

# Bi-hamiltonian Geometry and Separation of Variables for Gaudin Models: a case study

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## Abstract

We address the study of the classical Gaudin spin model from the bi-Hamiltonian point of view. We describe in details the  $sl(2)$  three particle case

## 1 Introduction

The Gaudin models are among the best known integrable lattice models [6, 7]. Physically, they consist of a system of  $N$   $sl(2)$ -valued “spins”  $A_i$ , possibly coupled to an external magnetic field whose strength is assumed to depend on the lattice site. Its phase space  $M$  can be identified with the  $n$ -fold product of  $sl(2)$ , and the Hamiltonian is

$$H_G = \frac{1}{2} \sum_{i \neq j=1}^N \text{Tr}(A_i A_j) + \sum_{i=1}^N \text{Tr}(a_i \sigma A_i), \quad \sigma, A_i \in sl(2). \quad (1.1)$$

On  $M$  there is a natural Hamiltonian structure, the direct product of the Poisson–Lie bracket on  $sl(2)$ . Complete integrability can be deduced by considering the Lax representation of such a model. One considers the Lax matrix  $L_{rat}(\lambda) = \sigma + \sum_{i=1}^N \frac{A_i}{\lambda - a_i}$ , and proves [9, 3, 4] that

- a) The Hamiltonian flow associated with  $H_G$  is a Lax flow;
- b) integrals of the motion are provided by the spectral invariants of  $L$ ;
- c) the Lax equations admits an  $R$ -matrix formulation, so that the integrals of the motion are in mutual involution.

In this note we want to address the problem from the standpoint of bi-Hamiltonian geometry, namely to frame these models within the so-called Gel’fand–Zakharevich (GZ) [8] set-up for bi-Hamiltonian integrable systems. Essentially, the core of such an approach relies in the extensive use of the Lenard recursion relation, associated with a bi-Hamiltonian structure, to generate mutually commuting constants of the motion, via the so-called method of the Casimirs of the Poisson pencil.

To equip the Gaudin model with a bi-Hamiltonian structure we will use a suitable strategy, to be discussed in Section 2, which consists in considering, along with the Lax matrix  $L_{rat}$  a *polynomial* Lax matrix  $L_{pol} = \prod_{i=1}^N (\lambda - a_i) L_{rat}$ , and recalling that on the space of polynomial matrices a family of mutually compatible Poisson structures is defined. The natural Poisson structure we referred above turns out to be a specific element of such a family, and so we can define a second Poisson structure for the Gaudin model simply choosing another element of such a family. Successively, we use this bi-Hamiltonian structure to recover complete integrability of the model following the GZ recursion procedure. Finally, we will briefly address the problem of Separation of variables, in the bi-Hamiltonian set-up discussed in[5].

Such an analysis can be performed for an arbitrary number of spins (or “particles”), and, *mutatis mutandis* letting the “spins” be  $sl(n)$ ,  $n \geq 3$  valued. A full account of this program is outside the size of this paper. In this note we will use the three particle  $sl(2)$  model for a case study. The above mentioned generalization of this approach will be published elsewhere.

## 2 The model and the bi-Hamiltonian structure

We consider the three-particle  $sl(2)$  Gaudin model, whose degrees of freedom are encoded in the Lax matrix with spectral parameter  $\lambda$

$$L_{rat} = \sigma + \sum_{i=1}^3 \frac{A_i}{\lambda - a_i}, \quad (2.1)$$

where the  $a_i$  are distinct constant parameters and

$$A_i = \begin{pmatrix} h_i/2 & f_i \\ e_i & -h_i/2 \end{pmatrix}, \quad i = 1, 2, 3, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The phase space  $M$  is<sup>1</sup> the Cartesian product of three copies of  $sl(2)$ ; on  $M$  one can define “standard” Poisson brackets, simply taking the Cartesian product of the Lie–Poisson brackets on  $sl(2)$ :

$$\{h_i, e_j\} = 2\delta_{ij}e_i \quad \{h_i, f_j\} = -2\delta_{ij}f_i \quad \{e_i, f_j\} = \delta_{ij}h_i$$

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<sup>1</sup>Hereinafter we will not distinguish between  $sl(2)$  and its dual, assuming implicitly to identify the two by means of the Killing form.

Here and in the sequel we will extensively use a “matrix” representation of these brackets (rather, of the Poisson tensor  $P$  associated with these brackets), which can be explained as follows.

