Real Jacobian Elliptic Function Parametrizations for a Genuinely Asymmetric Biquadratic Curve

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Abstract

We give real Jacobian elliptic function parametrizations for a genuinely asymmetric biquadratic curve where the variables and parameters are real.

1 Introduction

In a recent paper [3] a method was developed to simplify any real elliptic asymmetric biquadratic curve[1], i.e.

$$\bar{\alpha}X^2Y^2 + \bar{\beta}X^2Y + \bar{\delta}XY^2 + \bar{\gamma}X^2 + \bar{\kappa}Y^2 + \bar{\epsilon}XY + \bar{\xi}X + \bar{\lambda}Y + \bar{\mu} \,=\, 0, \tag{1}$$

to a canonical curve of the following form

$$x^2y^2 + \gamma(x^2 + y^2) + \epsilon xy \pm 1 = 0 \tag{2}$$

or

$$x^2y^2 + \gamma(x^2 - y^2) + \epsilon xy - 1 = 0, \tag{3}$$

using a real invertible transformation $T(X,Y) : (X,Y) \mapsto (x,y) = (f(X), g(Y))$ with f and g modular, and, correspondingly, to simplify the dynamics on it from being generated by

$$X' = -X - \frac{\bar{\delta}Y^2 + \bar{\epsilon}Y + \bar{\xi}}{\bar{\alpha}Y^2 + \bar{\beta}Y + \bar{\gamma}}\,, \quad Y' = -Y - \frac{\beta X'^2 + \bar{\epsilon}X' + \bar{\lambda}}{\bar{\alpha}X'^2 + \bar{\delta}X' + \bar{\kappa}} \tag{4}$$

to

$$x' = y\,, \quad y' = -x - \frac{\epsilon y}{y^2 + \gamma} \tag{5}$$

or

$$x' = -x - \frac{\epsilon y}{y^2 + \gamma}\,, \quad y' = -y - \frac{\epsilon x'}{x'^2 - \gamma}, \tag{6}$$

respectively.

Futhermore, a procedure was outlined which one could use to parametrize the *symmetric* canonical biquadratic curves given in equation (2), and consequently many of the asymmetric biquadratic curves

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[1]Following [3], we classify the biquadratic as *rational* if at least one of the square root signs disappears when y is solved as a function of x or x is solved as a function of y, meaning y is a rational function of x or x is a rational function of y. On the other hand, if neither square root sign disappears we call the biquadratic *elliptic*. We call a biquadratic symmetric when it is symmetric in the variables, otherwise we call it asymmetric.

(1), using Jacobian elliptic functions[2]. The results for the parametrization when the variables and parameters are real were given in [3, Table 3]. However, two cases were not parametrized. The fundamentally *asymmetric* canonical curve given in equation (3) was also not parametrized. As noted in [3, Section 5], however, if one allows complex variables and parameters then all elliptic biquadratics can be reduced to the canonical form given in equation (2) (with + sign) [3]. This then can be parametrized by the Jacobian elliptic function sn, see [1, pp. 471–473].

In this paper we give parametrizations for the fundamentally asymmetric normal form for many asymmetric biquadratics given in equation (3) using Jacobian elliptic functions when the variables and parameters are real. We also discuss the two outstanding cases of the symmetric canonical curve given in equation (2) (with + sign).

2 Asymmetric Case

Consider the asymmetric biquadratic given in equation (3) after making the replacement $\epsilon \to 2\epsilon$, that is

$$B(x,y) = x^2y^2 + \gamma(x^2 - y^2) + 2\epsilon xy - 1 = 0. \tag{7}$$

Writing y as a function of x, we get

$$y = \frac{-\epsilon x \pm \sqrt{-\gamma x^4 + (\gamma^2 + \epsilon^2 + 1)x^2 - \gamma}}{x^2 - \gamma}, \tag{8}$$

while writing x as a function of y, we get

$$x = \frac{-\epsilon y \pm \sqrt{\gamma y^4 + (\gamma^2 + \epsilon^2 + 1)y^2 + \gamma}}{y^2 + \gamma}. \tag{9}$$

