tag{27}$$

The codimension two bifurcation diagram, $(\lambda_0^r, \lambda_1^r)$, for these equations can be found in figure 1 in Ref. [López-Ruiz *et al.*, 1993]. In that figure, projections into the amplitude subspaces are shown for different regions of the parameter space. Only the real part of the amplitudes must be considered because the dynamics of the phases is trivial. Notice that the dimension of the amplitude space is three, that there are three invariant planes (each defined by ρ_i=0) and one fixed point in each of the three invariant lines. But heteroclinic connections cannot be established between these fixed points due to the residual symmetry that remains in the plane (ρ_1, ρ_2)

after projecting in the amplitude space. This symmetry is given by the invariance of the equations (25-27) under the variable change: $(\rho_0, \rho_1, \rho_2) \to (\rho_0, \rho_2, \rho_1)$. If the flow is projected on the plane (ρ_1, ρ_2) the dynamics derives towards one of the modes, ρ_1 or ρ_2 (or $\rho_1 = \rho_2$ if these amplitudes were the same at the beginning). In this situation one of theses modes interacts with ρ_0 and there is no possibility to construct a HC because the origin is a repulsor and there are only two degrees of freedom (fig. 2).

If one assumes a subcritical bifurcation for birth of the fundamental mode ($\lambda_0^r < 0$ in Eqs. (25-27), this allows to obtain heteroclinic connections between the three transverse modes considered here [Arecchi *et al.*, 1992]. Since however we are interested to study the CO_2 laser, this cannot be assumed; on the contrary, all modes are born supercritically from the origin and thus all λ's have a positive real part.

4.2 Four Mode Interaction

Once excluded the interaction among three modes that bifurcate supercritically we analyse the possibility of a heteroclinic connection among four modes with angular momenta ± 1, ± 2.

According to Eq. (17), the normal form describing the interaction between four modes of angular momenta ± 1, ± 2 are

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 - (A(z_1 z_1^* + 2 z_2 z_2^*) + B(z_3 z_3^* + z_4 z_4^*)) z_1, && (28)\\
\dot{z}_2 &= \lambda_1 z_2 - (A(2 z_1 z_1^* + z_2 z_2^*) + B(z_3 z_3^* + z_4 z_4^*)) z_2, && (29)\\
\dot{z}_3 &= \lambda_2 z_3 - (D(z_3 z_3^* + 2 z_4 z_4^*) + C(z_1 z_1^* + z_2 z_2^*)) z_3, && (30)\\
\dot{z}_4 &= \lambda_1 z_4 - (D(2 z_3 z_3^* + z_4 z_4^*) + C(z_3 z_3^* + z_4 z_4^*)) z_4. && (31)
\end{aligned}$$

The behaviour of this system can be analyzed setting $z_i = \rho_i e^{i\phi_i}$ and writing for the real part of the amplitudes the following equations,

$$\dot{\rho}_1 = \lambda_1^r \rho_1 - (A^r(\rho_1^2 + 2\rho_2^2) + B^r(\rho_3^2 + \rho_4^2))\rho_1, \qquad (32)$$

$$\dot{\rho}_2 = \lambda_1^r \rho_2 - (A^r(2\rho_1^2 + \rho_2^2) + B^r(\rho_3^2 + \rho_4^2))\rho_2, \quad (33)$$

$$\dot{\rho}_3 = \lambda_2^r \rho_3 - (C^r(\rho_3^2 + 2\rho_4^2) + D^r(\rho_1^2 + \rho_2^2))\rho_3, \quad (34)$$

$$\dot{\rho}_4 = \lambda_1^r \rho_4 - (C^r(2\rho_3^2 + \rho_4^2) + D^r(\rho_3^2 + \rho_4^2))\rho_4. \quad (35)$$

Let us observe the residual symmetry in the amplitude space: $(\rho_1, \rho_2, \rho_3, \rho_4) \to (\rho_2, \rho_1, \rho_4, \rho_3)$. The same line of reasoning of the preceding subsection can be followed: the symmetry in the plane $(\rho_1, \rho_2) \to (\rho_2, \rho_1)$ reduces the dynamics to one of these modes. But the interaction of this mode with (ρ_3, ρ_4) still presents the residual symmetry $(\rho_i, \rho_3, \rho_4) \to (\rho_i, \rho_4, \rho_3)$. Returning to the case of three interacting modes discussed in section 4.1 we also conclude that there is no way to construct a heteroclinic cycle among these modes. Let us remark that symmetry in these equations is crucial in avoiding a heteroclinic connection. Thus the dynamics allows to connect the plane (ρ_1, ρ_2) with the plane (ρ_3, ρ_4), but it dies there without any possibility to return to the first level. Therefore, at least, another mode must be active in order to allow the system to come back to the initial level (in some sense a 'pump' mode). In this form a heteroclinic cycle could be obtained with a minimun of five modes.