We identify the tangent and cotangent bundles to  $M$  with  $M$  itself, so that a tangent vector will be written as the vector  $\dot{X} = (\dot{A}_1, \dot{A}_2, \dot{A}_3)$  and the differential of a function  $F$  as  $dF = \left( \frac{\partial F}{\partial A_1}, \frac{\partial F}{\partial A_2}, \frac{\partial F}{\partial A_3} \right)$ , and notice that the Hamiltonian vector fields associated with the Lie–Poisson brackets are given by  $\dot{A}_i = [A_i, \frac{\partial F}{\partial A_i}]$ ,  $i = 1, 2, 3$ . We can recast in matrix form such a formula as:

$$\dot{X} = \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \end{pmatrix} = PdF = \begin{pmatrix} [A_1, \cdot] & 0 & 0 \\ 0 & [A_2, \cdot] & 0 \\ 0 & 0 & [A_3, \cdot] \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial F}{\partial A_1} \\ \frac{\partial F}{\partial A_2} \\ \frac{\partial F}{\partial A_3} \end{pmatrix} \quad (2.2)$$

Similarly, we will write in an analogous matrix form the linear Poisson brackets we are going to discuss in the sequel.

To endow  $M$  with a bi-Hamiltonian structure consider, along with the rational Lax matrix  $L_{rat}$  the *polynomial* Lax matrix

$$L_{poly} = \prod_{i=1}^3 (\lambda - a_i) L_{rat} = \lambda^3 \sigma + B_2 \lambda^2 + B_1 \lambda + B_0. \quad (2.3)$$

Explicitly, this change of coordinates on  $M$  is given by:

$$\begin{aligned} B_0 &= a_2 a_3 A_1 + a_1 a_3 A_2 + a_1 a_2 A_3 - s_3 \sigma \\ B_1 &= -(a_2 + a_3) A_1 - (a_1 + a_3) A_2 - (a_1 + a_2) A_3 + s_2 \sigma \\ B_2 &= A_1 + A_2 + A_3 - s_1 \sigma \end{aligned} \quad (2.4)$$

with

$$s_1 = (a_1 + a_2 + a_3), s_2 = a_1 a_2 + a_1 a_3 + a_2 a_3, s_3 = a_1 a_2 a_3. \quad (2.5)$$

The rationale for this “coordinate change” is that, on the space of polynomial pencils of matrices a family of mutually compatible Poisson brackets are defined[9] via R–matrix theory. In a nutshell, this family can be described (in our three particle case) by saying that there is a map  $\phi$  from the space of degree three polynomials in the variable  $\lambda$  to the set of Poisson structures on the manifold of polynomial Lax matrices of the form (2.3) which sends the monomials  $\lambda^0, \dots, \lambda^3$  into four fundamental Poisson brackets,  $\Pi_i, i = 0, \dots, 3$ . These fundamental brackets are

written as follows:

$$\begin{aligned} \Pi_0 &= \begin{pmatrix} [B_1, \cdot] & [B_2, \cdot] & [\sigma, \cdot] \\ [B_2, \cdot] & [\sigma, \cdot] & 0 \\ [\sigma, \cdot] & 0 & 0 \end{pmatrix} & \Pi_1 &= \begin{pmatrix} -[B_0, \cdot] & 0 & 0 \\ 0 & [B_2, \cdot] & [\sigma, \cdot] \\ 0 & [\sigma, \cdot] & 0 \end{pmatrix} \\ \Pi_2 &:= \begin{pmatrix} 0 & -[B_0, \cdot] & 0 \\ -[B_0, \cdot] & -[B_1, \cdot] & 0 \\ 0 & 0 & [\sigma, \cdot] \end{pmatrix} & \Pi_3 &= - \begin{pmatrix} 0 & 0 & [B_0, \cdot] \\ 0 & [B_0, \cdot] & [B_1, \cdot] \\ [B_0, \cdot] & [B_1, \cdot] & [B_2, \cdot] \end{pmatrix} \end{aligned}$$

A straightforward computation shows that under the map connecting the  $A_i$ 's with the  $B_j$ 's, the standard Poisson bracket (2.2) is sent into the combination:

$$\Pi_3 - s_1\Pi_2 + s_2\Pi_1 - s_3\Pi_0 \quad (2.6)$$

of the fundamental brackets. So we can regard  $P$  as being associated by  $\phi$  with the polynomial

$$p(\lambda) := (\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = \lambda^3 - s_1\lambda^2 + s_2\lambda - s_3$$

and we consider the Poisson tensor  $Q = \Pi_2 - s_1\Pi_1 + s_2\Pi_0 = \phi(q(\lambda))$ , where

$$q(\lambda) = (p(\lambda)/\lambda)_+ = \lambda^2 - s_1\lambda + s_2.$$

Translating back in the coordinates associated with the matrices  $A_i$  the tensor  $Q$ , we get that its components are given by expressions of the form:

$$Q_{i,j} = \left( \sum_{k=1}^3 c_{i,j}^k [A_k, \cdot] \right) + d_{i,j}[\sigma, \cdot] \quad (2.7)$$

where the  $c_{i,j}^k$  and the  $d_{i,j}$  are somewhat complicated rational functions of the parameters  $a_1, a_2, a_3$ , whose explicit expressions are irrelevant here.

