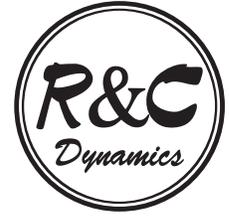


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THE ROLLING MOTION OF A RIGID BODY ON A PLANE AND A SPHERE. HIERARCHY OF DYNAMICS

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In this paper we consider cases of existence of invariant measure, additional first integrals, and Poisson structure in a problem of rigid body's rolling without sliding on plane and sphere. The problem of rigid body's motion on plane was studied by S. A. Chaplygin, P. Appel, D. Korteweg. They showed that the equations of motion are reduced to a second-order linear differential equation in the case when the surface of dynamically symmetric body is a surface of revolution. These results were partially generalized by P. Woronetz, who studied the motion of body of revolution and the motion of round disk with sharp edge on the surface of sphere. In both cases the systems are Euler–Jacobi integrable and have additional integrals and invariant measure. It turns out that after some change of time defined by reducing multiplier, the reduced system is a Hamiltonian system. Here we consider different cases when the integrals and invariant measure can be presented as finite algebraic expressions.

We also consider the generalized problem of rolling of dynamically nonsymmetric Chaplygin ball. The results of studies are presented as tables that describe the hierarchy of existence of various tensor invariants: invariant measure, integrals, and Poisson structure in the considered problems.

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1. Equations of Rigid Body Motion on Plane and Sphere without Sliding (Nonholonomic Rolling)

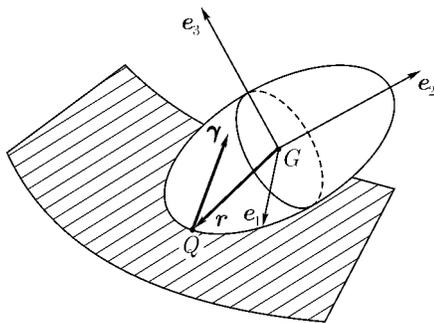


Fig. 1

In this paper we consider the equations of rigid body’s rolling on plane and sphere, because in these two cases, as opposed to a rolling on an arbitrary surface, the equations of motion are similar the Euler–Poisson equations. In both cases there are six first-order equations for six variables. In the potential field the equations have two integrals of motion: the energy integral and the geometrical integral (for Euler–Poisson equations there is also the area integral; its analog in problem of rolling is presented only under additional dynamical and geometrical restrictions).

Suppose that the rigid body rolls without sliding (i.e. the velocity of contact point Q is equal to zero) on the fixed surface represented by plane or sphere. The first part of equations of motion is the vector dynamical equation of kinetic moment \mathbf{M}

behavior in time with respect to the contact point Q (Fig. 1). This equation is represented for arbitrary shapes of body and surface in the form

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\omega} + m\dot{\mathbf{r}} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{M}_Q, \tag{1.1}$$

where \mathbf{M} , $\boldsymbol{\omega}$, $\mathbf{r} = GQ$, \mathbf{M}_Q are supposed to be projected on the principal central axes of inertia in the body; here $\boldsymbol{\omega}$ is the angular velocity, \mathbf{M}_Q is the moment of external forces with respect to the contact point, G is the center of mass. The second part of the motion equation is the vector kinetic equation of Poisson type different for the cases of plane a) and sphere b):

a)
$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{1.2}$$

where $\boldsymbol{\gamma}$ is the unit vector orthogonal to the plane,

b)
$$R_0(\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma}) = \dot{\mathbf{r}}, \tag{1.3}$$

where $\boldsymbol{\gamma}$ is the unit vector orthogonal to the sphere of radius R_0 (see Fig. 8).

In equations (1.1), (1.2), (1.3) we suppose that the radius vector \mathbf{r} is expressed as a function of normal vector $\boldsymbol{\gamma}$ with the help of equation

$$\boldsymbol{\gamma} = -\frac{\text{grad } f}{|\text{grad } f|} \tag{1.4}$$

that define the Gauss transformation, where $f(\mathbf{r}) = 0$ is the equation of the body’s surface in the main central frame of references connected with the body. We suppose that the body is everywhere convex

(to exclude the collisions during the motion), and equation (1.4) is uniquely solvable with respect to $\mathbf{r} = \mathbf{r}(\boldsymbol{\gamma})$. We assume that this condition is always fulfilled in the following text.

Using (1.4) we can represent kinematic equation (1.3) describing the dynamics of vector $\boldsymbol{\gamma}$ in the case of sphere in the form

$$\dot{\boldsymbol{\gamma}} = (1 + k(\mathbf{B} - k)^{-1}) \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad k = R_0^{-1}, \tag{1.5}$$

where $\mathbf{B} = \|b_{ij}\|$ is a degenerate matrix with the components $b_{ij} = -\frac{\partial}{\partial r_i} \left(\frac{1}{|\nabla f|} \frac{\partial f}{\partial r_j} \right)$.

The relation between \mathbf{M} and $\boldsymbol{\omega}$ is defined by the equation

$$\mathbf{M} = \mathbf{I}\boldsymbol{\omega} + m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad \boldsymbol{\omega} = \frac{\mathbf{A}\mathbf{M} - m\mathbf{r} \times (\mathbf{A}\mathbf{r} \times \mathbf{A}\mathbf{M})}{1 - m(\mathbf{r}, \mathbf{A}\mathbf{r})}, \tag{1.6}$$

where m is the mass of body, \mathbf{I} is the central tensor of inertia, $\mathbf{A} = (\mathbf{I} + m\mathbf{r}^2)^{-1}$.

If the potential $U = U(\boldsymbol{\gamma})$ depends only on the components of vector $\boldsymbol{\gamma}$, then we can present the moment of external forces in the form:

a) in the case of plane $\mathbf{M}_Q = \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}$;

b) in the case of sphere $\mathbf{M}_Q = \boldsymbol{\gamma} \times (1 + k(\mathbf{B}^T - k)^{-1}) \frac{\partial U}{\partial \boldsymbol{\gamma}}$, where \mathbf{B} is matrix (1.5).

For equations (1.1) and (1.2),(1.3) we always have the energy integral and the geometrical integral

$$H = \frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad F_1 = \boldsymbol{\gamma}^2 = 1. \tag{1.7}$$

REMARK. The proof is based on the formula

$$\frac{1}{2}(\mathbf{M}, \boldsymbol{\omega})' = (\mathbf{M}_Q, \boldsymbol{\omega}).$$

It follows only from equation (1.1) and does not depend on the shape of the surface, on which the body is rolling.

According to Euler–Jacobi theorem (the theory of last multiplier), to integrate these equations we need two more independent first integrals and an invariant measure [15]. Recall that the density of invariant measure ρ of the general system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_n)$$

satisfies the Liouville equation $\text{div}(\rho\mathbf{v}) = 0$. In the general case none of these objects exists; therefore, the system shows some interesting asymptotic and chaotic properties characteristic for the oscillation of Celtic stones. Such properties are not typical for conservative systems [13, 15]. We consider all the known cases of existence of additional first integrals (one or two at once) and the cases of existence of invariant measure.

Dynamical and geometrical bounds leading to the existence of the first integrals and of the invariant measure are independent in some sense. For some combinations of parameters there exist only the measure or only the additional integral. In some extreme cases two additional integrals and measure can exist at the same time; thus, the system became completely integrable.

The results of the study are presented separately for the cases of body’s rolling on plane and on sphere and collected in tables 1, 2. The following subsections are essentially the comments for these tables.

2. Body on a Plane

Body of revolution’s rolling on plane (S. A. Chaplygin [26], P. Appell [1, 31]). If both the surface of body and the central ellipsoid of inertia are coaxial surfaces of revolution, then for equations (1.1), (1.2) there exist two additional integrals and invariant measure. We assume that the potential U is an arbitrary function of $\gamma_3 = \cos \theta$, i. e. it depends only on the slope of revolution axis of the body to the vertical. In particular, in the case of the gravity field the center of mass must be situated on the axis of revolution.

The integrability of problem for an arbitrary body of revolution was shown by S. A. Chaplygin in 1897 [26]. He also demonstrated that it is possible to add a balanced uniformly revolving rotor along the axis of revolution (gyrostat) preserving the integrability of problem. More specific cases of this problem were studied by

- a. Routh (1884): the rolling of unbalanced dynamically symmetric ball on plane.
- b. Neumann, Carvallo (1898), Appell (1899) and Korteweg (1900): the rolling of round disk.

The results of Neumann and Carvallo mainly concern the deduction of motion equations and determination of stationary solutions. Note that Neumann during the deduction of equations of motion at first made the same mistake that occurred before in Lindelöf paper. He applied the Lagrangean formalism without the necessary "nonholonomic" modifications. In the subsequent studies he corrected this mistake, but did not solve the problem in quadratures. The Lindelöf mistake was analyzed in detail by S. A. Chaplygin (1897). He obtained a new form of equations of nonholonomic dynamics and was able to reduce the considered problem of rolling of revolution body to two linear first-order equations. In the case of round disk’s rolling, S. A. Chaplygin showed the possibility of reduction of these two equations to one linear (second-order) equation solvable in hypergeometric functions. We should also note that, before Chaplygin’s work, the equations of heavy revolution body motion were obtained in 1861 by G. Slesser, but their integrability was not indicated.

Somewhat later (in 1898), the analogous substitution (in the equations obtained by Carvallo in the paper presented for the Fourneyron prize) was used by Appell and in the slightly different form by Korteweg. They both did not know S. A. Chaplygin’s paper that was published in inaccessible journal only in Russian (the English translation of Chaplygin’s papers dedicated to this problem is published in this journal in 2002). This is the reason of the fact that in many modern textbooks and research papers (O’Reily [36]) the problem of round disk’s rolling is connected with the names of Appell and Korteweg, although the previous text show that this opinion is not completely correct.

