

# Supersymmetric Extensions of the Harry Dym Hierarchy

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## Abstract

We study the supersymmetric extensions of the Harry Dym hierarchy of equations. We obtain the susy-B extension, the doubly susy-B extension as well as the  $N=1$  and the  $N=2$  supersymmetric extensions for this system. The  $N=2$  supersymmetric extension is particularly interesting, since it leads to new classical integrable systems in the bosonic limit. We prove the integrability of these systems through the bi-Hamiltonian formulation of integrable models and through the Lax description. We also discuss the supersymmetric extension of the Hunter-Zheng equation which belongs to the Harry Dym hierarchy of equations.

# 1 Introduction:

Supersymmetric extensions of a number of well know bosonic integrable models have been studied extensively in the past. The supersymmetric Korteweg-de Vries (sKdV) equation [1], the supersymmetric nonlinear Schrödinger (sNLS) equation [2] and the supersymmetric Two-Boson (sTB) equation [3] represent just a few in this category. A simple supersymmetric covariantization of bosonic integrable models, conventionally known as the B supersymmetrization (susy-B), has also attracted a lot of interest because of the appearance of such models in string theories. We have, for instance, the B extensions of the KdV (sKdV-B) equation [4], the supersymmetric TB (sTB-B) equation [5] and so on. Supersymmetric extensions of integrable models using a number  $N$  of Grassmann variables greater than one [6] and supersymmetric construction of dispersionless integrable models [7] have also been studied extensively in the past few years. The extended supersymmetric models are particularly interesting because, in the bosonic limit, they yield new classical integrable systems.

A classic bosonic integrable equation, the so called Harry Dym (HD) equation [8], has attracted much interest recently. The proprieties of this equation are discussed in detail in Ref. [9], and we simply emphasize that this equation shares the properties typical of solitonic equations, namely, it can be solved by the inverse scattering transform, it has a bi-Hamiltonian structure and infinitely many symmetries. In fact, the HD equation is one of the most exotic solitonic equations and the hierarchy to which it belongs, has a very rich structure [10]. In this hierarchy we also have nonlocal integrable equations such as the Hunter-Zheng (HZ) equation [11], which arises in the study of massive nematic liquid crystals as well as in the study of shallow water waves. The HD equation, on the other hand, is relevant in the study of the Saffman-Taylor problem which describes the motion of a two-dimensional interface between a viscous and a non-viscous fluid [12].

An earlier attempt to supersymmetrize the HD equation is discussed in [13]. However, this study of  $N=1$  supersymmetrization introduces a bosonic as well as an independent fermionic superfield, yielding a pair of coupled equations, and, consequently, is not in the conventional spirit of minimal supersymmetrization. In this paper we intend to study the question of supersymmetrization of the HD hierarchy systematically. The paper is organized as follows. In section 2, we review some of the essential results for the HD equation and its hierarchy. The simpler susy-B extension (sHD-B) and the doubly B extension (sHD-BB) of the HD hierarchy as well their bi-Hamiltonian formulation and Lax pairs are described in section 3. In section 4, we derive the  $N=1$  supersymmetric extensions of the HD (sHD) equation. We find that, in this case, there exist two nontrivial  $N=1$  extensions. In the case of one of them, we have a bi-Hamiltonian description (we have not found a Lax representation yet) while in the second case, we have a Lax description (we have not found a Hamiltonian structure yet that satisfies the Jacobi identity). We also describe the supersymmetric extension for the HZ equation. In section 5, we describe the  $N=2$  supersymmetrization of the HD hierarchy which yields four possibilities and we discuss their properties. We end with a brief conclusion in section 6.

## 2 The Harry Dym Hierarchy:

The Harry Dym equation

$$w_t = (w^{-1/2})_{xxx} , \quad (1)$$

appears in many disguised forms, namely,

$$\begin{aligned} v_t &= \frac{1}{4}v^3v_{xxx} , \\ u_t &= \frac{1}{4}u^{3/2}u_{xxx} - \frac{3}{8}u^{1/2}u_xu_{xx} + \frac{3}{16}u^{-1/2}u_x^3 , \\ r_t &= (r_{xx}^{-1/2})_x , \end{aligned} \quad (2)$$

where  $v = -2^{1/3}w^{-1/2}$ ,  $u = v^2$  and  $r_{xx} = w$ , respectively. In this paper, as in [10], we will confine ourselves, as much as is possible, to the form of the HD equation as given in (1).