Summing up, we have equipped the phase space  $M = (sl(2))^3$  of the Gaudin model with the Poisson pencil

$$P_\lambda = Q - \lambda P.$$

Quite clearly, the specific choice of the polynomial  $q(\lambda)$  is somewhat arbitrary, and different choices could be considered. However, in this paper we will stick to this choice, and, in the next Section we will study the GZ geometry of such a pencil.

### 3 The GZ analysis of the Poisson pencil

The Gel'fand–Zakharevich method is a “recipe” to associate with a Poisson pencil  $Q - \lambda P$  an integrable system (that is, a family of mutually commuting vector fields). It is particularly suited for studying Poisson pencils of non–maximal rank. Its core can be described as follows:

Let  $M$  be a  $2n + k$  dimensional bi-Hamiltonian manifold, and let us suppose that the rank of  $P$  and  $Q$  equals  $2n$ . One starts fixing a basis  $\{H_i^0\}_{i=1,\dots,k}$  of the Casimir functions of  $P$ , that is functions satisfying  $PdH_i^0 = 0$ . Applying the second tensor  $Q$  to one of such functions one gets (in general) a non trivial vector field  $X_i^1 = QdH_i^0$ , and tries to find another function  $H_i^1$  such that  $QdH_i^0 = PdH_i^1$ . If such an equation can be solved for  $H_i^1$ , then one generates a new vector field applying  $Q$  to  $dH_i^1$ , and so on and so forth. Supposing one can iterate this process, one finds, for each independent Casimir of  $P$ , a Lenard chain of vector fields  $X_j^a$ . As a consequence of the bi-Hamiltonian recursion relations, all functions  $H_i^j$  commute (w.r.t. both Poisson brackets), even if they do not pertain to the same Lenard chain. Complete integrability (of every vector field of the chains) is recovered whenever  $n + k$  elements of the family of functions  $H_k^j$  are functionally independent, noticing that  $k$  elements of such “fundamental” Hamiltonians are provided by the  $k$  Casimirs  $H_i^0$ . Indeed, all vector fields of the Lenard chains are tangent to the generic symplectic leaf of  $P$ , and, when restricted to such a leaf, the remaining  $n$  functions of the fundamental Hamiltonians provide the required  $n$  mutually commuting integrals of the motion.

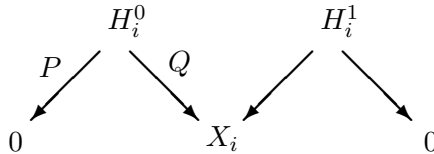
To apply these ideas to the three particle Gaudin model, we first notice that  $\dim M = 9$  and the rank both of  $P$  and  $Q$  equals 6. Actually, a basis  $H_i^0$  of Casimirs is given by the three functions

$$H_i^0 = \text{Tr} A_i^2 = \text{res}_{\lambda=a_i} (\lambda - a_i) \text{Tr}(L_{rat}^2)$$

Furthermore, applying  $Q$  to such functions we get three (independent) vector fields  $X_i$ . Finally, a long but straightforward computation shows that it is possible to find three additional functions  $H_i^1$  such that:

$$PdH_i^1 = X_i, \quad QdH_i^1 = 0$$

In other words, we can arrange these six functions in three Lenard chains of the form



The new Hamiltonians  $H_i^1$  are a linear combinations of the Casimirs  $H_i^0$  and of

the other independent spectral invariants  $K_i$  of the Lax matrix, given by

$$K_i = \text{res}_{\lambda=a_i} \text{Tr}(L_{rat}^2) = 2 \left( \sum_{j \neq i} \frac{\text{Tr}(A_i A_j)}{a_i - a_j} + a_i \text{Tr}(\sigma A_i) \right).$$

For instance one has

$$H_1^1 = \frac{(-a_1 g_1 H_1^0 + a_2 a_3 (a_2 - a_3)(a_2 H_2^0 - a_3 H_3^0) + t_1 K_1)}{a_1 (a_1 - a_2)^2 (a_1 - a_3)^2} \quad (3.1)$$

with

$$g_1 = a_1 (s_1^2 - 3s_3), t_1 = a_1 a_2 a_3 (a_1 - a_2)(a_1 - a_3).$$

( $H_2^1$  and  $H_3^1$  are obtained by cyclic permutations of the indexes  $(1, 2, 3)$ ). Notice that there is an invertible linear relation (with constant coefficients) between the basis of spectral invariants  $\{H_i^0, K_i\}$  and the basis of GZ Hamiltonians  $\{H_i^0, H_i^1\}$