Here we present the results obtained by S. A. Chaplygin in the modern algebraic form that let us to show the invariant measure in the explicit form and also to obtain the simplest forms of the first integrals. It turns out that we can generalize these results to dynamically nonsymmetric situation.

In the case of body of revolution we can find the solutions of equation of surface (1.6) in the explicit form

$$r_1 = f_1(\gamma_3)\gamma_1, \quad r_2 = f_1(\gamma_3)\gamma_2, \quad r_3 = f_2(\gamma_3), \tag{2.1}$$

where $f_i(\gamma_3)$, $i = 1, 2$ are function subjected to the differential equation that defines the meridional section

$$\frac{df_2}{d\gamma_3} = f_1 - \frac{1 - \gamma_3^2}{\gamma_3} \frac{df_1}{d\gamma_3}. \tag{2.2}$$

If we denote the main central tensor of inertia as $\mathbf{I} = \text{diag}(I_1, I_1, I_3)$, ($I_1 = I_2$), then we can explicitly calculate the density of invariant measure of equations (1.1), (1.2). It exists for arbitrary functions $f_1(\gamma_3)$, $f_2(\gamma_3)$ that define the surface

$$\rho = \frac{1}{\sqrt{I_1 I_3 + m(\mathbf{r}, \mathbf{I} \mathbf{r})}} = \frac{1}{\sqrt{I_1 I_3 + I_1 m f_1^2 (1 - \gamma_3^2) + I_3 m f_2^2}}. \tag{2.3}$$

REMARK. For the equations motion in variables ω, γ the density of invariant measure differs from (2.3) by factor $\det \mathbf{I}_Q$, where $\mathbf{I}_Q = \mathbf{I} + m(\mathbf{r}^2 \mathbf{E} - \mathbf{r} \otimes \mathbf{r})$ is the tensor of inertia with respect to the point of contact.

In the case $I_1 = I_2$ we have $\det \mathbf{I}_Q = (I_1 + m\mathbf{r}^2)(I_1 I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))$. It is interesting that measure (2.3) contain one of these factors.

It is easy to verify that under the above conditions the equations of motion also have the *symmetry field* \mathbf{v} defined by the differential operator

$$\hat{\mathbf{v}} = M_1 \frac{\partial}{\partial M_2} - M_2 \frac{\partial}{\partial M_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_1}. \tag{2.4}$$

It corresponds to the invariance of the system with respect to rotations about the axis of dynamical symmetry. Using this field we can reduce the order of system. For that we should choose the integrals of vector fields (2.4) as reduced variables to present the equations in the simplest form. After a number of tries, we choose the following reduced variables

$$\begin{aligned} K_1 &= \frac{(\mathbf{M}, \mathbf{r})}{f_1} = M_1 \gamma_1 + M_2 \gamma_2 + \frac{f_2}{f_1} M_3 \\ K_2 &= \frac{\omega_3}{\rho} = \rho \left(m f_1 f_2 (M_1 \gamma_1 + M_2 \gamma_2) + (I_1 + m f_2^2) M_3 \right) \\ K_3 &= \frac{M_2 \gamma_1 - M_1 \gamma_2}{\sqrt{(1 - \gamma_3^2)(I_1 + m\mathbf{r}^2)}}. \end{aligned} \tag{2.5}$$

In these variables the equations of motion of the reduced system have the following form

$$\begin{aligned} \dot{\gamma}_3 &= k K_3 \\ \dot{K}_1 &= -k K_3 \rho I_3 \left(1 - \left(\frac{f_2}{f_1} \right)' \right) K_2, \\ \dot{K}_2 &= -k K_3 \rho m f_1 (f_1 - f_2') K_1, \\ \dot{K}_3 &= -\frac{k}{I_1^2 f_1^2 (1 - \gamma_3^2)^2} \left(f_2 (f_1 (1 - \gamma_3^2) + \gamma_3 f_2) (m f_1^2 K_1^2 + I_3 K_2^2) + \right. \\ &\quad \left. + \gamma_3 f_1^2 I_1 K_1^2 - f_1 (f_1 (1 - \gamma_3^2) + 2\gamma_3 f_2) \frac{K_1 K_2}{\rho} + \right. \\ &\quad \left. + m f_1^2 \rho f_2 (1 - \gamma_3^2) (\gamma_3 f_1 I_1 - f_2 I_3) K_1 K_2 \right) - k \frac{\partial U(\gamma_3)}{\partial \gamma_3}, \end{aligned} \tag{2.6}$$

where $k = \sqrt{\frac{1 - \gamma_3^2}{I_1 + m\mathbf{r}^2}}$.

It is easy to show that these equations have the invariant measure with density $\rho = k^{-1}$ and integral of energy

$$\begin{aligned} H &= \frac{1}{2} (\mathbf{M}, \boldsymbol{\omega}) + U(\gamma_3) = \\ &= \frac{1}{2 I_1 (1 - \gamma_3^2)} \left(K_1^2 - \frac{I_3}{m f_1^2} K_2^2 + \frac{m f_2^2}{I_1} \left(K_1 - \frac{K_2}{\rho m f_1 f_2} \right)^2 \right) + \frac{1}{2} K_3^2 + U(\gamma_3). \end{aligned} \tag{2.7}$$

Moreover, for system (2.6) we have the following theorem

Theorem. *After the change of time $k dt = d\tau$ vector field (2.6) become Hamiltonian*

$$\frac{dx_i}{d\tau} = \{x_i, H\}, \quad \mathbf{x} = (\gamma_3, K_1, K_2, K_3)$$

with Hamiltonian (2.7) and degenerate Poisson bracket:

$$\begin{aligned} \{\gamma_3, K_3\} &= 1, \quad \{K_1, K_3\} = -I_3\rho\left(1 - \left(\frac{f_2}{f_1}\right)'\right)K_2, \\ \{K_2, K_3\} &= -m\rho f_1(f_1 - f_2')K_1 \end{aligned} \tag{2.8}$$

(all the other brackets are equal to zero).

The proof of theorem is obtained by the direct verification of equations and of Jacobi identity.

It turns out that equation (2.6) can be written in antisymmetric, almost Hamiltonian form (that sometimes is referred to as antigradient form)

$$\frac{dx_i}{d\tau} = J_{\lambda ij} \frac{\partial H}{\partial x_j}, \quad \mathbf{J}_\lambda = -\mathbf{J}_\lambda^T, \tag{2.9}$$

where

$$\mathbf{J}_\lambda = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda & -I_3\rho\left(1 - \left(\frac{f_2}{f_1}\right)'\right)K_2 - \lambda u \\ 0 & -\lambda & 0 & -m\rho f_1(f_1 - f_2')K_1 - \lambda v \\ -1 & I_3\rho\left(1 - \left(\frac{f_2}{f_1}\right)'\right)K_2 + \lambda u & m\rho f_1(f_1 - f_2')K_1 + \lambda v & 0 \end{pmatrix} \tag{2.10}$$

$$u = \frac{1}{f_1^2 I_1^2 (1 - \gamma_3^2) K_3} \left(\frac{(\mathbf{r}, \mathbf{I}\mathbf{r})}{m} K_2 - \frac{f_1 f_2}{\rho} K_1 \right),$$

$$v = \frac{1}{I_1^2 (1 - \gamma_3^2) K_3} \left(\frac{f_2}{f_1 \rho} K_2 - (I_1 + m f_2^2) K_1 \right),$$

and λ is an arbitrary function of $(K_1, K_2, K_3, \gamma_3)$. At $\lambda = 0$ we obtain the degenerate tensor \mathbf{J}_0 corresponding to bracket (2.8); although for $\lambda \neq 0$ tensor (2.10) is nondegenerate, it does not satisfy the Jacobi identity, i. e. it does not define a Poisson bracket.

If λ is chosen in the form

$$\lambda = \alpha(\gamma_3)K_3, \tag{2.11}$$

then the tensor $\tilde{\mathbf{J}} = (\lambda^{-1}\mathbf{J}_\lambda)$ satisfies the Jacobi identity, and the corresponding vector field

$$\mathbf{v} = (\lambda^{-1}\mathbf{J}_\lambda)\nabla H \tag{2.12}$$

is Hamiltonian; at the same time the divergence of field (2.12) is nonzero. Thus, the considered nonholonomic system generate an example of Hamiltonian vector field with nontrivial measure $\rho = \alpha(\gamma_3)K_3$. Note also that the function $\rho = \alpha(\gamma_3)K_3$ is a reducing multiplier in Chaplygin's terminology, and in this case it differs from invariant measure (2.3). The close example of Poisson structure for the problem of ball's rolling on body of revolution was presented by Hermans [32].

Bracket (2.8) has two Casimir functions [5] that are integrals of motion; therefore, system (2.6) is integrable. The integrability and existence of linear integrals can be established by the different classical method: we divide the second and the third equation of system (2.6) by $\dot{\gamma}_3$ and obtain the system of two linear non-autonomous first-order equations

$$\frac{dK_1}{d\gamma_3} = -\rho I_3 \left(1 - \left(\frac{f_2}{f_1}\right)'\right) K_2, \quad \frac{dK_2}{d\gamma_3} = -m\rho f_1(f_1 - f_2')K_1. \tag{2.13}$$

In somewhat different variables connected with semifixed axes, equations (2.13) were obtained by S. A. Chaplygin. Equations (2.13) do not contain the potential, which is presented only in the expression of energy integral (2.7). Using this equation we determine the dependence of nutation angle

on time after the solving of linear system (2.13). In the general case this dependence has periodic oscillating character.

Because the equations (2.13) are linear with respect to γ_3 , the general solution can be obtained as the linear superposition

$$K_i = c_1 g_i^{(1)} + c_2 g_i^{(2)}, \quad i = 1, 2, \tag{2.14}$$

where $g_i^{(1)}(\gamma_3)$, $g_i^{(2)}(\gamma_3)$ are elementary solutions of (2.13). Inverting expressions (2.14) with respect to c_i we obtain the expressions for the lacking first integrals, which are expressed in the general case in terms of real analytic, but nonalgebraic (for example, hypergeometric) functions. Nevertheless, they are always linear with respect to M_i (i. e. with respect to generalized velocities).

