Generalized hierarchy of matrix Burgers type and n-wave equations

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Abstract

This article concerns the dressing method for solving of multidimensional nonlinear Partial Differential Equations. In particular, we join hierarchy of matrix Burgers type equation with hierarchies of equations integrable by the Inverse Spectral Transform (IST). Example of resonance interaction of wave packets in (3+1)-dimensions is given.

1 Introduction

It is well known that different versions of the dressing method [1, 2, 3, 4, 5] are very successful tools for solving of so called completely integrable nonlinear Partial Differential Equations (PDE). These equations, in turn, have wide application area in different branches of physics, such as hydrodynamics, plasma physics, superconductivity, nonlinear optics. Although till recently dressing methods have been used only for PDE integrable by the Inverse Spectral Transform (IST), it has been shown in [6] that there is another type of equations (maybe not integrable by IST) which admit properly modified dressing procedure for construction of large manifold of their solutions. But the technique proposed there left opened many questions. For instance, it is not clear whether derived nonlinear PDE can be linearized by some substitution [7, 8]. Also it was difficult to characterize the manifold of available solutions.

In this paper we replace algebraic operator with integral one, generalize system of equations introducing the set of additional parameters (independent variables of nonlinear PDE) and modify significantly the algorithm given in [6]. This allows us to simplify description of PDE's properties, exhibit more information about solution manifold as well as relations to the classical solvable (both linearizable and integrable by IST) PDE. Although all statements following hereafter can be proved, we omit most of the proofs for the sake of brevity. They will be given in different paper.

Thus, the basic object is the following $N \times N$ matrix integral equation

$$\Phi \equiv \Phi(\lambda, \mu; t) = \int\limits_{D_\nu} \Psi(\lambda, \nu; t) U(\nu, \mu; t) d\nu \equiv \Psi * U, \tag{1}$$

where $\lambda = (\lambda_1, \dots, \lambda_{N_\lambda})$, $\mu = (\mu_1, \dots, \mu_{N_\mu})$, $\nu = (\nu_1, \dots, \nu_{N_\nu})$ are vector variables with different length in general; integration is over whole space D_ν of the appropriate vector parameter; Φ, Ψ and U are $N \times N$ matrix functions of arguments. Star means integration over space of inner variable: $f * g \equiv \int\limits_{D_\nu} f(\lambda_1, \dots, \nu) g(\nu, \dots, \mu_n) d\nu$. We require that Ψ is invertible operator, i.e. equation (1) can be solved uniquely for the function U. By definition, operator $\mathcal{A}(\lambda, \mu)$ is invertible, if there are operators $\mathcal{A}_L^{-1}(\lambda, \mu)$ and $\mathcal{A}_R^{-1}(\lambda, \mu)$ such that $\int\limits_{D_\nu} \mathcal{A}(\lambda, \nu) \mathcal{A}_R^{-1}(\nu, \mu) d\nu = \int\limits_{D_\nu} \mathcal{A}_L^{-1}(\lambda, \nu) \mathcal{A}(\nu, \mu) d\nu = \delta(\lambda - \mu)$. The functions Φ and Ψ are related by means of the *compatible* system of linear integral-differential equations, which, on the other hand, introduce (infinite) set of additional parameters $t = (t_1, t_2 \dots)$ (independent variables of nonlinear PDE):

$$M_i * \Psi = \sum_k L_{ik} * \Phi * C_{ki}, \quad C_{ki} = C_{ki}(\lambda, \mu; t), \quad i = 1, 2, \dots \tag{2}$$

where $M_j = M_j(\lambda, \mu; \partial_{t_1}, \partial_{t_2}, \dots)$ are first order and $L_{jk} = L_{jk}(\lambda, \mu; \partial_{t_1}, \partial_{t_2}, \dots)$ are arbitrary order linear differential operators with matrix coefficients depending on λ and μ. This overdetermined system together with its compatibility condition defines Ψ and Φ. Finally, the same compatibility condition with substitution Φ from the eq.(1) results in nonlinear PDE whose solution is expressed in terms of U.