4.3 Five Mode Interaction

Thus, we analyse the interaction between modes with angular momenta 0, ± 1, ± 2 in a system with perfect $O(2)$ symmetry. In this case symmetries could induce heteroclinic cycles. The bifurcation equations for this case read as follow:

$$\dot{z}_0 = \lambda_0 z_0 - (A(z_1 z_1^* + z_2 z_2^*) + B(z_3 z_3^* + z_4 z_4^*) + C z_0 z_0^*) z_0, \quad (36)$$

$$\dot{z}_1 = \lambda_1 z_1 - (D(z_1 z_1^* + 2 z_2 z_2^*) + F(z_3 z_3^* + z_4 z_4^*) + E z_0 z_0^*) z_1, \quad (37)$$

$$\dot{z}_2 = \lambda_1 z_2 - (D(2 z_1 z_1^* + z_2 z_2^*) + F(z_3 z_3^* + z_4 z_4^*) + E z_0 z_0^*) z_2, \quad (38)$$

$$\dot{z}_3 = \lambda_2 z_3 - (G(z_3 z_3^* + 2 z_4 z_4^*) + I(z_1 z_1^* + z_2 z_2^*) + H z_0 z_0^*) z_3, \quad (39)$$

$$\dot{z}_4 = \lambda_2 z_3 - (G(2 z_3 z_3^* + z_4 z_4^*) + I(z_1 z_1^* + z_2 z_2^*) + H z_0 z_0^*) z_4. \quad (40)$$

A complete study of this set of equations is beyond the scope of this paper. We focus our attention to the possibility of obtaining the situation displayed in figure

3. Let us first find the restrictions in the coefficients to have a connection between the rotating waves (RW) of two interacting modes (Fig. 4):

$$\dot{z}_1 = \lambda_1 z_1 - (A_1 z_1 z_1^* + A_1' z_2 z_2^*) z_1, \tag{41}$$

$$\dot{z}_2 = \lambda_2 z_2 - (A_2 z_2 z_2^* + A_2' z_1 z_1^*) z_2. \tag{42}$$

After projecting in the amplitudes space ($z_i = \rho_i e^{i\phi_i}$) these lead to:

$$\dot{\rho}_1 = \lambda_1^r \rho_1 - A_{1R}\rho_1^3 - A_{1R}'\rho_2^2\rho_1, \tag{43}$$

$$\dot{\rho}_2 = \lambda_2^r \rho_2 - A_{2R}\rho_2^3 - A_{2R}'\rho_1^2\rho_2. \tag{44}$$

To obtain a connection as in Fig. 4 the unstable manifold of $RW_1 \equiv (\sqrt{\frac{\lambda_1^r}{A_{1R}}}, 0)$ must coincide with the stable manifold of $RW_2 \equiv (0, \sqrt{\frac{\lambda_2^r}{A_{2R}}})$. This is reached by imposing that the eigenvalues of the Jacobian matrix are one positive (in RW_1) and another negative (in RW_2). As the flow is bounded and there exist no fixed point in the interior of the plane (ρ_1, ρ_2), the unstable manifold of RW_1 is the stable manifold of RW_2. The following conditions must hold:

$$\frac{\lambda_2^r}{\lambda_1^r} > \frac{A_{2R}'}{A_{1R}}, \tag{45}$$

$$\frac{\lambda_1^r}{\lambda_2^r} < \frac{A_{1R}'}{A_{2R}}. \tag{46}$$

These considerations are now extended to the five mode interaction in order to find the conditions to have a global heteroclinic connection of the type (Fig. 5):

$$\rho_0 \to plane(\rho_3, \rho_4) \to plane(\rho_1, \rho_2) \to \rho_0 \tag{47}$$