The HD equation is a member of the bi-Hamiltonian hierarchy of equations given by

$$w_t^{(n+1)} = \mathcal{D}_1 \frac{\delta H_{n+1}}{\delta w} = \mathcal{D}_2 \frac{\delta H_n}{\delta w} , \quad (3)$$

for  $n = -2$ , where the bi-Hamiltonian structures are

$$\begin{aligned} \mathcal{D}_1 &= \partial^3 , \\ \mathcal{D}_2 &= w\partial + \partial w , \end{aligned} \quad (4)$$

and the Hamiltonians for the HD equation are

$$\begin{aligned} H_{-1} &= \int dx \left( 2w^{1/2} \right) , \\ H_{-2} &= \int dx \left( \frac{1}{8}w^{-5/2}w_x^2 \right) . \end{aligned} \quad (5)$$

We note here that the second structure in (4) corresponds to the centerless Virasoro algebra while

$$\mathcal{D} = \mathcal{D}_2 + c\mathcal{D}_1 \quad (6)$$

represents the Virasoro algebra with a central charge  $c$ . We note also that the recursion operator following from (4),  $R = \mathcal{D}_2\mathcal{D}_1^{-1}$ , can be explicitly inverted to yield

$$R^{-1} = \frac{1}{2}\partial^3w^{-1/2}\partial^{-1}w^{-1/2} . \quad (7)$$

Also, the conserved charges

$$H_0 = - \int dx w ,$$

$$\begin{aligned}
H_0^{(1)} &= \int dx (\partial^{-1}w) , \\
H_0^{(2)} &= \int dx (\partial^{-2}w) ,
\end{aligned} \tag{8}$$

are Casimirs (or distinguished functionals) of the Hamiltonian operator  $\mathcal{D}_1$  (namely, they are annihilated by the Hamiltonian structure  $\mathcal{D}_1$ ). As a consequence of this, it is possible to obtain, in an explicit form, equations from (3) for integers  $n$  both positive and negative, i.e.,  $n \in \mathbb{Z}$ . As shown in [10], for  $n > 0$ , we have three classes of nonlocal equations. However, in this paper we will only study the hierarchy associated with the local Casimir  $H_0$  in (8). In this way, for  $n = 1$ , we obtain from (3), with the conserved charges

$$\begin{aligned}
H_1 &= \int dx \frac{1}{2}(\partial^{-1}w)^2 , \\
H_2 &= \int dx \frac{1}{2}(\partial^{-2}w)(\partial^{-1}w)^2 ,
\end{aligned} \tag{9}$$

the Hunter-Zheng (HZ) equation

$$w_t = -(\partial^{-2}w)w_x - 2(\partial^{-1}w)w , \tag{10}$$

which is also an important equation that belongs to the Harry Dym hierarchy.

The integrability of the HD equation (1) also follows from its nonstandard Lax representation

$$\begin{aligned}
L &= \frac{1}{w}\partial^2 , \\
\frac{\partial L}{\partial t} &= -2[B, L] ,
\end{aligned} \tag{11}$$

where

$$B = \left( L^{3/2} \right)_{\geq 2} = w^{-3/2}\partial^3 - \frac{3}{4}w^{-5/2}w_x\partial^2 . \tag{12}$$

Conserved charges, for  $n = 1, 2, 3, \dots$ , are obtained from

$$H_{-(n+1)} = \text{Tr} L^{\frac{2n-1}{2}} . \tag{13}$$

A Lax representation for the HZ equation (10) is also known and is given by (11) with

$$B = \frac{1}{4}(\partial^{-2}w)\partial + \frac{1}{4}\partial^{-1}(\partial^{-2}w)\partial^2 . \tag{14}$$

However, in this case, the operator  $B$  is not directly related to  $L$ , and, consequently, the Lax equation is not of much direct use (in the construction of conserved charges etc).