After this preliminary discussion we derive some general equations using the following simplified version of the system (2):

$$\Psi_{t_i} = S_i * \Phi * C_i, \quad i = 1, 2, \dots, \tag{3}$$

where $S_i(\lambda, \mu; t)$ and $C_i(\lambda, \mu; t)$ are known functions of t, which will be seen later. The compatibility condition for the system (3) has the form

$$\begin{aligned} S_{it_j} * \Phi * C_i - S_{jt_i} * \Phi * C_j + S_i * \Phi * C_{it_j} - S_j * \Phi * C_{jt_i} + \\ S_i * \Phi_{t_j} * C_i - S_j * \Phi_{t_i} * C_j = 0, \end{aligned} \tag{4}$$

which is linear system of compatible integral-differential equations for the function Φ. Solving this equation, substituting result in (3) and integrating it, we obtain the expression for Ψ: $\Psi(\lambda, \mu; t) = \partial_{t_1}^{-1}(S_1 * \Phi * C_1)(\lambda, \mu; t) + E(\lambda, \mu) + F(\lambda, \mu; t_2, t_3, \dots)$. Here E is invertible operator, function F provides compatibility of the system (3). Being invertible, operator Ψ provides unique solution to the eq.(1).

On the other hand, eq.(4) may be given another form after substitution eq.(1) for Φ and (3) for Ψ_{t_i}

$$\begin{aligned} S_{it_j} * \Psi * U * C_i - S_{jt_i} * \Psi * U * C_j + S_i * \Psi * U * C_{it_j} - S_j * \Psi * U * C_{jt_i} + \\ S_i * (S_j * \Psi * U * C_j * U + \Psi * U_{t_j}) * C_i - S_j * (S_i * \Psi * U * C_i * U + \Psi * U_{t_i}) * C_j = 0, \end{aligned} \tag{5}$$

which is nonlocal equation quadratic in U. It may result in nonlinear PDE for the dependent variables, expressed in terms of U, S_i and C_i. To provide this possibility we must impose specific dependence of the functions S_i and C_i on their arguments, for instance, in accordance to the following set of relations:

$$S_i(\lambda,\mu;t) = S(\lambda,\mu), \;\; \Phi = S*\Phi + \chi, \;\; \chi = \chi(\lambda,\mu;t), \tag{6}$$

$$C_i(\lambda,\mu;t) = \int\limits_{D_\nu} A_i(\lambda,\nu)p_1(\nu;t)p_2(\mu)d\nu + c_1(\lambda)B_i c_2(\mu;t) \equiv A_i * p_1(t)p_2 + c_1 B_i c_2(t), \tag{7}$$

$$A_i * c_1 = c_1 B_i, \tag{8}$$

where A is invertible operator, $A_i * A_j = A_j * A_i$, $[B_i, B_j] = 0$ and $[*, *]$ means commutator of two matrices. Eqs.(6-8) split eq.(4) into the following set of three integral-differential equations for Φ, p_1 and c_2 :

$$S*\Phi_{t_j}*A_1 - S*\Phi_{t_1}*A_j = 0, \tag{9}$$

$$A_1*(p_1)_{t_j} - A_j*(p_1)_{t_1} = 0, \tag{10}$$

$$B_1 c_{2t_j} - B_j c_{2t_1} = 0. \tag{11}$$

Then one has the following nonlinear equation instead of (5)

$$\begin{aligned} \Psi*(U_{t_j}*A_1 - U_{t_1}*A_j + U*C_j*U*A_1 - U*C_1*U*A_j)+ \\ \chi_{t_1}*A_j - \chi_{t_j}*A_1 + \chi*(C_1*U*A_j - C_j*U*A_1) \;\; = \;\; 0. \end{aligned} \tag{12}$$