Summing up the bi-Hamiltonian structure we consider gives rise to three Lenard chains of “length” one each (that is each one comprising one independent vector field). Since the “physical” Gaudin Hamiltonian (1.1) is a linear combination of the  $K_i$ ,  $H_g = \frac{1}{2} \sum_{i=1}^3 a_i K_i$  and hence of the GZ Hamiltonians, we have recovered complete integrability of the model. Finally, notice that we can collect the Hamiltonians  $\{H_i^0, H_i^1\}$  into the three polynomials

$$F_i(\lambda) = \lambda H_i^0 + H_i^1,$$

and algebraically represent the short Lenard chain(s) depicted above by means of the formula  $(Q - \lambda P) dF_i(\lambda) = 0$ .

## 4 On Separation of Variables

In this last Section we will briefly address the problem of Separation of Variables for the inhomogeneous Gaudin models. Separation of variables (SoV) for this model was proven [12, 13, 3, 4], who defined separated variables as coordinates of the poles of a suitably normalized Baker–Akhiezer function associated with the Lax matrix (2.1). A geometrical approach to the SoV problem, based on bi-Hamiltonian geometry, has quite recently been discussed in the literature [11, 1, 2, 5]. Here we want to show that SoV for the Gaudin models falls within this scheme.

The basic geometrical object underlying the bi-Hamiltonian approach to SoV is a so-called  $\omega N$  manifold, that is, the datum of a symplectic manifold  $(V, \omega)$  endowed with an additional  $(1, 1)$ -tensor  $N$  with vanishing torsion satisfying a suitable compatibility condition with respect to the symplectic 2-form  $\omega$  (a *Nijenhuis* tensor for short) The content of the “bi-Hamiltonian” SoV theorem (see [5]) can be stated as follows:

**I)** On (suitable open sets of) of the  $\omega N$  manifold  $V$  a family distinguished canonical coordinates (called Darboux–Nijenhuis (DN) coordinates) is intrinsically defined through the spectral properties of its Nijenhuis tensor  $N$  [10].

**II)** A Liouville integrable system, characterized by a complete set of mutually (w.r.t. the Poisson tensor  $\omega^{-1}$ ) integrals of the motion  $\{H_i\}_{i=1,\dots,N}$  is separable in DN coordinates if and only if these Hamiltonians commute also w.r.t. the second bracket defined by

$$Q = N \cdot \omega^{-1}. \quad (4.1)$$

As we have seen, the Gaudin models (as it is the case for a number of integrable systems related with R–matrix theory and/or reductions of soliton equations) admit a (reasonably) natural bi-Hamiltonian formulation on a bi-Hamiltonian manifold where none of the brackets of the pencil is non degenerate. So, the first problem to be tackled in such an instance is the so–called reduction problem, that is to concoct from the (degenerate) bi-Hamiltonian structure of the problem a structure of  $\omega N$  manifold on suitable submanifolds of  $M$ , in such a way to satisfy condition **II** above.

To show that this can be successfully achieved in the Gaudin models we will follow a recipe discussed in [2, 5], specifically suited for GZ systems. We consider the Poisson pencil  $Q - \lambda P$ , and fix a basis  $\{H_1^0, \dots, H_k^0\}$  of Casimirs of  $P$ , where we assume that  $k = \text{corank}(P) = \text{corank}(Q)$ . The symplectic leaves  $S_c$  of  $P$  are thus characterized by the equations  $H_i^0 = \text{const}_i$ , and come equipped with a natural symplectic structure. It holds[5] the

**Proposition:** In the above geometrical set–up, let  $Z_1, \dots, Z_h$  be a family of vector fields on  $M$ , transversal to the symplectic leaves of  $P$  such that the functions vanishing along the distribution  $\langle Z_i \rangle$  generated by the  $Z_i$ 's are a Poisson subalgebra both w.r.t.  $P$  and  $Q$ . Then the generic symplectic leaf of  $P$  has the structure of an  $\omega N$  manifold. The Nijenhuis tensor  $N$  is defined, via Eq. (4.1) by the restriction of the following (modified) Poisson tensor  $\tilde{Q} = Q - \sum_{i=1}^h Q(d(H_i^{\text{top}})) \wedge \tilde{Z}_i$ , where  $\tilde{Z}_i$  are a normalized basis (possibly defined on an open subset of  $S_c$ ) for  $\langle Z_i \rangle$ , i.e.  $\tilde{Z}_j(H_i^0) = \delta_{ij}$ . Condition **II** above is automatically satisfied.  $\square$