### 3 The Susy-B Harry Dym (sHD-B, sHD-BB) Equations:

The most natural generalization of an equation to a supersymmetric one is achieved simply by working in a superspace. We note, from the HD equation (1), that by a simple dimensional analysis, we can assign the following canonical dimensions to various quantities

$$[x] = -1, \quad [t] = 3, \quad \text{and} \quad [w] = 4. \quad (15)$$

The  $N=1$  supersymmetric equations are best described in the superspace parameterized by the coordinates  $z = (x, \theta)$ , where  $\theta$  represents the Grassmann coordinate ( $\theta^2 = 0$ ). In this space, we can define

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}, \quad (16)$$

representing the supercovariant derivative. From (16) it follows that

$$D^2 = \partial, \quad (17)$$

which determines the dimension of  $\theta$  to be

$$[\theta] = -\frac{1}{2}. \quad (18)$$

Let us introduce the fermionic superfield

$$W = \psi + \theta w, \quad (19)$$

which has the canonical dimension

$$[W] = [\psi] = \frac{7}{2}. \quad (20)$$

A simple supersymmetrization of a bosonic system, conventionally known as the B supersymmetric (susy-B) extension [4], is obtained by simply replacing the bosonic variable  $w$ , in the original equation, by

$$(DW) = w + \theta \psi', \quad (21)$$

where  $W$  represents a fermionic superfield. This leads to a manifestly supersymmetric equation and following this for the case of the equation (1), we obtain the susy-B HD (sHD-B) equation

$$W_t = \partial^2 D \left( (DW)^{-1/2} \right), \quad (22)$$

where  $W$  is the fermionic superfield (19).

This system is bi-Hamiltonian with the even Hamiltonian operators

$$\begin{aligned}\mathcal{D}_1 &= \partial^2, \\ \mathcal{D}_2 &= D(DW)D^{-1} + D^{-1}(DW)D,\end{aligned}\tag{23}$$

and the odd Hamiltonians (which follow from (5) under the substitution  $w \rightarrow (DW)$ )

$$\begin{aligned}H_{-1} &= \int dz 2(DW)^{1/2}, \\ H_{-2} &= \int dz \frac{1}{8}(DW)^{-5/2}(DW_x)^2.\end{aligned}\tag{24}$$

The Casimirs of  $\mathcal{D}_1$  can be easily identified with the ones following from (8).

The sHD-B equation (22) has two possible nonstandard Lax representations. Let

$$L = (DW)^{-1}D^4 + cW_x(DW)^{-2}D^3.\tag{25}$$

Then, it can be easily checked that the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = \left[ (L^{3/2})_{\geq 3}, L \right],\tag{26}$$

leads to the sHD-B equation (22) for  $c = 0, -1$ . Here the projection  $(\ )_{\geq 3}$  is defined with respect to the powers of the supercovariant derivative  $D$ .

For any given integrable bosonic equation, we can also define a doubly susy-B extension as follows. Just as we defined a superspace in the case of  $N = 1$  supersymmetry, let us define a superspace parameterized by  $z = (x, \theta_1, \theta_2)$ , where  $\theta_1, \theta_2$  define two Grassmann coordinates (anticomuting and nilpotent, namely,  $\theta_1\theta_2 = -\theta_2\theta_1$ ,  $\theta_1^2 = \theta_2^2 = 0$ ). In this case, we can define two supercovariant derivatives

$$\begin{aligned}D_1 &= \frac{\partial}{\partial \theta_1} + \theta_1 \frac{\partial}{\partial x}, \\ D_2 &= \frac{\partial}{\partial \theta_2} + \theta_2 \frac{\partial}{\partial x},\end{aligned}\tag{27}$$

which satisfy

$$D_1^2 = D_2^2 = \partial, \quad D_1 D_2 + D_2 D_1 = 0.\tag{28}$$

Such a superspace naturally defines a system with  $N = 2$  supersymmetry. Let us consider a bosonic superfield,  $W$ , in this space which will have the expansion (we denote it by the same symbol as in the case of  $N = 1$ )

$$W = w_0 + \theta_1 \chi + \theta_2 \psi + \theta_2 \theta_1 w_1.\tag{29}$$

Then, we can simply replace the bosonic variable in the original equation by  $(D_1 D_2 W)$  which leads

to the doubly susy-B extension of a given equation. For the HD equation (1), this leads to

$$W_t = \partial D_1 D_2 \left( (D_1 D_2 W)^{-1/2} \right), \quad (30)$$

which defines the sHD-BB equation. This procedure can, of course, be generalized to any  $N$  extended supersymmetry and we do not pursue this any further. We simply point out that eq. (30) is bi-Hamiltonian, as we would expect. For example, it is Hamiltonian with

$$H = \int dz (D_1 D_2 W_x)^2 (D_1 D_2 W)^{-5/2}, \quad (31)$$

and

$$\begin{aligned} \mathcal{D} = & -\partial W \partial^{-2} D_1 D_2 - D_1 D_2 \partial^{-2} W \partial + D_1 \partial^{-1} W D_2 - D_2 W \partial^{-1} D_1 \\ & + D_1 D_2 \partial^{-1} W - W D_1 D_2 \partial^{-1} + D_1 W \partial^{-1} D_2 - D_2 \partial^{-1} W D_1. \end{aligned} \quad (32)$$

The second Hamiltonian structure can also be easily obtained.