These integrals in some sense generalize the area integral (corresponding to the cyclic angle of precession) and the cyclic Lagrange integral (corresponding to the cyclic angle of proper rotation) [5] that exist in the classical problem of heavy symmetric top's motion about a fixed point (the Lagrange case). The presence of such integrals causes the great similarity of qualitative researches of these problems.

Let's consider all known situations when integrals are algebraic or can be expressed through members of some known classes of special functions.

Round disk (S. A. Chaplygin, P. Appell, D. Korteweg). Generally speaking, we consider a disk with the center of mass displaced along the axis of dynamical symmetry (Fig. 2). In this case functions (2.1) are explicitly expressed as

$$f_1 = \frac{R}{\sqrt{1 - \gamma_3^2}}, \quad f_2 = a, \tag{2.15}$$

where R is a radius of coin, a is the displacement of the center of mass along the axis of dynamical symmetry (Fig. 2).

The interesting fact in this case is the independence of measure (2.3) from the phase variables $\rho = \text{const}$. For variables (2.5) we obtain the equations

$$\frac{dK_1}{d\theta} = \frac{\rho m R^2}{\sin \theta} K_2, \quad \frac{dK_2}{d\theta} = I_3 \rho (\sin \theta + \frac{a}{R} \cos \theta) K_1, \tag{2.16}$$

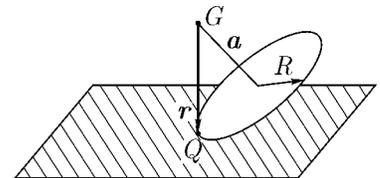


Fig. 2

where $\rho = (I_1 I_3 + I_1 m R^2 + I_3 m a^2)^{-1/2}$. These two linear equations are reduced to one linear second-order equation with respect to ω_3

$$\frac{d^2 \omega_3}{d\theta^2} - \text{ctg } \theta \frac{d\omega_3}{d\theta} + \rho^2 m R (R + a \text{ctg } \theta) I_3 \omega_3 = 0. \tag{2.17}$$

At $a = 0$ with the help of substitution $\cos \theta = 1 - 2x$ equation (2.17) is transformed to the hypergeometric type equation [1]

$$x(1 - x) \frac{d^2 \omega_3}{dx^2} + (1 - 2x) \frac{d\omega_3}{dx} - \rho^2 I_3 m R^2 \omega_3 = 0.$$

In the papers [17, 24] the following result was shown: the disk for almost all initial conditions do not fall on the plane. In [2] the similar result was obtained for the nonintegrable problem of heavy disk's rolling on a slopping plane.

The results concerning the stability of stationary motions and the qualitative analysis of motion see in the papers [19, 24] and also in the book [22].

Dynamically symmetric ball with the displaced center of mass (E. Routh, S. A. Chaplygin). In this case

$$f_1 = R, \quad f_2 = R\gamma_3 + a, \quad (2.18)$$

where R is the radius of ball, a is the distance from the center of mass to the geometrical center. The measure ρ is not constant any more

$$\rho = (I_1 I_3 + I_1 m R^2 (1 - \gamma_3^2) + I_3 m (R\gamma_3 + a)^2)^{-1/2}, \quad (2.19)$$

and the equations for K_1, K_2 become trivial: $\dot{K}_1 = 0, \dot{K}_2 = 0$, i. e. the expressions

$$\begin{aligned} K_1 &= \omega_3 \rho^{-1} = \rho^{-1} (m R^2 \gamma_3 (\mathbf{M}, \boldsymbol{\gamma}) + I_1 M_3 + m a R ((\mathbf{M}, \boldsymbol{\gamma}) + M_3 \gamma_3) + m a^2 M_3) = \text{const}, \\ K_2 &= \frac{1}{R} (\mathbf{M}, \mathbf{r}) = (\mathbf{M}, \boldsymbol{\gamma}) + \frac{a}{R} M_3 = \text{const}. \end{aligned} \quad (2.20)$$

are integrals of motion.

The integral K_2 represents *the Jellett integral*. This integral is also present under the arbitrary law of friction at the point of contact [22]. The integral K_1 was found by E. Routh in 1884 [23] and its form is a little bit mysterious. It was also indicated by S. A. Chaplygin in the paper [26]. Once again we shall note that both integrals are linear with respect to the velocities. They are the immediate generalizations of the cyclic integrals corresponding to the precession ψ and to the proper rotation φ , but have no such natural dynamical origin. The integral K_2 sometimes is referred to as the *Chaplygin integral*.

REMARK. For axisymmetric bodies we can also indicate the other cases of existence of simple quadratic integral of the form

$$F = a K_1^2 + b K_2^2, \quad a, b = \text{const}.$$

Obviously, we have to require in addition to condition (2.2) the following one

$$\frac{K_2^{-1} \dot{K}_1}{K_1^{-1} \dot{K}_2} = \frac{I_3 \left(1 - \left(\frac{f_2}{f_1}\right)'\right)}{m f_1 (f_1 - f_2')} = \lambda = \text{const}. \quad (2.21)$$

The general solution of equations (2.21) and (2.2) is expressed in hypergeometric functions. Among the axisymmetric figures of the second order only the ball with displaced center satisfies these equations. The closed bounded curves satisfying (2.21) and (2.2) and different from ball are similar to the ovals. Below we show that in this case the simple quadratic integral exists in the totally symmetric case. ($I_1 \neq I_2 \neq I_3 \neq I_1$).

Three-dimensional point maps in nonholonomic mechanics. Before we consider the following cases of the body's motion, we shall present some general construction that let us to establish relations between equations (1.1), (1.2), (1.3) to some point one-to-one map in three-dimensional space. We present the computer analysis of this map using the numerical integration of the indicated system at the fixed value of energy. Using this method we can find out and give a visual interpretation to various possibilities of existence of measure and integrals in their various combinations.

To construct the three-dimensional map we use the Andoyer–Deprit variables (L, G, H, l, g, h) , which were regularly used in our book [5] for computer (and analytical) research of Euler–Poisson, Kirchhoff and other Hamiltonian equations. As against to nonholonomic situation, in the classical case these variables are canonical, and by virtue of the fact that the area integral is always present in the Euler–Poisson type equations, we can limit ourselves in the classical case to two-dimensional maps. The problems described above require two additional integrals of motion; therefore, it is necessary to use three-dimensional maps, and such maps are not necessarily possess an invariant measure (as against to Hamiltonian mechanics). Using the known formulas we make the transition from the

variables (\mathbf{M}, γ) to the Andoyer–Deprit variables

$$\begin{aligned} M_1 &= \sqrt{G^2 - L^2} \sin l, & M_2 &= \sqrt{G^2 - L^2} \cos l, & M_3 &= L, \\ \gamma_1 &= \left(\frac{H}{G} \sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g \right) \sin l + \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \cos l, \\ \gamma_2 &= \left(\frac{H}{G} \sqrt{1 - \left(\frac{L}{G}\right)^2} + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g \right) \cos l - \sqrt{1 - \left(\frac{H}{G}\right)^2} \sin g \sin l, \\ \gamma_3 &= \left(\frac{H}{G}\right) \left(\frac{L}{G}\right) - \sqrt{1 - \left(\frac{L}{G}\right)^2} \sqrt{1 - \left(\frac{H}{G}\right)^2} \cos g, \end{aligned} \tag{2.22}$$

in which we can express the energy $E = E(L, G, H, l, g)$.

In the Euler–Poisson equations the value of $H = (\mathbf{M}, \gamma)$ is constant, but for equations (1.1), (1.2), (1.3) this is not the case any more. We fix the level of energy $E = E_0$, then choose the intersecting plane, for example, as $g = g_0 = \text{const}$, and obtain the three-dimensional map induced by sequential intersections of the phase trajectory with the chosen intersecting plane. We present the map in the variables $(L/G, H/G, l)$, because of its compactness by virtue of the fact that $|\frac{L}{G}| \leq 1, |\frac{H}{G}| \leq 1$. Typical examples of such three-dimensional maps are presented in Figs. 3, 5, 6, 7. It is obvious that, because of the presence of one additional integral, the trajectories are situated

on two-dimensional invariant manifolds of the point map, and the presence of two additional integrals imply the stratification of the three-dimensional space on invariant curves (Fig. 3). In general case when both the integrals and the measure are absent, the complicated behavior of trajectories is possible. In this case random motions alternate with the asymptotic attracting properties typical for dissipative systems. Note also that as against to (\mathbf{M}, γ) the variables L, G, H, l, g, h are more convenient for the analysis of three-dimensional map, because in this variables the linear and angular components are separated, and they have the obvious geometrical meaning (see [5]).

One of examples of three-dimensional map is the well-known Smale–Williamson map. It does not preserve the measure, but is expressed by analytical formulas. Other examples can be obtained by the study of general (nonconservative) perturbations of two-degree Hamiltonian systems.