These conditions are:

$$\rho_0 \to (\rho_3, \rho_4) \quad \begin{cases} \frac{\lambda_2^r}{\lambda_0^r} > \frac{H^r}{C^r} \\ \frac{\lambda_0^r}{\lambda_2^r} < \frac{B^r}{G^r} \end{cases} \tag{48}$$

$$(\rho_3, \rho_4) \to (\rho_1, \rho_2) \quad \begin{cases} \frac{\lambda_1^r}{\lambda_2^r} > \frac{F^r}{G^r} \\ \frac{\lambda_2^r}{\lambda_1^r} < \frac{I^r}{D^r} \end{cases} \tag{49}$$

$$(\rho_1, \rho_2) \to \rho_0 \quad \begin{cases} \frac{\lambda_0^r}{\lambda_1^r} > \frac{A^r}{D^r} \\ \frac{\lambda_1^r}{\lambda_0^r} < \frac{E^r}{C^r} \end{cases} \tag{50}$$

Two different situations can be distinguished. The first one corresponds to $A^r < 0, H^r < 0$ and $F^r < 0$. In such a case, condition (45) is automatically verified for the three connections and those relationships (48-50) reduce simply to

$$1 < \frac{B^r}{G^r}\frac{I^r}{D^r}\frac{E^r}{C^r}. \tag{51}$$

Let us call this case "strong" HC among transverse modes. The physical meaning of this case clearly appears looking at Eqs. (36-40). Indeed, selecting $A^r < 0, H^r < 0$ and $F^r < 0$, any growth of ρ_0 produces a source term in the equations for (ρ_3, ρ_4); any growth of (ρ_3, ρ_4) produces a source term in the (ρ_1, ρ_2) equations and any growth of (ρ_1, ρ_2) gives rise to a source term in the ρ_0 equation. So the cycle can close itself.

Guided by such a suggestion, the values of nonlinear coefficients of Eqs. (36-40) are calculated directly from Maxwell-Bloch equations following the procedure described in Sec. 3, and more explicit in the appendix of Ref. [López-Ruiz *et al.*, 1994], with the physical parameters of a CO_2 laser. Calculations show that any choice of those physical parameters leading to a negative value of A'_{2R} leads also to a negative value of A'_{1R}. Therefore, the conditions of a "strong" HC in a CO_2 laser cannot be fulfilled.

The second possibility will be called "weak" heteroclinic connection. This corresponds to a positive real part for all nonlinear coefficients of Eqs. (36-40). In such a case, the set of conditions (48-50) written above are equivalent to:

$$M \equiv Min_0 \; Min_1 \; Min_2 > 1, \tag{52}$$

where

$$Min_0 \equiv Min(\frac{C^r}{H^r}, \frac{B^r}{G^r}), \tag{53}$$

$$Min_1 \equiv Min(\frac{G^r}{F^r}, \frac{I^r}{D^r}), \tag{54}$$

$$Min_2 \equiv Min(\frac{D^r}{A^r}, \frac{E^r}{C^r}). \tag{55}$$

Calculations with realistic parameters under usual conditions in a CO_2 laser with a perfect $O(2)$ symmetry show numerical evidence that M has a maximum value equal to 1. Nevertheless, symmetry arguments do not exclude the possibility of finding heteroclinic cycles in modified setups, provided that condition (52) be satisfied.

5 Conclusions and discussion

The conditions for the existence of heteroclinic cycles in a CO_2 laser with a cylindrical symmetry have been obtained. From general symmetry arguments we proved that this kind of cycles are not possible when there are only three or four modes -arising from a supercritical bifurcation- interacting into the cavity.

At least five modes must interact to produce that kind of cycles. In this case the relationships that the coefficients must satisfy for having heteroclinic connections are found. Calculations for CO_2 laser systems show that those requirements can not be fulfilled under usual conditions.

Some modification in the nonlinear coupling of the Maxwell-Bloch equations is required, such as introducing a nonlinear passive device into the cavity that modifies the nonlinear coefficients.

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Figure Captions

1. Hipotetic sketch of a possible heteroclinic connection $\{Q_1, Q_2, Q_3\}$ for a 3D-symmetric system of ODEs. Each line $Q_i(t)$ connects the fixed points (P_i, P_{i+1}) and lies on the correspondent invariant plane Π_i.

2. 2D-projections of the flow in the amplitude space for the interaction in a CO_2 laser of modes 0 and ± 1.

3. A fifth mode, ρ_0, connecting the two planes of symmetry, $(\rho_1, \rho_2) - (\rho_3, \rho_4)$, of equations (32-35) is needed in order to build a heteroclinic connection.

4. A generic heteroclinic connection between the rotating waves of two interacting modes.

5. Heteroclinic connection established with five interacting modes in a CO_2 laser: $\rho_0 \to plane\,(\rho_3, \rho_4) \;\to\; plane\,(\rho_1, \rho_2) \;\to\; \rho_0$.