#### 4 The Supersymmetric $N=1$ Harry Dym (sHD) and Hunter-Zheng (sHZ) Equations:

As we have seen, the susy-B extension of a system is a very simple supersymmetrization. However, to obtain nontrivial supersymmetrizations, we can follow one of the following two approaches. In this section, we will discuss  $N=1$  supersymmetrization of the system and correspondingly, it is appropriate to work in the superspace defined in (16)–(19).

With the superfield (19) as our basic variable, the first approach is to write the most general local equation in superspace which is consistent with all canonical dimensions and which reduces to (1) in the bosonic limit. This involves a free parameter and the equation takes the form

$$\begin{aligned} W_t = & \frac{1}{8} \left[ -8(5a - 2)W_{xxx}(DW)^{-3/2} + 2(65a - 6)W_x(DW_{xx})(DW)^{-5/2} \right. \\ & + 30(5a + 2)W_{xx}(DW_x)(DW)^{-5/2} - 15(21a + 2)W_x(DW_x)^2(DW)^{-7/2} \\ & + W \left\{ 8(5a - 6)(DW_{xxx})(DW)^{-5/2} + 10(a - 6)W_{xxx}W_x(DW)^{-7/2} \right. \\ & + 35(6 - a)W_{xx}W_x(DW_x)(DW)^{-9/2} + 40(6 - 7a)(DW_{xx})(DW_x)(DW)^{-7/2} \\ & \left. \left. + 105(3a - 2)(DW_x)^3(DW)^{-9/2} \right\} \right], \end{aligned} \quad (33)$$

where  $a$  is the arbitrary parameter. In the case of the HD equation, it is possible to supersymmetrize the two Hamiltonian structures in (3), which is easily seen from the fact that the second Hamiltonian structure is the centerless Virasoro algebra. Thus, the supersymmetrized Hamiltonian structures follow to be

$$\mathcal{D}_1 = D\partial^2,$$

$$\mathcal{D}_2 = \frac{1}{2} [W\partial + 2\partial W + (DW)D] . \quad (34)$$

Requiring eq. (33) to be bi-Hamiltonian with respect to (34), namely, requiring

$$W_t = \mathcal{D}_1 \frac{\delta H_{-1}}{\delta W} = \mathcal{D}_2 \frac{\delta H_{-2}}{\delta W} , \quad (35)$$

determines the parameter to be  $a = 6$ . The Hamiltonians in (35), in this case have the forms ( $dz = dx d\theta$  with  $\int d\theta = 0$  and  $\int d\theta \theta = 1$ )

$$\begin{aligned} H_{-1} &= \int dz 2W(DW)^{-1/2} , \\ H_{-2} &= \int dz \frac{1}{8} \left[ W_x(DW_x)(DW)^{-5/2} - 15WW_xW_{xx}(DW)^{-7/2} \right] , \end{aligned} \quad (36)$$

and the  $N=1$  sHD equation assumes the simple form

$$W_t = D\partial^2 \left( 2(DW)^{-1/2} - 3WW_x(DW)^{-5/2} \right) . \quad (37)$$

It is worth noting here that this equation differs from the sHD-B equation (22) in the presence of the second term inside the parenthesis on the right hand side, which vanishes in the bosonic limit. (We would like to point out parenthetically that we do not generate the sHD-B equation in this approach because of our requirement that the equation be bi-Hamiltonian with respect to the structures in (34).)

It is easy to check that the Hamiltonian  $H_{-1}$  is a Casimir of  $\mathcal{D}_2$  and the conserved charge

$$H_0 = - \int dz W \quad (38)$$

is a Casimir of  $\mathcal{D}_1$ . Furthermore, the Hamiltonian structure  $\mathcal{D}_2$  can be written in the form

$$\mathcal{D}_2 = \frac{1}{2} (DW)^{1/2} D(1 + X)(DW)^{1/2} , \quad (39)$$

where

$$X \equiv \frac{3}{2} \left( D \frac{W}{(DW)} D^{-1} + D^{-1} \frac{W}{(DW)} D \right) D , \quad (40)$$

and therefore can be formally inverted. Thus, in this case also the associated recursion operator has a formal inverse.