Rolling of balanced, dynamically nonsymmetric ball (Chaplygin ball [25]). The equations of motion of dynamically nonsymmetric ball with the center of mass coinciding with the geometrical center can be written in the form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, & \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \mathbf{M} &= \mathbf{I} \boldsymbol{\omega} + D \boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}), & D &= ma^2, \end{aligned} \tag{2.23}$$

where $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$ is the central tensor of inertia, $U = U(\boldsymbol{\gamma})$ is the potential energy. Equations (2.23) always have the measure with density ρ and the first integrals of the form

$$\begin{aligned} \rho &= \frac{1}{\sqrt{1 - D(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})}}, & \mathbf{A} &= (\mathbf{I} + D\mathbf{E})^{-1}, & \mathbf{E} &= \|\delta_{ij}\|, \\ H &= \frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), & F_1 &= \boldsymbol{\gamma}^2 = 1 & F_2 &= (\mathbf{M}, \boldsymbol{\gamma}). \end{aligned} \tag{2.24}$$

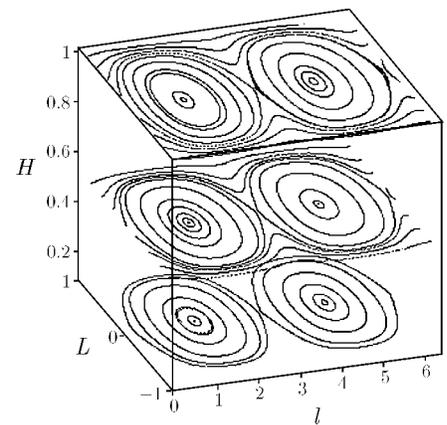


Fig. 3. The three-dimensional map described in subsection 2 for the case of Chaplygin ball. The figure shows very clearly that all trajectories are situated on joint level surfaces of two integrals $H = \text{const}$ and $\mathbf{M}^2 = \text{const}$ ($I_1 = 1, I_2 = 2, I_3 = 3$)

At $U = 0$ there exists the additional integral $F_3 = \mathbf{M}^2$ and the problem become integrable (S. A. Chaplygin, 1903 [25]); the corresponding three-dimensional map is presented in Fig. 3.

It was shown in the paper [15] that this problem is still integrable with a Brun potential

$$U = \frac{1}{2}k(\mathbf{I}\boldsymbol{\gamma}, \boldsymbol{\gamma}).$$

The integral F_3 in this case has the form

$$F_3 = \mathbf{M}^2 - \frac{k}{\det \mathbf{A}}(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})$$

The authors indicated in [6] that for any potential U the change of time $d\tau = \rho dt$ in equations (2.23) makes them Hamiltonian with a Poisson bracket, which is nonlinear with respect to the phase variables $(\mathbf{M}, \boldsymbol{\gamma})$ and has the form

$$\begin{aligned} \{M_i, M_j\} &= \varepsilon_{ijk}\rho^{-1}(M_k - g\gamma_k), \quad \{M_i, \gamma_j\} = \varepsilon_{ijk}\rho^{-1}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \\ g &= D(\boldsymbol{\omega}, \boldsymbol{\gamma}) = \frac{D(\boldsymbol{\gamma}, \mathbf{A}\mathbf{M})}{1 - D(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})}. \end{aligned} \tag{2.25}$$

Bracket (2.25) is degenerated; its Casimir functions are integrals F_1, F_2 (2.24). The Hamiltonian corresponding to bracket (2.25) is obtained from energy (2.24) expressed as a function of the moments by the formula

$$H = \frac{1}{2}(\mathbf{M}, \mathbf{A}\mathbf{M}) + \frac{1}{2}g(\mathbf{A}\mathbf{M}, \boldsymbol{\gamma}) + U(\boldsymbol{\gamma}). \tag{2.26}$$

After the change of variables $\mathbf{K} = \rho\mathbf{M}$ the Poisson bracket and the Hamiltonian are represented in the form

$$\begin{aligned} \{K_i, K_j\} &= \varepsilon_{ijk}(K_k - D\rho^2(\mathbf{K}, \boldsymbol{\gamma})a_k\gamma_k), \quad \{K_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \\ H &= \frac{1}{2}\rho^{-2}(\mathbf{K}, \mathbf{A}\mathbf{K}) + \frac{1}{2}D(\mathbf{A}\mathbf{K}, \boldsymbol{\gamma})^2 + U(\boldsymbol{\gamma}). \end{aligned} \tag{2.27}$$

Thus, at the zero level $(\mathbf{K}, \boldsymbol{\gamma}) = 0$ bracket (2.27) passes to the bracket described by algebra $e(3)$. on which we can write the Euler–Poisson and Kirchhoff equations [5].

Note that for the considered problem the density of measure ρ is the reducing multiplier (by Chaplygin [27]). With its help the nonholonomic equations are reduced to the Hamiltonian system. Chaplygin himself used such reduction integrating the equations of motion of nonsymmetric ball; as a preliminary he introduced a nonholonomic analog of spherocon variables. It is possible to do these operations in inverse order [25]: one makes at first the change of time $d\tau = \rho dt$ to receive a Hamiltonian system, and then introduces the usual spherocon variables and use the Hamilton–Jacobi method.

As against to Poisson structure (2.8) related to the system reduced on the field of a symmetry corresponding to proper rotation, structure (2.12) is related to complete system (1.1), (1.2). Unfortunately, we were unable to generalize (to lift) reduced structure (2.8) to such complete system. Possibly it is either too difficult or some dynamic effects prevent such generalization. Unfortunately, the dynamic effects preventing the reduction to Hamiltonian form are very purely investigated [4].

Rolling of unbalanced, dynamically nonsymmetric ball on plane. In this case equations (1.1), (1.2) can be written in the following convenient form

$$\begin{cases} \dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\omega} + m\dot{\mathbf{r}} \times (\boldsymbol{\omega} \times \mathbf{r}), \\ \dot{\mathbf{r}} = \mathbf{r} \times \boldsymbol{\omega} - \mathbf{a} \times \boldsymbol{\omega} = (\mathbf{r} - \mathbf{a}) \times \boldsymbol{\omega} \\ \mathbf{M} = \mathbf{I}\boldsymbol{\omega} + m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}), \end{cases} \tag{2.28}$$

where \mathbf{a} is the vector connecting the center of mass with the geometrical center $\mathbf{r} = R\boldsymbol{\gamma} + \mathbf{a}$ (see Fig. 4). It turns out that in the case $\mathbf{a} \neq 0$ the integral $F_3 = M^2$ of system (2.23) can be directly generalized and written in the form

$$F = M^2 - m\mathbf{r}^2(M, \boldsymbol{\omega}) = M^2 - 2m\mathbf{r}^2 H, \quad (2.29)$$

where $H = \frac{1}{2}(M, \boldsymbol{\omega})$ is the energy integral. Though this integral is simple enough, but probably it was not known earlier. Note only that under the additional requirement of the dynamical symmetry the Jellett integral and the Chaplygin integrals were found (see subsection 2). Integral (2.29) can be considered as their generalization for dynamically nonsymmetric situation.

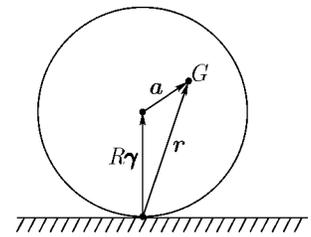


Fig. 4

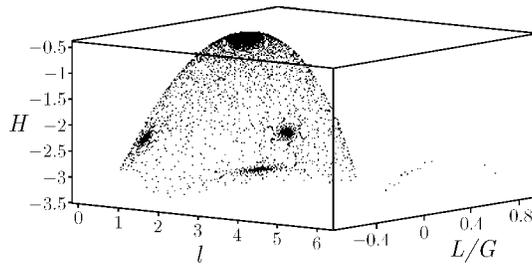


Fig. 5. One of trajectories in the problem of rolling of unbalanced ball on plane. The figure shows clearly that all points are situated on some surface; the condensations of points correspond to asymptotic approximations of the trajectory to periodic solutions. The trajectory goes out from the top and approaches to the three points in lower part of surface

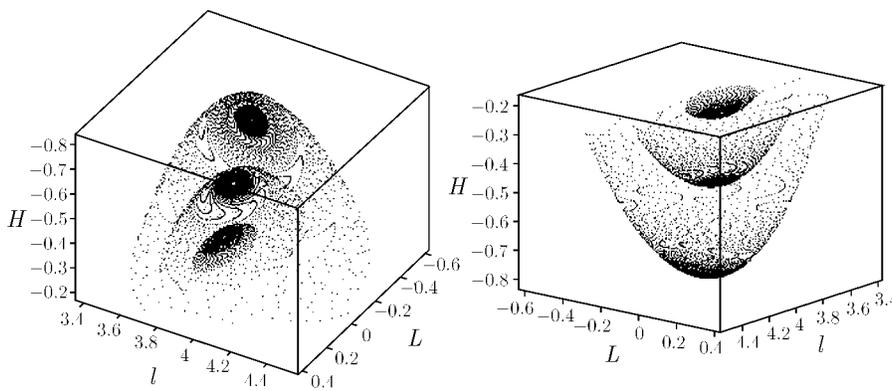


Fig. 6. Three trajectories in the problem of rolling of unbalanced ball on plane. The figure shows very clearly that points are situated on two-dimensional surfaces (corresponding to the level of integral (2.29)). The condensation of points corresponds to an asymptotic approximation to some periodic solution.

We were unable to obtain generalizations of this integral for cases with gyrostat and Brun field. Note also that for $\mathbf{a} \neq 0$ there is probably no measure. This is illustrated in Figs. 5, 6. These figures shows the asymptotic trajectories of the point map situated on the two-dimensional surface of integral (2.29).

An arbitrary body with a spherical central ellipsoid of inertia. $I_1 = I_2 = I_3 = \mu$, $\mu = \text{const}$.

This problem in general case requires both additional integrals, nevertheless, there is always an invariant measure. For the first time this fact was noted by V. A. Yaroschuk [28]. In this case the equations written in variables ω, γ are more convenient. They have form

$$\begin{aligned}
 (\mu + m r^2) \dot{\omega} &= m(\dot{r} + \omega \times r)(r, \omega) - m\omega(r, \dot{r}) + \gamma \times \frac{\partial U}{\partial \gamma}, \\
 \dot{\gamma} &= \gamma \times \omega.
 \end{aligned}
 \tag{2.30}$$

Equations (2.30) have an invariant relation $(\dot{\omega}, r) = 0$, which is used for simplification of some calculations.