Note, that reduction leading to the equations introduced in [6] will be discussed in different paper. Here we consider two other examples of multidimensional systems. First of them (Sec.2.) represents combination of linearizable (Burgers type) and completely integrable (n-wave) $(3+1)$-dimensional systems, having solutions depending on arbitrary functions of three variables. Second example (Sec.3) is another generalization of the matrix n-wave system [9]. Properly introduced multiple scales expansion of this system results in the multidimensional ((3+1)-dimensional in our case) equation describing resonance interaction of wave packets. Its solutions may depend on arbitrary functions of two variables. Both examples have extension into $(n+1)$-dimensions with arbitrary n.

2 Generalized hierarchy of linearizable (Burgers type) and integrable by IST (n-wave) systems

In this section $S(\lambda,\mu) = \delta(\lambda-\mu)$, $\chi = 0$, $A_j = \underbrace{A*\cdots*A}_{j} \equiv A^j$, $B_j = B^j$, where $A(\lambda,\mu)$ is invertible operator and B is nondegenerate constant matrix. Thus $\Psi_{t_i} = \Phi * C_i$ with C_i given by (7). After applying operator Ψ^{-1} from the left to the eq.(12) one results in

$$\begin{aligned} E_j = U_{t_j}*A + U*A^j*p_1p_2*U*A + U*c_1B^jc_2*U*A- \\ (U_{t_1}*A^j + U*A*p_1p_2*U*A^j + U*c_1Bc_2*U*A^j) \;\; = \;\; 0 \end{aligned} \tag{13}$$

We may derive nonlinear system for the functions

$$u = p_2*U*c_1, \;\; q_n = p_2*U*A^n*p_1, \;\; v_n = \partial^n c_2*U*c_1, \;\; w_{nm} = \partial^n c_2*U*A^m*p_1, \tag{14}$$

which has the following "short" form

$$p_2 * E_j * c_1 = 0, \;\; p_2 * E_j * A^n * p_1 = 0, \;\; \partial^n c_2 * E_j * c_1 = 0, \;\; \partial^n c_2 * E_j * A^m * p_1 = 0, \tag{15}$$

or, extended form

$$\begin{aligned}
u_{t_j} - u_{t_1} B^{j-1} + q_j u - q_1 u B^{j-1} + u B^j v_0 - u B v_0 B^{j-1} &= 0, && (16)\\
q_{n t_j} - q_{n+j-1 t_1} + q_j q_n - q_1 q_{n+j-1} + u B^j w_{0n} - u B w_{0(n+j-1)} &= 0, &&\\
v_{n t_j} - v_{n t_1} B^{j-1} + w_{nj} u - w_{n1} u B^{j-1} - B^{j-1} v_{n+1} + v_{n+1} B^{j-1} + & && (17)\\
v_n B^j v_0 - v_n B v_0 B^{j-1} &= 0, &&\\
w_{mn t_j} - w_{m(n+j-1)_{t_1}} + w_{mj} q_n - w_{m1} q_{n+j-1} - B^{j-1} w_{(m+1)n} + & && (18)\\
w_{(m+1)(n+j-1)} + v_m B^j w_{0n} - v_m B w_{0(n+j-1)} &= 0. && (19)
\end{aligned}$$

The complete system of pure PDE is represented by the following set: eq. (16) with $j = 2$, eqs. (17) and (18) with $j = 2,\ 3$, eq. (19) with $j = 2,\ 3,\ 4$. Thus, this system is (3+1)-dimensional. It may be given the compact form if one introduces column of matrices u, q_n, v_n, w_{nm}: $\chi = [u, q_1, q_2, \ldots, v_1, v_2, \ldots, w_{00}, w_{10}, w_{01}, \ldots]^T$:

$$\sum_{l=1}^{4} \sum_{mn} V_{lijmn} \partial_{t_l} \chi_{mn} + \sum_{klmn} T_{ijklmn} \chi_{kl} \chi_{mn} = 0, \tag{20}$$

where V_{lijmn} and T_{ijklmn} are constants expressed in terms of the elements of the matrix B.