It can be easily checked that the following charges

$$H_1 = \int dz \frac{1}{4} (D^{-1}W)(D^{-2}W) ,$$



$$H_2 = \int dz \frac{1}{2} (D^{-1}W)(D^{-2}W)(D^{-3}W), \quad (41)$$

are conserved and reduce to (9) in the bosonic limit. From

$$W_t = \mathcal{D}_1 \frac{\delta H_2}{\delta W} = \mathcal{D}_2 \frac{\delta H_1}{\delta W}, \quad (42)$$

we obtain the  $N=1$  supersymmetric HZ (sHZ) equation

$$W_t = -\frac{3}{2}W(D^{-1}W) - W_x(D^{-3}W) - \frac{1}{2}(DW)(D^{-2}W). \quad (43)$$

Both the sHD and the sHZ equations are bi-Hamiltonian systems and the infinite set of commuting conserved charges can be constructed recursively. As a result, they describe supersymmetric integrable systems.

The second approach to finding a nontrivial  $N=1$  supersymmetrization of the HD equation is to start with the Lax operator in (11) and generalize it to superspace. Let us start with the most general Lax operator involving non-negative powers of  $D$ ,

$$L = a_0^2 D^4 + \alpha_1 D^3 + a_1 D^2 + \alpha_2 D + a_2, \quad (44)$$

with the identification

$$a_0 = (DW)^{-1/2}, \quad (45)$$

where Roman coefficients are bosonic and Greek ones are fermionic. It is easy to verify that, in this case, there are only three projections,  $(\ )_{\geq 0,1,3}$  (with respect to powers of  $D$ ), that can lead to a consistent Lax equation. Using this ansatz, we have not yet been able to obtain the sHD equation (22) using fractional powers of the Lax operator (44). The Lax pair for this system, therefore, remains an open question.

On the other hand, when

$$\alpha_1 = cW_x(DW)^{-2}, \quad a_1 = a_2 = 0 = \alpha_2, \quad (46)$$

where  $c$  is an arbitrary parameter, the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = [(L^{3/2})_{\geq 3}, L], \quad (47)$$

yields consistent equations only for  $c = 0, -1, -\frac{1}{2}$ . As we have pointed out in the last section, for the values of the parameter,  $c = 0, -1$ , we have the sHD-B equation. The third choice of the parameter, therefore, leads to a new nontrivial  $N=1$  supersymmetrization of the HD equation. Namely, with

$$L = (DW)^{-1} D^4 - \frac{1}{2} W_x (DW)^{-2} D^3, \quad (48)$$

the Lax equation

$$\frac{\partial L}{\partial t} = \left[ (L^{3/2})_{\geq 3}, L \right], \quad (49)$$

leads to a second  $N = 1$  supersymmetrization of the HD equation of the form

$$W_t = \frac{1}{16} \left[ 8D^5((DW)^{-1/2}) - 3D(W_{xx}W_x(DW)^{-5/2}) \right. \\ \left. + \frac{3}{4}(DW_x)^2W_x(DW)^{-7/2} - \frac{3}{4}D^{-1} \left( (DW_x)^3(DW)^{-7/2} \right) \right]. \quad (50)$$

This is manifestly a nonlocal susy generalization in the variable  $W$  which, however, is a completely local equation in the variable  $(DW)$ .

Since this system of equations has a Lax description, it is integrable and the conserved charges can be calculated in a standard manner and the first few charges take the forms

$$H_1 = \int dz W_x(DW_x)(DW)^{-5/2}, \quad (51)$$

$$H_2 = \int dz W_x \left[ 16(DW_{xxx})(DW)^{-7/2} - 84(DW_{xx})(DW_x)(DW)^{-9/2} + 77(DW_x)^3(DW)^{-11/2} \right],$$

and so on. However, we have not yet succeeded in finding a Hamiltonian structure which satisfies Jacobi identity (it is clear that the Hamiltonian structure is nonlocal, since the Hamiltonian is local).