The density of invariant measure for these equations indicated in [28] can be represented in the form

$$\rho = (\mu + r^2)^{3/2}.
 \tag{2.31}$$

We were unable to reproduce the generalization of this measure for the case of body's rolling on sphere indicated in the paper [28]. Probably, this generalization does not exist. Also it is not known, what nontrivial integrable cases can be obtained with the help of measure (2.31), and whether the system has any Hamiltonian origin, possibly after the appropriate change of time.

Gyrostatic generalizations. Following mainly the paper by S. A. Chaplygin [26], we present the generalizations of the indicated problems for the case with additional uniformly rotating balanced rotor. The corresponding system can be interpreted as a nonholonomic gyrostat. The gyrostatic effect can be also obtained by an addition of multiply connected cavities completely filled by the ideal incompressible liquid possessing nonzero circulation into the body [5]. In the described case, equation for the moment (1.1) can be presented as

$$\dot{M} = (M + S) \times \omega + m \dot{r} \times (\omega \times r) + M_Q,$$

where S is the constant three-dimensional vector of gyrostatic moment. It is easy to verify that the addition of rotor does not influence the existence of invariant measure with the density depending on the positional variables γ .

a) Body of revolution. The equations of type (2.13) for the rotor with gyrostatic moment $S = (0, 0, s)$ directed along the axis of revolution in variables (2.5) have the form

$$\frac{dK_1}{d\gamma_3} = -I_3 \rho \left(1 - \left(\frac{f_2}{f_1} \right)' \right) K_2 - s, \quad \frac{dK_2}{d\gamma_3} = -m \rho f_1 ((f_1 - f_2') K_1 + f_2 s).
 \tag{2.32}$$

Equations (2.32) were obtained in less convenient form by S. A. Chaplygin [26]. The density of invariant measure is also defined by equation (2.3).

Let's consider sequentially the generalizations of the indicated earlier cases of disk, ellipsoid, and ball with the displaced center.

b) Round disk. Now equation (2.17) have to the following form

$$\begin{aligned}
 \frac{d^2 \omega_3}{d\theta^2} - \text{ctg } \theta \frac{d\omega_3}{d\theta} + m R I_3 (R + a \text{ctg } \theta) \rho^2 \omega_3 &= s m R \rho^2 (R + a \text{ctg } \theta), \\
 \rho &= (I_1 I_3 + I_1 m R^2 + I_3 m a^2)^{-1/2},
 \end{aligned}
 \tag{2.33}$$

and at $a = 0$ in general case it is reduced to non-homogeneous (for $s \neq 0$) hypergeometric equation.

c) Ball with the displaced center of mass. Here system (2.32) has the form

$$\frac{dK_1}{d\gamma_3} = -s, \quad \frac{dK_2}{d\gamma_3} = -m \rho R (R \gamma_3 + a) s,
 \tag{2.34}$$

where ρ is defined by relation (2.19). We can immediately show the first integral that generalizes the Jellett integral

$$F = K_1 + s\gamma_3 = \text{const.} \tag{2.35}$$

The second integral generalizing Routh (Chaplygin) integral has more complicated nonalgebraic form

$$(I_1 - I_3)\rho^{-1}\omega_3 - s \left\{ \rho^{-1} - I_1 \sqrt{\frac{ma^2}{I_1 - I_3}} \operatorname{arctg} \left(\sqrt{\frac{m}{I_1 - I_3}} \rho(R\gamma_3(I_1 - I_3) - aI_3) \right) \right\} = \text{const.} \tag{2.36}$$

In integral (2.36) we assume $I_1 > I_3$. For $I_1 < I_3$ the integral contain hyperbolic functions. Integral (2.36) was explicitly presented by A. S. Kuleshov [18]. It is essentially simplified at $I_1 = I_3 = \mu$ and has the form

$$\rho^{-1} \left(3\mu\omega_3 - s \frac{\mu + mR^2 - 2ma^2 - maR\gamma_3}{ma^2} \right) = \text{const.} \tag{2.37}$$

The form of integral is even simpler for the case of balanced homogeneous ball ($a = 0$):

$$\omega_3 + \frac{1}{2} \frac{mR^2\gamma_3^2}{\sqrt{\mu(\mu + mR^2)}} = \text{const.}$$

This simple integrable generalization was indicated by D. K. Bobilev [3] (some additional simplifications in the case of explicit integration were also indicated by N. E. Zhukovsky [12]).

d) Dynamically nonsymmetric ball. The most general gyrostatic generalization for the case of Chaplygin ball was suggested by A. P. Markeev [20]. The equations of motion and the integrals are

$$\begin{aligned} \dot{\mathbf{M}} &= (\mathbf{M} + \mathbf{S}) \times \boldsymbol{\omega}, \quad \boldsymbol{\gamma} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ H &= \frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}), \quad F_1 = \boldsymbol{\gamma}^2 = 1, \quad F_2 = (\mathbf{M} + \mathbf{S}, \mathbf{M} + \mathbf{S}), \quad F_3 = (\mathbf{M} + \mathbf{S}, \boldsymbol{\gamma}), \end{aligned} \tag{2.38}$$

where \mathbf{S} is the constant three-dimensional vector of gyrostatic moment. Note that we were unable to generalize Poisson structure (2.25) to system (2.38) for $\mathbf{S} \neq 0$.

Rolling of ellipsoid on plane. It turns out that in the problem of rolling of balanced ellipsoid, which axes are principal axes of inertia, there are cases of existence of specific invariant measure, that are defined by restrictions on ratios of moments of inertia and semiaxes of the ellipsoid of the surface. This measure has found by V. A. Yaroschuk in [29].

For the problem of ellipsoid’s rolling, two cases of existence of the invariant measure were already indicated in subsections 2 and 2. They are accordingly measures of balanced, dynamically nonsymmetric ball (2.24) and of arbitrary body with the spherical central ellipsoid of inertia (2.31). It is interesting that no obstacles to existence of the analytical invariant measure of the general rolling problem of balanced ellipsoid, which principal axes are principal axes of inertia, are not found yet (as against to the case of Celtic stone [15]). It is possible, that this measure exists (at least in this situation there is no asymptotical behavior typical for the Celtic stones), but is complicated and nonalgebraic.

For the surface of ellipsoid $(\mathbf{r}, \mathbf{B}^{-1}\mathbf{r}) = 1$, where $\mathbf{B} = \text{diag}(b_1, b_2, b_3)$, b_i are squares of larger semiaxes, we have the explicit expression

$$\mathbf{r} = \frac{\mathbf{B}\boldsymbol{\gamma}}{\sqrt{(\mathbf{B}\boldsymbol{\gamma}, \boldsymbol{\gamma})}}. \tag{2.39}$$

If the cental tensor of inertia has the form

$$\mathbf{I} = \mu\mathbf{E} + \lambda m\mathbf{B}, \tag{2.40}$$

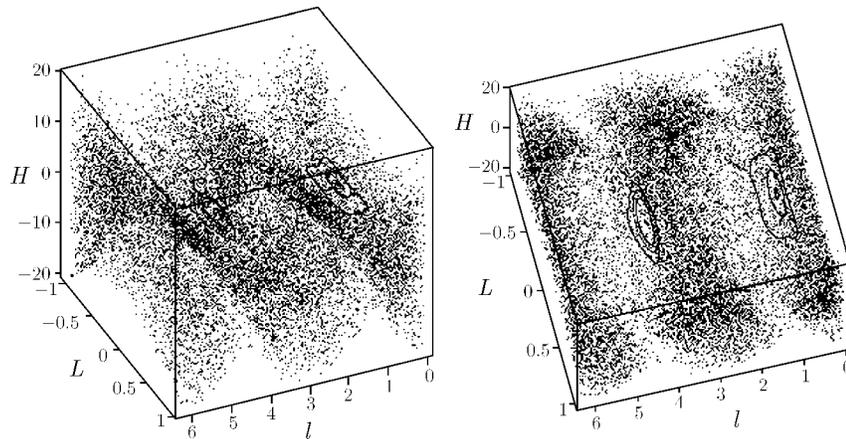


Fig. 7. Some trajectories in the problem of rolling of ellipsoid with the spherical tensor of inertia on plane. The figure shows clearly the regular trajectories filling some curves that enclose stable periodic solutions (permanent rotations), which in this case are degenerated as it was stated above. A random layer (which is obtained from one trajectory) in this case is not situated on any surface.

then equations (1.1), (1.2) have a measure only in the case $\lambda = 0$ and $\lambda = 1$ with the density (in variables \mathbf{M}, γ)

$$\rho = (\mu + m\mathbf{r}^2)^{-1/2} = ((\mu + m\mathbf{B})\mathbf{r}, \mathbf{B}^{-1}\mathbf{r})^{-1/2}. \tag{2.41}$$

REMARK. For the case $\lambda = 0$, measure (2.41) was already indicated in subsection 2. It is defined by formula (2.31) and is present for any surface of the body. The differences in powers of expressions (2.41) and (2.31) are connected with the various systems of variables (\mathbf{M}, γ) and $(\boldsymbol{\omega}, \gamma)$ and with the corresponding transformations of densities of invariant measures.

Note that if equality (2.40) is fulfilled, then the motion equations have a two-parameter set of vertical permanent rotations at arbitrary real λ (in other cases this set is one-parameter). A. V. Karapetyan in [14] showed that the conditions of existence of such sets are even a little bit wider and have the form

$$\sum_{ijk} I_i(b_i - b_k) = 0, \tag{2.42}$$

where $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$. In addition to tensor (2.40) conditions (2.42) are fulfilled in the case of non-holonomic Chaplygin ball ($b_1 = b_2 = b_3 = R^2$) when there exist both the measure and integral (2.24), and also the conditions hold in the axially symmetric situation $I_1 = I_2, b_1 = b_2$. Unexpectedly, equality (2.40) is also the necessary (but, generally speaking, insufficient) condition of the integrability of equations of motion for the case of ellipsoid on ideally smooth plane [8]. Note the interesting fact that for $\lambda \neq 0$ and $\lambda \neq 1$, both the measure and obstacles to its existence are not found. The three-dimensional section in the case $\lambda = 0$ is presented on Fig. 7.