Physical application of the eqs. (16-19) is not found yet. In particular, it reduces into the following (2+1)-dimensional systems:

1. *Matrix Burgers type system* (i.e. linearizable) for the function q_0, if $c_1 = 0$ or $N = 1$.

2. *Matrix n-wave equation* ($n = N(N-1)/2$) for the function v_0, if $p_1 = 0$.

2.1 Construction of solutions

First, one needs to solve the system (8-10) for the functions c_1, Φ, p_1, c_2:

$$\begin{aligned}
A * c_1 &= c_1 B, && (21)\\
\Phi(\lambda, \mu; t) &= \int\limits_{\Omega_k} \int\limits_{D_\nu} \Phi_0(\lambda, \nu; k) e^{\eta_1(\nu; k, t)} \phi_0(\nu, \mu; k) dk d\nu, && (22)\\
p_1(\lambda; t) &= \int\limits_{\Omega_k} \int\limits_{D_\nu} p_0(\lambda, \nu; k) e^{\eta_2(\nu; k, t)} p_{10}(\nu; k) dk d\nu, && (23)\\
c_2(\lambda; t) &= \int\limits_{\Omega_k} e^{k \sum_i B^i t_i} c_{20}(\lambda; k) dk && (24)
\end{aligned}$$

where $\eta_i(\mu; k, t) = \sum_{j=1}^{4} \eta_{ij}(\mu; k) t_j$, $i = 1, 2$, $[\eta_{ij}, \eta_{ik}] = 0$, $\det(\eta_{ij}) \neq 0$. Parameter k is complex in general, integration is over whole complex plane Ω_k of this parameter. Function c_{20} is arbitrary and functions ϕ_0 and p_0 solve the following system:

$$\eta_{1j}(\nu;k)\phi_0(\nu,\mu;k) = \int\limits_{D_{\nu_1}} \eta_{1(j-1)}(\nu;k)\phi_0(\nu,\nu_1;k)A(\nu_1,\mu)d\nu_1, \tag{25}$$

$$p_0(\lambda,\nu;k)\eta_{2j}(\nu;k) = \int\limits_{D_{\nu_1}} A(\lambda,\nu_1)p_0(\nu_1,\nu;k)\eta_{2(j-1)}(\nu;k)d\nu_1, \quad j=2,3,\dots. \tag{26}$$

Functions η_{j1} $(j=1,2)$ are arbitrary, while η_{jn} with $n>1$ provide compatibility of eqs. (25) and (26).

Now one can integrate (3) to get Ψ $(j=1)$:

$$\Psi(t) = E + \partial_{t_1}^{-1}\left[\Phi(t) * A * p_1(t)p_2 + \Phi(t) * c_1 c_2(t)\right], \tag{27}$$

where $E = E(\lambda,\mu)$ is invertible operator independent on t. For instance, $E(\lambda,\mu) = \delta(\lambda-\mu)$. Next, find U from (1): $U = \Psi^{-1} * \Phi$. In general, operator Ψ^{-1} can be constructed only numerically, unless Φ_0 is degenerate ($\Phi_0(\lambda,\mu;k) = \sum_n \Phi_{01}(\lambda)\Phi_{02}(\mu;k)$) and explicit form for E_L^{-1} is known. In this case Ψ^{-1} may be found analytically, following the procedure proposed, for instance, in [10], where $\bar\partial$-problem with degenerate kernel has been solved. Similarly, eqs. (25) and (26) can be solved numerically, unless A has the following structure: $A(\lambda,\mu) = A_0(\lambda,\mu) + \sum_j A_{j1}(\lambda)A_{j2}(\mu)$, where operator A_0 is invertible with known analytical form for A_{0R}^{-1}. For instance, $A_0(\lambda,\mu) = \delta(\lambda-\mu)$. Then compatibility condition of the system (25) and (26) produces dispersion relations in the form $\eta_{1n}(\mu;k) = \eta_{11}(\mu;k)\; F_1[(\phi_0 * A_{i1})(\mu;k),\; i=1,2,\dots]$, $\eta_{2n}(\mu;k) = F_2[(A_{i2} * p_0)(\mu;k),\; i=1,2,\dots]\; \eta_{21}(\mu;k)$, where F_i are given matrix functions of matrix arguments.