## 5 The $N=2$ Supersymmetric Harry Dym Hierarchy:

The most natural way to discuss the  $N = 2$  supersymmetric extension of the HD equation is in the  $N = 2$  superspace introduced earlier in (27)–(29). Looking at the bosonic superfield  $W$  in (29), we note that it has two bosonic components as well as two fermionic components. In the bosonic limit, when we set the fermions to zero, the  $N = 2$  equation would reduce to two bosonic equations. Since we have only the single HD equation (1) to start with, the construction of such a system is best carried out in the Lax formalism. This also brings out the interest in such extended supersymmetric systems, namely, they lead to new bosonic integrable systems in the bosonic limit.

As in (44), let us consider the most general  $N = 2$  Lax operator which contains differential operators in this superspace of the following form (taking a more general Lax involving only differential operators does not lead to equations which reduce to the HD equation),

$$L = W^{-1}\partial^2 + (D_1W^{-1})(\kappa_1D_1 + \kappa_2D_2)\partial + (D_2W^{-1})(\kappa_3D_1 + \kappa_4D_2)\partial \\ + (\kappa_5(D_1D_2W)W^{-2} + \kappa_6(D_1W)(D_2W)W^{-3})D_1D_2, \quad (52)$$

where  $\kappa_i, i = 1, 2, \dots, 6$  are arbitrary constant parameters. The  $N = 2$  supersymmetry corresponds to an internal  $O(2)$  invariance that rotates  $\theta_1 \rightarrow \theta_2, \theta_2 \rightarrow -\theta_1$  and correspondingly  $D_1 \rightarrow D_2, D_2 \rightarrow$

$-D_1$  (thereby rotating the fermion components of the superfield into each other). This invariance, imposed on the Lax operator, identifies

$$\kappa_4 = \kappa_1, \quad \kappa_3 = -\kappa_2 . \quad (53)$$

Using the computer algebra program REDUCE [14] and the special package SUSY2 [15], we are able to study systematically the hierarchy of equations following from the Lax equation

$$\frac{\partial L}{\partial t} = \left[ (L^{3/2})_{\geq 2}, L \right] . \quad (54)$$

Here, the projection (which is the highest consistent projection as is also the case with the bosonic HD equation in (12)) is understood as follows. Let us recall that a general pseudodifferential operator in  $N = 2$  superspace has the form

$$P = \sum_{n=-\infty}^{n=\infty} (P_0^n + P_1^n D_1 + P_2^n D_2 + P_3^n D_1 D_2) \partial^n . \quad (55)$$

For such a pseudodifferential operator, the projection in (54) is defined as

$$\begin{aligned} P_{\geq 2} &= P_3^0 D_1 D_2 + (P_1^1 D_1 + P_2^1 D_2 + P_3^1 D_1 D_2) \partial \\ &+ \sum_{n \geq 2} (P_0^n + P_1^n D_1 + P_2^n D_2 + P_3^n D_1 D_2) \partial^n . \end{aligned} \quad (56)$$

The consistency of the equation (54) leads to four possible solutions for the values of the arbitrary parameters

1.  $\kappa_1 = \kappa_2 = \kappa_5 = \kappa_6 = 0$  ,
2.  $\kappa_2 = 0, \kappa_1 = \kappa_5 = -\frac{\kappa_6}{2} = 1$  ,
3.  $\kappa_2 = \kappa_5 = \kappa_6 = 0, \kappa_1 = \frac{1}{2}$  ,
4.  $\kappa_2 = 0, \kappa_1 = \kappa_5 = \frac{1}{2}, \kappa_6 = \frac{3}{4}$  .

We will now discuss the various cases separately in some detail.

The first and the second cases can be discussed together since they lead to the same dynamical equation. Namely, in this case, the two Lax operators take the forms

$$\begin{aligned} L^{(1)} &= W^{-1} \partial^2 , \\ L^{(2)} &= W^{-1} \partial^2 + (D_1 W^{-1}) D_1 \partial + (D_2 W^{-1}) D_2 \partial - (D_1 D_2 W^{-1}) D_1 D_2 \\ &= -D_1 D_2 W^{-1} D_1 D_2 . \end{aligned} \quad (57)$$

It can be checked that both these Lax operators lead to the same dynamical equation which is

nothing other than the sHD-BB equations we have discussed earlier and, therefore, we do not study this any further.