All the results described above that are connected to the rolling of body on plane are presented in table 1.

3. Body on a Sphere

Now we consider systematically the situations analogous to one in the case of plane that originate in the problem of rolling of body on a sphere. First of all we shall note that kinematic equation (1.3) can be written as

$$\dot{\gamma} = \gamma \times \boldsymbol{\omega} \mp k\dot{r}, \quad k = 1/R_0, \tag{3.1}$$

Table 1. Rolling of body on plane

tensor of inertia	dynamically nonsymmetric case $I_1 \neq I_2 \neq I_3 \neq I_1$			axial dynamical symmetry $I_1 = I_2, U = U(\gamma_3)$			total dynamical symmetry $I_1 = I_2 = I_3 = \mu$
surface of body	ball		ellipsoid	an arbitrary body of revolution	round disk with sharp edge	unbalanced ball	arbitrary
geometrical and dynamical restrictions	the center of mass coincides with the geometrical center	the center of mass does not coincide with the geometrical center	the axes of dynamical and geometrical ellipsoid coincide $\mathbf{I} = \mu\mathbf{E} + m\mathbf{B}$	the geometrical and dynamical axes coincide and contain the center of mass			—
measure	$(1-D(\gamma, \mathbf{A}\gamma))^{-1/2}$	unknown	$(\mu + m\mathbf{r}^2)^{-1/2}$	$(I_1 I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))^{-1/2}$	const	$(I_1 I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))^{-1/2}$	$(\mu + m\mathbf{r}^2)^{3/2}$ (for variables ω and γ)
additional integrals	$M^2 = \text{const}$ $(M, \gamma) = \text{const}$ (two integrals)	$M^2 - m\mathbf{r}^2(M, \omega) = \text{const}$ (one integral)	none of integrals are found	two integrals are obtained from the solution of system of two linear equations (2.13)	two integrals are obtained from the solution of hypergeometric equation (2.17)	$\omega_3/\rho = \text{const}$ $(M, \mathbf{r}) = \text{const}$	none of integrals are found
integrable addition of gyrostat	possible (A. P. Markeev 1986)	it seems to be impossible	the measure is preserved	S. A. Chaplygin (1897)	S. A. Chaplygin (1897)	at $I_1 = I_2 = I_3$ the gyrostat was added by D. K. Bobylev, at $I_1 = I_2 \neq I_3$ by A. S. Kuleshov (2000)	the measure is preserved
Hamiltonian form	the system is Hamiltonian after the change of time (A. V. Borisov, I. S. Mamaev, 2001)	it seems that the system is not Hamiltonian	unknown	the reduced system is Hamiltonian after the change of time, defined by the reducing multiplier (A. V. Borisov, I. S. Mamaev, 2001)			unknown
authors	S. A. Chaplygin (1903)	A. V. Borisov, I. S. Mamaev (2001)	V. A. Yaroschuk (1995)	S. A. Chaplygin (1897)	S. A. Chaplygin (1897), P. Appell, D. Korteweg (1898)	E. J. Routh (1884), S. A. Chaplygin (1897)	V. A. Yaroschuk (1992.)
generalizations and remarks	the integrable addition of Brun field is possible (V. V. Kozlov, 1985) the Hamiltonian form is preserved for arbitrary fields with the loss of one integral	the Brun field can not be added (preserving the integral)	—	—	—	—	—

Remark. The cases of existence of the corresponding (tensor) invariants are indicated by gray color in the table. The partial filling corresponds to the incomplete set of integrals.

where the "minus" sign means the rolling on interior of surface of sphere (Fig. 8 a), and the "plus" sign means the rolling on exterior of surface of sphere (Fig. 8 b).

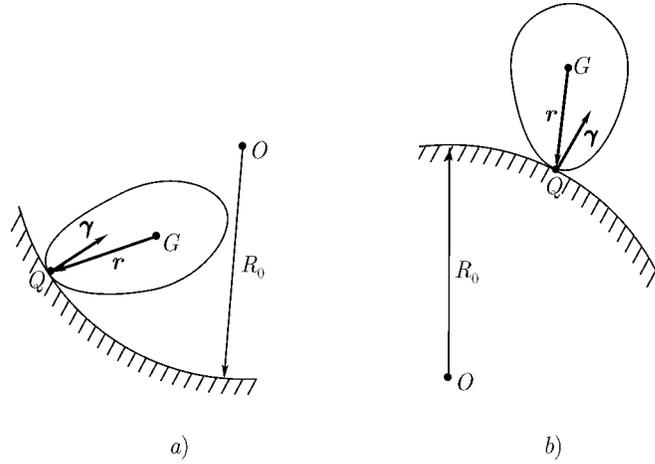


Fig. 8.

Rolling of body of revolution. The body of revolution is defined analogously by formulas (2.1), and we can obtain the explicit form of the density of invariant measure

$$\begin{aligned} \rho(\gamma_3) &= \rho_0(1 - kf_1)^3(1 - kf'_2), \\ \rho_0 &= (I_1I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))^{-1/2} = (I_1I_3 + mI_1f_1^2(1 - \gamma_3^2) + mI_3f_2^2\gamma_3^2)^{-1/2} \end{aligned} \quad (3.2)$$

and establish the presence of the symmetry field \mathbf{v} with operator (2.4).

Reduced system (2.13) in the variables of type (2.5)

$$K_1 = (\mathbf{M}, \mathbf{r})(1 - kf_1)f_1^{-1}, \quad K_2 = \omega_3\rho_0^{-1}, \quad (3.3)$$

has the form

$$\begin{aligned} \frac{1}{\rho_0} \frac{dK_1}{d\gamma_3} &= - \left[I_3 \left(1 - \left(\frac{f_2}{f_1} \right)' \right) + kf_1(I_1 - I_3)(1 - kf'_2) \right] K_2, \\ \frac{1 - kf_1}{\rho_0} \frac{dK_2}{d\gamma_3} &= -mf_1(f_1 - f'_2 - kf_1f'_2)K_1 + mk\rho_0I_1f_1^2 \left(\gamma_3f'_2 + \sqrt{1 - \gamma_3^2} \left(f_1\sqrt{1 - \gamma_3^2} \right)' \right) K_2. \end{aligned} \quad (3.4)$$

Equations (3.4) in somewhat different variables connected with the semifixed axes were obtained by P. V. Woronetz [11], who generalized the Chaplygin arguments to the case of sphere.

Integrating the equations of system (3.4) we determine the dependence on time of the nutation angle $\theta = \arccos(\gamma_3)$ using the quadrature of the energy integral

$$\begin{aligned} H &= \frac{1}{2} \frac{I_1 + m\mathbf{r}^2}{1 - \gamma_3^2} \dot{\gamma}_3^2 + \\ &+ \frac{1}{2I_1(1 - \gamma_3^2)} \left(\frac{K_1^2}{(1 - kf_1)^2} - \frac{I_3}{mf_1^2} K_2^2 + \frac{mf_2^2}{I_1} \left(\frac{K_1}{1 - kf_1} - \frac{K_2}{m\rho f_1 f_2} \right)^2 \right) + U(\gamma_3). \end{aligned}$$

For the problem of rolling of the body of revolution on sphere we can describe the reduced system in variables (2.5) and the corresponding Poisson structure similarly to the case of plane. We do not present the corresponding calculations because of their bulkiness.

Similarly to the problem of rolling on a plane we shall consider some special cases.

The problem of rolling of round disk with the center of masses displaced along the axis of symmetry in general case is not reduced to the hypergeometric equation any more, but the system nevertheless becomes simpler; thus, the measure is not constant any more. Same as above, f_1, f_2 are defined by relations (2.15), and the density of invariant measure has the following form

$$\rho = \rho_0 \left(1 - \frac{kR}{\sqrt{1 - \gamma_3^2}} \right)^3, \quad \rho_0 = \left(I_1 I_3 + m(I_1 R^2 + I_3 a^2) \right)^{-1/2} = \text{const},$$

where R is the disk radius and a is the displacement of the center of mass.

The second-order equation has the form

$$\frac{d^2 \omega_3}{d\theta^2} + \text{ctg } \theta \left(1 + \frac{kR}{\sin \theta - kR} \right) \omega_3 - \frac{\rho_0^2 m R I_3}{1 - \frac{kR}{\sin \theta}} \left(R + a \text{ctg } \theta + \frac{kR^2(I_1 - I_3)}{I_3 \sin \theta} \right) \omega_3 = 0, \quad (3.5)$$

where $\gamma_3 = \cos \theta$.

This equation in the particular case $I_3 = 2I_1, a = 0$ (the homogeneous balanced disk) was obtained by P. Woronetz [40], and at $a = 0$ it was investigated in [33] by methods of the qualitative analysis. In particular, the stability of stationary motions and the probability of falling of disk on sphere was investigated and this probability is found to be equal to zero. In the paper [34] it was also shown that, as against to the nonholonomic problem, the the Hamilton system describing the motion of disk on a sphere with absolute (ideal) sliding that seems to be simpler is not integrable any more, and its behavior has random properties. The number of degrees of freedom for this system is increased in comparison with the case of plane, where the indicated system is integrable because of the preservation of impulse's horizontal component, and this is the reason for such result.