The nature of the curves given by (7), e.g. whether the range of x and y is bounded or unbounded, can be determined by looking at the quartics

$$\Delta_x(x) = -\gamma x^4 + (\gamma^2 + \epsilon^2 + 1)x^2 - \gamma \tag{10}$$

and

$$\Delta_y(y) = \gamma y^4 + (\gamma^2 + \epsilon^2 + 1)y^2 + \gamma. \tag{11}$$

The possibilities are listed in Figure 1, where we introduce

$$\mathcal{B}_x = \frac{(\gamma^2 + \epsilon^2 + 1)}{\gamma} \tag{12}$$

and $\mathcal{B}_y = -\mathcal{B}_x$. Note that for $\gamma > 0$ we get

$$\gamma + \frac{1}{\gamma} \geq 2 \Rightarrow \gamma + \frac{1}{\gamma} + \frac{\epsilon^2}{\gamma} = \frac{\gamma^2 + \epsilon^2 + 1}{\gamma} \geq 2. \tag{13}$$

Similarly, for $\gamma < 0$ we get

$$\frac{\gamma^2 + \epsilon^2 + 1}{\gamma} \leq -2. \tag{14}$$

Also note that $\Delta_x(x)$ ($\Delta_y(y)$) has three (one) extrema if $\mathcal{B}_x \geq 2$ ($\mathcal{B}_y \leq -2$) and one (three) if $\mathcal{B}_x \leq -2$ ($\mathcal{B}_y \geq 2$). The functions $\Delta_x(x)$ and $\Delta_y(y)$ each take two different possible forms

[2] We refer the reader throughout this section to e.g. [2, pp. 18-31,284-285] for definitions and properties of Jacobian elliptic functions.

[3] This complex normal form for biquadratics was also found independently in [4].

Case	γ	$\mathcal{B}_x$	$\mathcal{B}_y$	$\Delta_x(0)$	$\Delta_y(0)$	$\Delta_x(x)$	$\Delta_y(y)$	$B(x,y)=0$
1	>0	≥ 2	≤ -2	$-\gamma$	γ			
2	<0	≤ -2	≥ 2	$-\gamma$	γ			

Figure 1: The possible forms of $\Delta_x(x)$ and $\Delta_y(y)$, see (10) and (11), where $\gamma \neq 0$, and corresponding representative asymmetric biquadratics $B(x,y)=0$ of (7).

$$\gamma\left(-x^4 + \frac{\gamma^2+\epsilon^2+1}{\gamma}x^2 - 1\right), \quad \gamma > 0, \tag{15}$$

$$\gamma\left(y^4 + \frac{\gamma^2+\epsilon^2+1}{\gamma}y^2 + 1\right), \quad \gamma > 0 \tag{16}$$

and

$$-\gamma\left(x^4 + \frac{\gamma^2+\epsilon^2+1}{-\gamma}x^2 + 1\right), \quad \gamma < 0, \tag{17}$$

$$-\gamma\left(-y^4 + \frac{\gamma^2+\epsilon^2+1}{-\gamma}y^2 - 1\right), \quad \gamma < 0. \tag{18}$$

Comparing the shape of $\Delta_x(x)$ and $\Delta_y(y)$ given in Figure 1 above with those given in [3, Figure 7] suggests the following parametrizations:

Case 1. $\mathrm{dn}/\sqrt{k'}$ or $\sqrt{k'}\mathrm{nd}$ for x and $\mathrm{cs}/\sqrt{k'}$ or $\sqrt{k'}\mathrm{sc}$ for y, and
Case 2. $\mathrm{cs}/\sqrt{k'}$ or $\sqrt{k'}\mathrm{sc}$ for x and $\mathrm{dn}/\sqrt{k'}$ or $\sqrt{k'}\mathrm{nd}$ for y.