For the third choice of parameters, the Lax operator can be written in the simple form

$$\begin{aligned} L^{(3)} &= W^{-1}\partial^2 + \frac{1}{2}((D_1W^{-1})D_1 + (D_2W^{-1})D_2)\partial \\ &= \frac{1}{2}(D_1W^{-1}D_1 + D_2W^{-1}D_2)\partial. \end{aligned} \quad (58)$$

The Lax equation (54), in this case, leads to a nontrivial  $N = 2$  supersymmetric HD equation of the form

$$\begin{aligned} W_t &= \frac{1}{64} \left[ 2(W^{-1/2})_{xxx} - 12(D_1W_{xx})(D_1W)W^{-5/2} - 12(D_2W_{xx})(D_2W)W^{-5/2} \right. \\ &\quad + 36(D_1W_x)(D_1W)W_xW^{-7/2} + 36(D_2W_x)(D_2W)W_xW^{-7/2} \\ &\quad \left. + 6(D_1W)(D_2W)(D_1D_2W_x)W^{-7/2} - 9(D_1W)(D_2W)(D_1D_2W)W_xW^{-9/2} \right]. \end{aligned} \quad (59)$$

In the bosonic sector, where we set all the fermions to zero so that (see (29))

$$W = w_0 + \theta_2\theta_1w_1, \quad (60)$$

the equation (59) reduces to

$$\begin{aligned} w_{0,t} &= \frac{1}{2}(w_0^{-1/2})_{xxx}, \\ w_{1,t} &= \frac{1}{64} \left[ -16w_{1,xxx}w_0^{-3/2} + 96w_{1,xx}w_{0,x}w_0^{-5/2} + 72w_{1,x}w_{0,xx}w_0^{-5/2} \right. \\ &\quad - 258w_{1,x}w_{0,x}^2w_0^{-7/2} - 6w_{1,x}w_1^2w_0^{-7/2} + 9w_1^3w_{0,x}w_0^{-9/2} \\ &\quad \left. - 108w_1w_{0,xx}w_{0,x}w_0^{-9/2} + 219w_1w_{0,x}^3w_0^{-9/2} \right]. \end{aligned} \quad (61)$$

The first of the equations in (61) is, of course, the HD equation (1), but is decoupled from the second component. Consequently, even though this set of equations represents a new integrable system, it is not very interesting. Let us note that we can reduce the  $N = 2$  supersymmetry of this system to  $N = 1$  supersymmetry in the following way. Let us define

$$W(x, \theta_1, \theta_2) = U(x, \theta_1) + \theta_2F(x, \theta_1), \quad (62)$$

and set the fermionic superfield  $F(x, \theta_1) = 0$ . This would, therefore, make the superfield  $W$  independent of the Grassmann coordinate  $\theta_2$  leaving us with  $N = 1$  supersymmetry. Under such a reduction, it is straightforward to see that the Lax operator (58) and the equation (59) go over to the ones in (48) and (upto multiplicative factors) the corresponding equation (50) with the identification

$$\theta_1 = \theta, \quad U(x, \theta_1) = (DW(x, \theta)). \quad (63)$$

The conserved charges for this system can be obtained from the Lax operator  $L^{(3)}$  in a standard manner, but we do not go into the details of this.

The fourth case is probably the most interesting of all. Here, the Lax operator takes the form

$$\begin{aligned} L^{(4)} &= W^{-1}\partial^2 + \frac{1}{2}((D_1W^{-1})D_1 + (D_2W^{-1})D_2)\partial \\ &\quad - W^{-1/2}(D_1D_2W^{-1/2})D_1D_2 \\ &= -\left(W^{-1/2}D_1D_2\right)^2. \end{aligned} \quad (64)$$

Interestingly enough, this Lax operator possesses two nontrivial square roots, namely,

$$\begin{aligned} L_1^{1/2} &= iW^{-1/2}D_1D_2, \\ L_2^{1/2} &= W^{-1/2}\partial + \frac{1}{2}\left[(D_1W^{-1/2})D_1 + (D_2W^{-1/2})D_2 - (W^{-1/2})_x\right] + \dots. \end{aligned} \quad (65)$$

We note here that a similar situation also arises in the study of the  $N = 2$  sKdV hierarchy [16] (for the case of the parameter  $a = 4$ ). In such a case, the general hierarchy of equations can be obtained from the Lax equation

$$\frac{\partial L}{\partial t_n} = \left[ \left( L_1^{n/2} L_2^{1/2} \right)_{\geq 2}, L \right], \quad (66)$$

where  $n = 0, 1, 2, \dots$ . For example, the first two flows of the hierarchy take the forms