Ball with displaced center. Here functions f_1, f_2 are also defined by relations (2.18), and the expression for measure ρ is the same as in the case of plane:

$$\rho = \left(I_1 I_3 + I_1 m R^2 (1 - \gamma_3^2) + I_3 m (R \gamma_3 + a)^2 \right)^{-1/2}.$$

In variables (3.3) equations (3.4) have the form

$$\frac{1}{\rho} K_1' = -kR(I_1 - I_3)(1 - kR)K_2, \quad \frac{1}{\rho} K_2' = \frac{kmR^3}{1 - kR} K_1. \quad (3.6)$$

With the help of these equations we obtain two linear with respect to K_1, K_2 nonalgebraic integrals of the form

$$F_2 = \left(\sqrt{m(I_3 - I_1)} K_2 + \frac{mR}{1 - kR} K_1 \right) e^{\lambda\tau}, \quad F_3 = \left(\sqrt{m(I_3 - I_1)} K_2 - \frac{mR}{1 - kR} K_1 \right) e^{-\lambda\tau}, \quad (3.7)$$

where $\lambda^2 = mk^2 R^4 (I_3 - I_1), \tau = \int \rho_0(\gamma_3) d\gamma_3$, and the additional quadratic algebraic integral (dependent on F_2, F_3)

$$F = F_2 F_3 = \frac{mR^2}{(1 - kR)^2} K_1^2 + (I_1 - I_3) K_2^2. \quad (3.8)$$

The integrals F_2, F_3 are new and generalize Routh and Jellett integrals (2.20). Integral (3.8) was found by A. S. Kuleshov [18]. In the case of spherical tensor of inertia $I_3 = I_1$ we have

$$K_1 = (\mathbf{M}, \mathbf{r}) = M_1 \gamma_1 + M_2 \gamma_2 + M_3 \left(\gamma_3 + \frac{a}{R} \right) = \text{const}, \quad (3.9)$$

$$(1 - kR) K_2 - kmR^3 K_1 \int \rho(\gamma_3) d\gamma_3 = \text{const},$$

i. e. K_1 coincide with the classical Jellett integral.

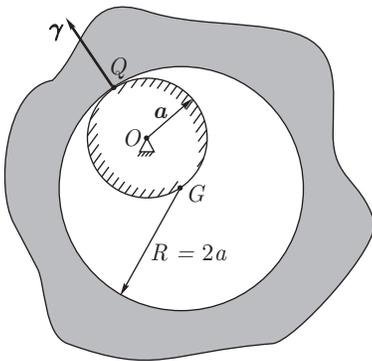


Fig. 9

Balanced, dynamically nonsymmetric ball on sphere.

We shortly describe an integrable case connected with the problem of rolling of balanced, dynamically nonsymmetric ball on sphere analogous to the motion of Chaplygin ball on plane (see (2.23), (2.24)). This system is defined by the equations

$$\begin{aligned} \dot{M} &= M \times \omega, \quad \dot{\gamma} = k\gamma \times \omega, \\ M &= (I + D)\omega - D\gamma(\omega, \gamma), \end{aligned} \tag{3.10}$$

where $k = \frac{R}{R-a}$, a is the radius of ball, m is its mass, $D = ma^2$, R is the radius of the fixed sphere (see Fig. 9). (In the case of plane $R \rightarrow \infty$ and $k = 1$.) Equations (3.10) always have the integrals

$$H = \frac{1}{2}(M, \omega), \quad F_1 = \gamma^2 = 1, \quad F_2 = M^2.$$

and the invariant measure with density ρ (2.24). We need one more integral for the complete integrability of system (such as the area integral at $k = 1$). It exists only under the additional condition $a = 2R$ found by A. V. Borisov in [7], which corresponds to the rolling of balanced, dynamically nonsymmetric ball on the interior of the fixed sphere (see Fig. 9). This integral have the form

$$F = (M, \bar{A}\gamma), \quad \bar{A} = E - 2(\text{tr}(I + D))^{-1}(I + D) \tag{3.11}$$

and can be generalized to case (3.10) with the additional Brun potential $U = \frac{1}{2}(I\gamma, \gamma)$ [7]. Using the transformations $\tilde{M} = \bar{A}M$, $\tilde{\gamma} = \gamma$ under condition $a = 2R$ we transform equations (3.10) into the equations describing the motion of Chaplygin ball on the horizontal plane. For arbitrary parameter k and potential U the Poisson structure of equations (3.10) similar to one indicated at $k = 1$ in subsection 2.2 (formula (2.25)) is still unknown.

Unbalanced, dynamically nonsymmetric ball on sphere. In this case there exists one (and only one) quadratic integral generalizing the corresponding result on plane. The equations are

$$\begin{cases} \dot{M} = M \times \omega + m\dot{r} \times (\omega \times r), \\ \dot{r} = \frac{1}{1-kR}(r - a) \times \omega, \end{cases} \tag{3.12}$$

where $r = R\gamma + a$, $M = I\omega + mr \times (\omega \times r)$. It has exactly the same form

$$F = M^2 - mr^2(M, \omega). \tag{3.13}$$

Note also that for $a \neq 0$ it seems that there is no invariant measure, and also generalizations for the cases with additional gyrostat or with the field of Brun problem are unknown. In the case of rotational symmetry $I_1 = I_2$, integral (3.13) was indicated by A. S. Kuleshov and the acquaintance with this result has induced authors to the analysis of dynamically nonsymmetric situation.

Gyrostatic generalizations. Let's briefly discuss the integrable gyrostatic generalizations. So to preserve the integrability for the body of revolution's case we should direct the balanced rotor with the moment S along the axis of dynamical symmetry. In variables (3.3) the analog of system (2.32) has the following form

$$\begin{aligned} \frac{1}{\rho_0}K'_1 &= - \left[I_3 \left(1 - \left(\frac{f_2}{f_1} \right)' \right) + kf_1(I_1 - I_3)(1 - kf'_2) \right] K_2 + \\ &\quad + s\rho_0^{-1}(1 - kf'_2)(1 - kf_1), \\ \frac{1 - kf_1}{\rho_0}K'_2 &= km\rho_0 I_1 f_1^2 \left[\gamma_3 f'_2 + \sqrt{1 - \gamma_3^2} \left(f_1 \sqrt{1 - \gamma_3^2} \right)' \right] K_2 - \\ &\quad - mf_1(f_1 - f'_2 - kf_1 f'_2) K_1 - smf_1 f_2 (1 - kf'_2)(1 - kf_1), \end{aligned} \tag{3.14}$$

where ρ_0 is defined by relation (3.2); thus, the linear system is non-homogeneous.

In the case of round disk, the integrals and equations (3.14) can not be essentially simplified. In the case of ball with the displaced center, f_1, f_2 are defined by equations (2.19), and equations (3.14) are

$$\begin{aligned} \frac{1}{\rho_0} K_1' &= -kR(I_1 - I_3)(1 - kR)K_2 - \frac{1}{\rho_0} R(1 - kR)^2 s, \\ \frac{1}{\rho_0} K_2' &= \frac{mR^3 k}{1 - kR} K_1 - mR(1 - kR)(R\gamma_3 + a)s. \end{aligned} \tag{3.15}$$

The first integrals in this case can not be written with the help of elementary functions.

In the case of total dynamical symmetry $I_1 = I_3$ with the help of equations (3.15) we obtain the explicit integral of Jellett type

$$F_2 = K_1 + (1 - kR)^2 \left(\gamma_3 + \frac{a}{R} \right) = (MR^{-1} + S(1 - kR), \mathbf{r})(1 - kR) = \text{const.}$$

Expressing from this integral K_1 and substituting the expression in the first equation of system (3.15), we obtain the explicit quadrature for $K_1(\gamma_3)$. The gyrostatic generalizations of integral (3.11) for the case of nonsymmetric ball are unknown.

Rolling of body with partially flat surface on sphere. Let's consider one more problem connected with the rolling of body on a sphere, which has no analog in the case of plane. We consider the rolling of the body with the flat foundation on the exterior surface of sphere (Fig. 11). This problem for the first time was considered by P. Woronetz, who indicated the integrability of the problem in the case of rotational symmetry. Let's write the equations of motion in the frame of references connected with the body in the case when the field of force is absent:

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\omega} + m\dot{\mathbf{r}} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad \mathbf{M} = \mathbf{I}\boldsymbol{\omega} + m\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}), \tag{3.16}$$

where \mathbf{I} is the central tensor of inertia, m is the mass of body.

For the flat part of surface we have $\mathbf{r} = (r_1, r_2, z)$, $z = \text{const}$, $\boldsymbol{\gamma} = (0, 0, 1)$, where r_1, r_2 are the projections of position of the center of mass on the flat foundation (see Fig. 11), and using the fact that $\dot{\boldsymbol{\gamma}} = 0$ we obtain for them using equation (1.3)

$$\dot{r}_1 = k^{-1}\omega_2, \quad \dot{r}_2 = -k^{-1}\omega_1, \quad k = R^{-1}. \tag{3.17}$$

We can specify two cases, when equations (3.16), (3.17) have invariant measure.

a) $z = 0$ — the center of mass is situated on the contact plane, and tensor of inertia $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$ is arbitrary. For the variables \mathbf{M}, \mathbf{r} the density of invariant measure is

$$\rho(\mathbf{M}, r_1, r_2) = (I_3 + m\mathbf{r}^2)^{-1/2}. \tag{3.18}$$

It can be also written for the equations in variables $\boldsymbol{\omega}, r_1, r_2$:

$$\rho(\boldsymbol{\omega}, r_1, r_2) = (I_1 I_2 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))(I_3 + m\mathbf{r}^2)^{1/2}. \tag{3.19}$$

REMARK. In this form under additional and not essential restriction $I_3 = I_1 + I_2$ (i. e. in the case of flat plate) the invariant measure was indicated by V. A. Yaroschuk [28].

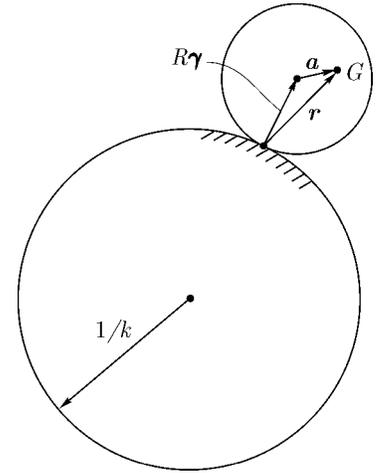


Fig. 10

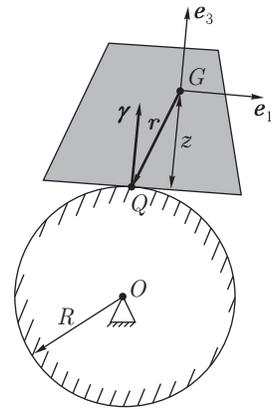


Fig. 11

In the considered case in addition to the measure, equations (3.16), (3.17) have one (and only one for $I_1 \neq I_2$) first integral

$$F = \mathbf{M}^2 - 2m\mathbf{r}^2H, \quad H = \frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}), \tag{3.20}$$

independent of energy integral. It was not known earlier.