We will now apply this scheme to the three particle  $sl(2)$  Gaudin model, endowed with the bi-Hamiltonian structure  $Q - \lambda P$  introduced in Section 2. As we have seen, the GZ structure of the problem is quite simple. We have three Casimir polynomials  $F_i(\lambda)$  of degree one, (meaning that, indeed  $\text{corank}(P) = \text{corank}(Q) = 3$ ) and so we have to find a three–dimensional distribution satisfying the properties of the Proposition recalled above.

The idea to solve this problem is very simple, and relies on the following observation on the (ordinary) Lie-Poisson brackets on a single copy of  $sl(2)$ . With the notations of Section 2 the Poisson bracket of two functions  $F, G$  on  $M$ , is given by  $\{F, G\} = \text{Tr}(\frac{\partial F}{\partial A} \cdot [A, \frac{\partial G}{\partial A}]) = -\text{Tr}(A \cdot [\frac{\partial F}{\partial A}, \frac{\partial G}{\partial A}])$ . Parametrizing the generic

element  $A$  of  $sl(2)$  as  $A = \begin{pmatrix} h/2 & f \\ e & -h/2 \end{pmatrix}$  we consider the vector field  $Z = \frac{\partial}{\partial e}$ .

We notice that differentials of functions vanishing along  $Z$  admit a very simple matrix representation. Indeed  $Z$  is matrixially represented as  $Z(A) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,

and so  $Z(F) = 0$  iff  $\left(\frac{\partial F}{\partial A}\right)_{1,2} = 0$ , i.e., iff  $\frac{\partial F}{\partial A}$  lies in the lower Borel subalgebra  $\mathfrak{b}_-$  of  $sl(2)$ .

Let  $F, G$  be functions such that  $Z(F) = Z(G) = 0$ , and let us compute  $Z(\{F, G\})$ . Thanks to the Leibniz property of the Lie derivative and the fact that  $Z$  is a constant vector field we have that

$$Z(\{F, G\}) = -\text{Tr}(Z(A) \cdot [\frac{\partial F}{\partial A}, \frac{\partial G}{\partial A}]) \quad (4.2)$$

which vanishes as well since  $\mathfrak{b}_-$  is indeed a Lie subalgebra of  $sl(2)$ . Finally we notice that the Casimir  $C$  of the Lie-Poisson bracket is given by  $C = h^2/2 + 2ef$ , so  $Z(C) = 2f$  and hence  $Z$  is generically transversal to the symplectic foliation of the Lie Poisson brackets; in particular, the normalized generator  $\tilde{Z}$  is given by  $1/2fZ$ .

To use this simple result in the case of the three particle  $sl(2)$  Gaudin model we consider the vector fields  $Z_i = \frac{\partial}{\partial e_i}, i = 1, 2, 3$ . The differentials of functions  $F$  vanishing along  $\langle Z_i \rangle$  are represented by triple of matrices  $dF = (\frac{\partial F}{\partial A_1}, \frac{\partial F}{\partial A_2}, \frac{\partial F}{\partial A_3})$  with  $\frac{\partial F}{\partial A_i} \in \mathfrak{b}_-, i = 1, 2, 3$ . The fact that such functions are a Poisson subalgebra for  $P$  is self evident. To ascertain that the same is true for  $Q$  one simply has notice that, using the explicit expressions (2.7), the brackets  $\{F, G\}_Q = \langle dF, QdG \rangle$  are given by the multiple sum

$$\{F, G\}_Q = \sum_{i,j} \text{Tr} \left( \frac{\partial F}{\partial A_i} \cdot \left( \sum_{k=1}^3 c_{i,j}^k [A_k, \frac{\partial G}{\partial A_j}] \right) + d_{i,j} [\sigma, \frac{\partial G}{\partial A_j}] \right).$$

Since  $\sigma$  is a constant and the vector fields  $Z_i$  admit the matrix representation  $Z_i(A_j) = \delta_{ij}Z(A)$  we see that the Lie derivatives  $Z_i(\{F, G\}_Q)$  is a multiple sum of terms like those of Eq. (4.2), and so vanish whenever  $Z(F) = Z(G) = 0$ . This proves that on the symplectic leaves of  $P$  the bi-Hamiltonian pencil  $Q - \lambda P$  induces an  $\omega N$  structure whose DN coordinates separate the Hamilton–Jacobi equations of the Gaudin models.

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