Following the procedure outlined in [3], we have obtained the various possible parametrizations given in Table 1. Note that $\mathcal{B}_x = -\mathcal{B}_y = 1/k + k \geq 2$ for Case 1 and $\mathcal{B}_x = -\mathcal{B}_y = -1/k - k \leq -2$ for Case 2 when $k \in [0,1]$. As a result the elliptic parametrizations given in Table 1 cover all possible cases. Also note that in Table 1 there are two possible parametrizations of x, y for each subcase. The second is related to the first by a shift of the argument $u \to u + \mathbf{K}(k)$, where

$$\mathbf{K}(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

is the complete elliptic integral of the first kind (and equals one quarter of the period of $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$).

Remark 1 The asymmetric biquadratic (7) is invariant under $(x,y) \mapsto (-x,-y)$. As a result the parametrizations given in Table 1 with $-x$ and $-y$ also parametrize the curve.

Remark 2 Using the parametrizations given in Table 1 and (6) (with $\epsilon \to 2\epsilon$), we generally obtain $(x', y') = (-p_x(\pm u + 2\eta), -p_y(\pm u + 3\eta))$ for Case 1 and $(x', y') = (-p_x(u \pm 2\eta), -p_y(u \pm 3\eta))$ for Case 2, where p_x and p_y are the appropriate elliptic functions from the Table.

The above parametrizations for x' and y' can be verified using the following fact: $B(x,y) = B(x',y) = B(x',y') = 0$, where $B(x,y)$ is given by equation (7). Assume x and y satisfy $B(x,y)=0$ and have the parametrizations p_x and p_y, respectively, where p_x and p_y are appropriate elliptic

Case	x	y	γ	ϵ
1	$\sqrt{k'}\,\mathrm{nd}(u)$ $\mathrm{dn}(u)/\sqrt{k'}$	$\mathrm{cs}(\pm u+\eta)/\sqrt{k'}$ $-\sqrt{k'}\,\mathrm{sc}(\pm u+\eta)$	$k'\,\mathrm{nd}^2(\eta)$	$k^2\,\mathrm{sd}(\eta)\mathrm{cd}(\eta)$
	$\sqrt{k'}\,\mathrm{nd}(u)$ $\mathrm{dn}(u)/\sqrt{k'}$	$\sqrt{k'}\,\mathrm{sc}(\pm u+\eta)$ $-\,\mathrm{cs}(\pm u+\eta)/\sqrt{k'}$	$\mathrm{dn}^2(\eta)/k'$	$k^2\,\mathrm{sn}(\eta)\mathrm{cn}(\eta)/k'$
2	$\sqrt{k'}\,\mathrm{sc}(u)$ $-\,\mathrm{cs}(u)/\sqrt{k'}$	$\pm\,\mathrm{dn}(u\pm\eta)/\sqrt{k'}$ $\pm\sqrt{k'}\,\mathrm{nd}(u\pm\eta)$	$-k'\,\mathrm{nd}^2(\eta)$	$k^2\,\mathrm{sd}(\eta)\mathrm{cd}(\eta)$
	$-\,\mathrm{cs}(u)/\sqrt{k'}$ $\sqrt{k'}\,\mathrm{sc}(u)$	$\mp\mathrm{dn}(u\pm\eta)/\sqrt{k'}$ $\mp\sqrt{k'}\,\mathrm{nd}(u\pm\eta)$	$-\mathrm{dn}^2(\eta)/k'$	$k^2\,\mathrm{sn}(\eta)\mathrm{cn}(\eta)/k'$

Table 1: Elliptic parametrization of the cases given in Figure 1 (the explicit dependence of functions on the modulus is suppressed for convenience).

functions for Case 1 (Case 2), say. The parametrization for x' can be determined using $B(x',y)=0$, i.e.

$$\begin{aligned} B(x',y) &= x'^2y^2+\gamma(x'^2-y^2)+2\epsilon x'y-1 \\ &= y^2x'^2-\gamma(y^2-x'^2)+2\epsilon yx'-1=0. \end{aligned} \tag{19}$$

The y variable, whose parametrization is known, is treated as the first variable in Table 1 and x', whose parametrization is to be determined, is the second variable. Case 2 (Case 1) is now being considered. Finally, $B(x',y')=0$ is used to determine the parametrization for y'. In this case x', whose parametrization is known, is treated as the first variable in Table 1 and y', whose parametrization is to be determined, is the second variable. Case 1 (Case 2) is now being considered again. Care needs to be taken to ensure that the same pair of parametrizations are chosen and that the right signs appear in the parametrizations for x' and y'. This can be achieved using x, y, x' or y' when required.