$$\begin{aligned} W_{t_1} &= \frac{1}{8} \left[ -4(D_1D_2W_x)W^{-1} + 6(D_1D_2W)W_xW^2 \right. \\ &\quad \left. + 6((D_1W)(D_2W_x) - 6(D_2W)(D_1W_x))W^{-2} - 15(D_1W)(D_2W)W_xW^{-3} \right], \\ W_{t_2} &= \frac{1}{16} \left[ 8(W^{-1/2})_{xxx} - 6(D_1D_2W)(D_1D_2W_x)W^{-5/2} + 9(D_1D_2W)^2W_xW^{-7/2} \right. \\ &\quad \left. + 3((D_1W)(D_1W_{xx}) + (D_2W)(D_2W_{xx}))W^{-5/2} \right. \\ &\quad \left. - 9((D_1W)(D_1W_x) + (D_2W)(D_2W_x))W_xW^{-7/2} \right. \\ &\quad \left. + 9((D_1W)(D_2W)(D_1D_2W))_xW^{-7/2} - 36(D_1W)(D_2W)(D_1D_2W)W_xW^{-9/2} \right]. \end{aligned} \quad (67)$$

In the bosonic sector, the second equation in (67) gives

$$\begin{aligned} w_{0,t_2} &= \frac{1}{16} \left[ 8(w_0^{-1/2})_{xxx} - 6w_{1,x}w_1w_0^{-5/2} + 9w_1^2w_{0,x}w_0^{-7/2} \right], \\ w_{1,t_2} &= \frac{1}{32} \left[ -8w_{1,xxx}w_0^{-3/2} + 48(w_{1,x}w_{0,x})_xw_0^{-5/2} - 144w_{1,x}w_{0,x}^2w_0^{-7/2} \right. \\ &\quad \left. - 6w_{1,x}w_1^2w_0^{-7/2} + 9w_1^3w_{0,x}w_0^{-9/2} + 12w_1w_{0,xxx}w_0^{-5/2} \right. \\ &\quad \left. - 126w_1w_{0,xx}w_{0,x}w_0^{-7/2} + 177w_{0,x}^3w_1w_0^{-9/2} \right]. \end{aligned} \quad (68)$$

This is a new bosonic system of coupled equations, which reduces on setting  $w_1 = 0$  to the HD

equation and is integrable.

The conserved charges for this last case of  $N = 2$  supersymmetrization can be constructed as follows.

$$\begin{aligned}
H_1 &= \int dz \text{sRes} L_2^{1/2} = \int dz (D_1 W)(D_2 W)W^{-5/2} , \\
H_2 &= \int dz \text{sRes}(L_1^{1/2} L_2^{1/2}) \\
&= \int dz [3(D_1 D_2 W)^2 W^{-3} + 3W_x^2 W^{-2} \\
&\quad + 2((D_1 W)(D_1 W_x) + (D_2 W)(D_2 W_x)) W^{-3} + (D_1 W)(D_2 W)(D_1 D_2 W)W^{-4}] , \\
H_3 &= \int dz \text{sRes} L_2^{3/2} \\
&= \int dz [128(D_1 D_2 W_x)W_x W^{-7/2} - 40(D_1 D_2 W)^3 W^{-9/2} + \dots] , \tag{69}
\end{aligned}$$

where  $dz = dx d\theta_1 d\theta_2$  and “sRes” is defined as the coefficient of the  $D_1 D_2 \partial^{-1}$  term in the pseudodifferential operator. We can also perform the  $N = 1$  reduction of this system. Requiring that the superfield  $W$  has no dependence on  $\theta_2$ , it is clear from the form of the Lax operator in (64) that it reduces to the one involving the second  $N = 1$  supersymmetrization (just as  $L^{(3)}$  does).

## 6 Conclusions:

In this paper, we have studied the question of supersymmetrization of the Harry Dym hierarchy systematically. We have used the simpler B supersymmetrization to derive the sHD-B and sHD-BB systems. The analysis of the nontrivial  $N = 1$  supersymmetrization leads to two such integrable systems. One has a natural bi-Hamiltonian description for which we have not been able to find the Lax description. On the other hand, the second has a natural Lax description for which we have not yet found a Hamiltonian structure that satisfies the Jacobi identity. Both these systems are integrable. The  $N = 2$  supersymmetrization from the Lax approach yields four possible Lax operators. Two of these describe the sHD-BB system while the other two give nontrivial  $N = 2$  supersymmetric extensions. In the bosonic limit, one of them leads to the HD equation decoupled from the second component while the other genuinely gives a coupled two component system of equations that is integrable.

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