Finally, one can show that solutions of our (3+1)-dimensional system (16-19) constructed in accordance with definitions (14) may depend on arbitrary functions of three real parameters, for instance, t_1, t_2 and t_3. This is owing to the factor $\Psi * A * p_1$.

3 Resonance wave interaction in (3+1)-dimensions

In this section we consider eqs.(6-11) with $S(\lambda,\mu) \neq \delta(\lambda-\mu)$, $\chi \neq 0$, $p_1 = 0$ and $\Psi_{t_i} = S*\Phi*C_i$. It is convenient to apply operator c_1 to the eqs.(1) and (9) from the right, giving them the form:

$$S * \tilde\Phi + \tilde\chi = \Psi * \tilde U, \quad \tilde U = U * c_1, \quad \tilde\chi = \chi * c_1, \tag{28}$$

$$(S * \tilde\Phi)_{t_j} = (S * \tilde\Phi)_{t_1} B_j, \quad B_1 = I, \tag{29}$$

I is identity matrix, B_i are diagonal matrices and $N \geq 4$. Let $\tilde\chi_{t_j}(\lambda;t) = \tilde\chi(\lambda;t)a_j$, where a_j are constant matrices. We will need the following notations: $b_j = a_1 B_j - a_j$, $V_0 = c_2 * \tilde U$ and $V_1 = c_{2t_1} * \tilde U$. Nonlinear eq.(12) gets the following form after applying operator c_1 from the right:

$$\Psi * (\partial_{t_j}\tilde U - \partial_{t_1}\tilde U B_j + \tilde U[B_j,V_0]) + \tilde\chi(b_j - [B_j,V_0]) = 0. \tag{30}$$

Now assume that $\det(b_j - [B_j,V_0]) \neq 0$ for all j and use two equations (30) with indexes j and k, $j \neq k$ to eliminate function $\tilde\chi$. After applying operator $c_2 * \Psi^{-1}$ from the left to the resulting equation, we receive:

$$(\partial_{t_k}V_0 - \partial_{t_1}V_0 B_k + [V_1,B_k] + V_0[B_k,V_0])(b_k - [B_k,V_0])^{-1} = \tag{31}$$
$$(\partial_{t_j}V_0 - \partial_{t_1}V_0 B_j + [V_1,B_j] + V_0[B_j,V_0])(b_j - [B_j,V_0])^{-1}.$$

Next, let us introduce different scales for variables t_k, V_0, V_1: $\partial_{t_k} \to \epsilon\partial_{t_k}$, $V_0 = \epsilon v$, $V_1 = \epsilon^2 v_1$. Keeping only leading terms, we get from the eq.(31):

$$E_k \equiv v_{t_1}(B_j b_j^{-1} - B_k b_k^{-1}) + v_{t_k} b_k^{-1} - v_{t_j} b_j^{-1} + \\ [v_1, B_k]b_k^{-1} - [v_1, B_j]b_j^{-1} + v[v, B_j]b_j^{-1} - v[v, B_k]b_k^{-1} = 0. \tag{32}$$

Thus the complete system is represented by the pair of equations (32), E_k and E_n, $k \neq n$. One can see that the following combination of these equations has no function v_1 and contains only off-diagonal elements of v:

$$E_k \left(B_n b_n^{-1} - B_j b_j^{-1}\right) - E_n \left(B_k b_k^{-1} - B_j b_j^{-1}\right) + \\ B_j(E_k - E_n)b_j^{-1} - B_n E_k b_n^{-1} + B_k E_n b_k^{-1} \;\; = \;\; 0. \tag{33}$$