Besides, it turns out that integral (3.20) *is transferred without modifications to the case $z \neq 0$, for which the measure is not known* (and probably does not exist). Note that we already see the similar situation in the problem of rolling of *unbalanced*, dynamically nonsymmetric ball (see subsection 3). The lack of measure for $z \neq 0$ obviously prevent the existence of Hamiltonian form. For $z \neq 0$ though the measure exists, the system is probably also not Hamiltonian even after the appropriate change of time.

b) $I_1 = I_2, z \neq 0$, *i. e. the center of mass is situated on the axis of dynamical symmetry*. In variables (\mathbf{M}, \mathbf{r}) the density of invariant measure is

$$\rho = (I_1I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))^{-1/2}. \tag{3.21}$$

This system is already integrable. Indeed, in variables

$$K_1 = (\mathbf{M}, \mathbf{r}) = M_1r_1 + M_2r_2 + M_3r_3, \\ K_2 = \frac{\omega_3}{\rho} = \frac{mz(M_1r_1 + M_2r_2) + (I_1 + mz^2)M_3}{\sqrt{I_1I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r})}}$$

the equations of motion of system have linear form

$$\frac{dK_1}{du} = \frac{I_1 - I_3}{2R\sqrt{I_3(I_1 + mz^2) + I_1mu}}K_2, \quad \frac{dK_2}{du} = \frac{m}{2R\sqrt{I_3(I_1 + mz^2) + I_1mu}}K_1. \tag{3.22}$$

P. Woronetz in the paper [38] noted that their explicit solution can be obtained in elementary functions. This solution defines two linear with respect to \mathbf{M} additional first integrals.

There are two different methods of solving of (3.22).

1) $I_1 < I_3$.

$$K_1 = \sqrt{I_1 - I_3}(-c_1 \cos \varphi(u) + c_2 \sin \varphi(u)), \quad K_2 = \sqrt{m}(c_1 \sin \varphi(u) + c_2 \cos \varphi(u)), \\ \varphi^2 = \frac{(I_1 - I_3)(I_1I_3 + I_3mz^2 + I_1mu)}{I_1^2mR^2}, \quad c_1, c_2 = \text{const.} \tag{3.23}$$

2) $I_1 > I_3$, In this case we should use hyperbolical instead of trigonometrical functions

In this case there is also simple, but dependent quadratic integral

$$F = mK_1^2 + (I_1 - I_3)K_2^2, \tag{3.24}$$

that evidently is a particular case of (3.20). The energy integral can be presented in the form

$$H = \frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}) + U(\mathbf{r}) = \frac{1}{8} \frac{I_1 + mz^2 + mu}{R^2u} \dot{u}^2 + \\ + \frac{1}{2I_1^2u} \left((I_1 + mz^2)K_1^2 + (I_1u + I_3z^2)K_2^2 - \frac{z}{I_1^2u} \sqrt{I_1I_3 + I_3mz^2 + I_1mu} K_1K_2 \right) + U(u), \tag{3.25}$$

and using this form we obtain after the integration of system (3.22) the explicit quadrature for \dot{u} .

In the axisymmetric case we can add rotor with the moment $\mathbf{S} = (0, 0, s)$ along the axis of dynamical symmetry; measure (3.21) is preserved, and equations (3.22) are

$$2R \frac{dK_1}{du} = \frac{I_1 - I_3}{\sqrt{I_3(I_1 + mz^2) + I_1 mu}} K_2 - s, \quad 2R \frac{dK_2}{du} = -\frac{m(K_1 + zs)}{\sqrt{I_3(I_1 + mz^2) + I_1 mu}}. \quad (3.26)$$

The general solution of system (3.26) is presented as a superposition of the solution of homogeneous equation at $s = 0$ (3.23) and the particular solution of non-homogeneous equation (3.26) of the following form

$$K_1^{(part)} = \frac{sz(I_3 + I_1(1 + R/z))}{I_1 - I_3}, \quad K_2^{(part)} = \frac{s\sqrt{I_3(I_1 + mz^2) + I_1 mu}}{I_1 - I_3}.$$

Thus quadratic integral (3.24) in this case is presented as

$$F = m(K_1 - K_1^{(part)})^2 + (I_1 - I_3)(K_2 - K_2^{(part)})^2.$$

REMARK. We have noted in the paper two nontrivial cases of non-existence of one quadratic integral. The similar integral is present in the case of homogeneous ball's motion on the triaxial ellipsoid. The origin of such integrals is connected with the presence of similar integrals in the axisymmetric situation, when the system is completely integrable, and there are two linear integrals. However, in the general case the dependence on the positional variables in these integrals is complicated and is expressed in special functions. In three cases that we have found, the linear integrals are expressed in elementary functions and generate the quadratic integral, which is represented by rational function. This integral can be also generalized to the nonsymmetric situation.

4. Conclusions

In this paper we collect all the known at present cases of existence of invariant measure, integrals, Poisson structure for the equations of nonholonomic rolling of rigid body on plane and sphere. In all conceivable assortment of situations we have not found out any case when *there are two integrals, but no measure*. Possibly it is connected with the specificity of the equations of nonholonomic mechanics.

Depending on the presence of this or that set of invariants there are qualitative distinctions in the behavior of system. The system can exhibit both typically Hamiltonian properties and the properties of conservative systems realized on the example of Celtic stones' problem. For the analysis of the indicated situations we use the method of *three-dimensional Poincaré maps* and exactly this method at first let us to find out integrals (2.29), (3.13), (3.20) numerically, and then to obtain their explicit form. Undoubtedly, the research of three-dimensional point maps in the cases of presence and lack of the measure both from analytical and from the computational point of view allows to find out many remarkable effects in nonholonomic systems.

From the point of view of this approach the problem of global evolution of the Celtic stone [14, 21] is the most interesting. In this problem we have to study the invariant and asymptotic manifolds on the level surface of three-dimensional maps (not preserving an area) realizing under various restrictions on the parameters of system.

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Table 2. Rolling of the body on sphere

tensor of inertia	dynamically nonsymmetric case $I_1 \neq I_2 \neq I_3 \neq I_1$					dynamical symmetry $I_1 = I_2, U = U(\gamma_3)$			
surface of the body	ball		body with partially flat surface			an arbitrary body of revolution	round disk	unbalanced ball	body with partially flat surface
geometrical and dynamical restrictions	the center of mass coincides with the geometrical center		the center of mass does not coincide with the geometrical center	the center of mass is situated on the contact plane ($z = 0$)	the center of mass is not situated on the contact plane	geometrical and dynamical axes coincide and contain the center of mass			
		$a = 2R$							
measure	$(1 - D(\gamma, \mathbf{A}\gamma))^{-1/2}$ (V. A. Yaroschuk, 1992)		unknown	$(I_3 + mr^2)^{-1/2}$ (V. A. Yaroschuk, 1992, A. V. Borisov, I. S. Mamaev, 2001)	unknown	$\frac{(1 - kf_1)^3(1 - k(\gamma_3 f_2)')}{\sqrt{I_1 I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r})}}$ (A. V. Borisov, I. S. Mamaev, 2001)			$(I_1 I_3 + m(\mathbf{r}, \mathbf{I}\mathbf{r}))^{-1/2}$
integrals	$M^2 = \text{const}$ one integral	$M^2 = \text{const},$ $(M, \overline{\mathbf{A}}\gamma) = \text{const}$ (two integrals) (A. V. Borisov, 1994)	$M^2 - mr^2(M, \omega) = \text{const}$ one integral (A. V. Borisov, I. S. Mamaev, 2001)	$M^2 - mr^2(M, \omega) = \text{const}$ (A. V. Borisov, I. S. Mamaev, 2001)		two integrals are obtained from the solution of system of two linear equations (3.4) (P. V. Woronetz, 1909)	two integrals are obtained from the solution of second order equation (3.5) (P. V. Woronetz, 1909)	two integrals are expressed in elementary functions (3.7) (A. S. Kuleshov, 2000, A. V. Borisov, I. S. Mamaev, 2001)	two integrals are expressed in elementary functions (3.23) (P. V. Woronetz, 1911)
integrable addition of gyrostat	possible without loss of integral and measure	not found	not found	not found	not found	possible along the axis of dynamical symmetry			
Hamiltonian form	probably, the system is not Hamiltonian	the system is Hamiltonian after the change of time (A. V. Borisov, I. S. Mamaev, 2000)	probably, the system is not Hamiltonian	probably, the system is not Hamiltonian		the reduced system is Hamiltonian after the change of time defined by the reducing multiplier (A. V. Borisov, I. S. Mamaev, 2001)			
generalizations and remarks	the addition of Brun field is possible preserving one integral and measure	the integrable addition of Brun field is possible (A. V. Borisov, Yu. N. Fedorov (1994))	the Brun field could not be added (preserving the integral)	V. A. Yaroschuk found the measure under additional nonessential restriction $I_3 = I_1 + I_2$	—	—	—	A. S. Kuleshov found only one quadratic integral dependent on two present linear integrals	P. V. Woronetz found the solution in quadratures under additional nonessential restriction $I_1 = I_2 = \frac{1}{2}I_3,$ $z = 0$

Remark. The cases of existence of the corresponding (tensor) invariants are indicated by gray color in the table. The partial filling corresponds to the incomplete set of integrals.

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