Remark 3 The parametrizations given in Table 1 allow an action-angle variable description of the dynamics on each of the asymmetric biquadratics of Figure 1 under their corresponding asymmetric maps (6). Identifying $(x,y)\to(x_n,y_n)$ and $(x',y')\to(x_{n+1},y_{n+1})$, $n\in\mathbb{Z}$ being discrete time, (7) becomes

$$x_n^2y_n^2+\gamma(x_n^2-y_n^2)+2\epsilon x_ny_n-1=0, \tag{20}$$

whereas (6) (with $\epsilon\to 2\epsilon$) becomes

$$x_{n+1}=-x_n-\frac{2\epsilon y_n}{y_n^2+\gamma}\,,\quad y_{n+1}=-y_n-\frac{2\epsilon x_{n+1}}{x_{n+1}^2-\gamma}. \tag{21}$$

Using Remark 2 above, the mapping (21) has the solution

$$\begin{aligned} x_n &= (-1)^n p_x(u_{2n},k)=(-1)^n p_x(2n\,\eta+u_0,k), \\ y_n &= (-1)^n p_y(u_{2n+1},k)=(-1)^n p_y((2n+1)\,\eta+u_0,k). \end{aligned} \tag{22}$$

This now allows us to describe the dynamics on each canonical asymmetric biquadratic curve in terms of the modulus k and argument u of the elliptic functions:

$$\begin{aligned} k_{n+1} &= k_n \\ u_{n+1} &= u_n+\eta. \end{aligned}$$

Finally, recall that (7), equivalently (20), is a normal form for many elliptic asymmetric biquadratics. In this case the original elliptic asymmetric biquadratic (1), written in terms of (X_n,Y_n) (after the identification $(X,Y) \to (X_n, Y_n)$ and $(X', Y') \to (X_{n+1}, Y_{n+1})$), can be related to (20) using an asymmetric modular transformation

$$(X_n, Y_n) = (f(x_n), g(y_n)), \tag{23}$$

where f and g are modular. It now follows from above that successive points on the original biquadratic can be written:

$$\begin{aligned} (X_n, Y_n) &= (f(x_n), g(y_n)) \\ &= ((f \circ ((-1)^n p_x))\,(2n\,\eta + u_0, k), (g \circ ((-1)^n p_y))\,((2n+1)\,\eta + u_0, k)). \end{aligned} \tag{24}$$

3 Symmetric Case

Consider the biquadratic

$$B(x,y) = x^2y^2 + \gamma(x^2 + y^2) + 2\epsilon xy + 1 = 0. \tag{25}$$

Writing y as a function of x, we get

$$y = \frac{-\epsilon x \pm \sqrt{-\gamma x^4 + (\epsilon^2 - \gamma^2 - 1)x^2 - \gamma}}{x^2 + \gamma}. \tag{26}$$

The quartic under the square root sign can be written as

$$-\gamma \left(x^4 + \frac{\epsilon^2 - \gamma^2 - 1}{-\gamma} x^2 + 1 \right), \quad -\gamma > 0. \tag{27}$$

The two cases of $B(x,y) = 0$ which were not parametrized. using (ratios of) Jacobian elliptic functions in [3] correspond to $0 < \mathcal{B} < 2$ (Case 6 of [3, Figure 7]) and $-2 < \mathcal{B} < 0$ (Case 4 of [3, Figure 7] when $-2 < \mathcal{B} < 0$), where $\mathcal{B}$ is defined as

$$\mathcal{B} = \frac{(\epsilon^2 - \gamma^2 - 1)}{\gamma}. \tag{28}$$

The two cases can be combined into one, i.e. $-2 < \mathcal{B} < 2$. In what follows, we show that this case can be transformed into one where a parametrization has been found.