Let $j = 2$, $k = 3$, $n = 4$ and write this equation in the following form

$$\sum_{n=1}^{4} s_{nij}\partial_{t_n} v_{ij} + \sum_{k:k\neq i\neq j} T_{ikj} v_{ik} v_{kj} = 0, \;\; i \neq j, \tag{34}$$

where s_{kij}, and T_{ikj} are constants, expressed in terms of the elements of the matrices B_j and b_j. If v_{ij} are real, then this equation describes resonance interaction of wave packets.

Reduction $t_k \to it_k$, $v_{ij} = \bar{v}_{ji}$, with real s_{nij} and T_{ikj} , $s_{nij} = s_{nji}$, $T_{ikj} = T_{jki}$ (bar means complex conjugated value) transforms the (3+1)-dimensional eq.(34) into (2+1)-dimensional n-wave equation with independent variables $\tau_k = t_k + t_1$, $k = 2, 3, 4$.

3.1 Construction of solutions

In this section we give the algorithm for construction the solution V_0 to the eq.(31).

Solutions of the eq.(29) and expression for $\tilde{\chi}$ have the form:

$$S * \tilde{\Phi}(\lambda) = \int_{\Omega_k} \tilde{\Phi}_0(\lambda, k) e^{k\sum_n B_n t_n} dk, \;\; c_2(\lambda) = \int_{\Omega_k} e^{k\sum_n B_n t_n} c_{20}(\lambda, k) dk, \tag{35}$$

$$\tilde{\chi}(\lambda) = \chi_0(\lambda) e^{\sum_n a_n t_n}. \tag{36}$$

To find Ψ we integrate eq.(3) ($j = 1$, remember that $B_1 = I$):

$$\Psi(\lambda, \mu) = \int_{\Omega_k}\int_{\Omega_q} \tilde{\Phi}_0(\lambda, k) e^{(k+q)\sum_n B_n t_n} c_{20}(\mu, q) \frac{dkdq}{k + q} + \delta(\lambda - \mu). \tag{37}$$

Thus

$$\tilde{U}(\lambda) = S * \tilde{\Phi}(\lambda) - \int_{\Omega_k}\int_{\Omega_q} \tilde{\Phi}_0(\lambda, k) e^{\sum_n B_n t_n (k+q)} \phi(q) \frac{dkdq}{k + q} + \tilde{\chi}(\lambda), \;\; \phi = c_{20} * \tilde{U} \tag{38}$$

and

$$V_0 = c_2 * \tilde{U}. \tag{39}$$

Unknown function ϕ, related with $\tilde{U}$, can be found only numerically in general case, unless functions $\tilde{\Phi}(\lambda, k)$ is degenerate [10]. Eq.(39) shows that V_0 may depend on $N \times N$ matrix function of two real variables, for instance, t_1 and t_2.

Regarding the multi-scale expansion given by eqs.(32), one should replace $t_n \to \epsilon t_n$, in formulae (35-39) and take arbitrary functions $\tilde{\Phi}_0$ and χ_0 proportional to ϵ. Thus $V_0 \sim \epsilon$.

4 Conclusions

Working with dressing methods we underline two directions: (a) increase of dimension of solvable nonlinear PDE and (b) provide rich class of their solutions. Nonlinear PDE derived with our algorithm admit infinite set of commuting flows corresponding to different parameters t_j. Since general equations are ruther complicated (see (16-19), (31)), the reasonable problem is construction of their reductions, which would exhibit physical application of these systems. Another way is multi-scale expansion of general systems, which in our case reveals (3+1)-dimensional equation describing resonance interaction of wave packets (see eq.(32) and (34)).

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