Consider the modular transformation[4] $(x,y) = (\frac{1-\bar{x}}{1+\bar{x}}, \frac{1-\bar{y}}{1+\bar{y}})$. Applying this transformation to the biquadratic (25) we obtain

$$\bar{B}(\bar{x}, \bar{y}) = (1 + \gamma + \epsilon)(\bar{x}^2\bar{y}^2 + 1) + (1 + \gamma - \epsilon)(\bar{x}^2 + \bar{y}^2) + 4(1 - \gamma)\bar{x}\bar{y} = 0. \tag{29}$$

Finally, dividing through by the coefficient of $(\bar{x}^2\bar{y}^2 + 1)$ we obtain

$$\hat{B}(\hat{x}, \hat{y}) = \hat{x}^2\hat{y}^2 + \left(\frac{1 + \gamma - \epsilon}{1 + \gamma + \epsilon} \right) (\hat{x}^2 + \hat{y}^2) + 4 \left(\frac{1 - \gamma}{1 + \gamma + \epsilon} \right) \hat{x}\hat{y} + 1 = 0. \tag{30}$$

In doing the division, we note that the coefficient of $(\bar{x}^2\bar{y}^2 + 1)$ is necessarily non-zero. This follows since the coefficient of $(\bar{x}^2\bar{y}^2 + 1)$ being zero implies the biquadratic, and the initial curve from which it is transformed, is rational, as can easily be checked.

Writing y as a function of x, we get

$$y = \frac{2(\gamma - 1)x \pm \sqrt{(\epsilon^2 - (\gamma + 1)^2)(x^4 + 1) - 2(\epsilon^2 - (\gamma + 1)^2 + 8\gamma)x^2}}{(1 + \gamma + \epsilon)x^2 + 1 + \gamma - \epsilon}. \tag{31}$$

[4]We note that for all parametrizable cases given in [3], the quartic is factorizable in the form $(ax^2 + b)(cx^2 + d)$. For the outstanding (combined) case, however, the quartic takes the form $x^4 + \alpha x^2 + 1$, where $-2 < \alpha < 2$, which is factorizable in the form $(x^2 + \sqrt{2 - \alpha}\,x + 1)(x^2 - \sqrt{2 - \alpha}\,x + 1)$. Looking for a modular transformation that produces a quartic with no odd terms in the quadratics, we obtain $x = (1 - \bar{x})/(1 + \bar{x})$.

The quartic under the square root sign can be rewritten as

$$(\epsilon^2 - (\gamma+1)^2)\left\{x^4 - \left(2 + \frac{16\gamma}{\epsilon^2 - (\gamma+1)^2}\right)x^2 + 1\right\}, \quad -\gamma > 0. \tag{32}$$

Note that the term $\epsilon^2 - (\gamma+1)^2$ is positive, see (34) below. To determine the range of $-2 - 16\gamma/(\epsilon^2 - (\gamma+1)^2)$ we can use $-2 < \mathcal{B} < 2$ from above, i.e.

$$-2 < \frac{(\epsilon^2 - \gamma^2 - 1)}{\gamma} < 2. \tag{33}$$

Subracting 2 and then dividing by -16 we obtain

$$0 < \frac{\epsilon^2 - (\gamma^2+1)^2}{-16\gamma} < \frac{1}{4}. \tag{34}$$

Inverting and then subtracting 2 we obtain

$$2 < -2 - \frac{16\gamma}{\epsilon^2 - (\gamma^2+1)^2} < +\infty. \tag{35}$$

I.e. Case 4 of [3, Figure 7]) when $\mathcal{B} < -2$, where $\mathcal{B}$ is defined as $\mathcal{B} = 2 + 16\gamma/(\epsilon^2 - (\gamma+1)^2)$, which is parametrized by $\sqrt{k'}\,\mathrm{sc}$ or $\mathrm{cs}/\sqrt{k'}$, see [3, Table 3]).

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