

On Whitham's Averaging Method

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UDC 517.944

Introduction. Whitham's averaged equations [1] for a nonlinear evolution system describe slow modulations of parameters on a family of periodic traveling wave solutions (or on families of multiphase quasiperiodic solutions, which are so far known to exist only for integrable equations) and are a system of hydrodynamic type [2, 3], that is, of the form

$$U_T^i = V_j^i(\bar{U})U_X^j, \quad i, j = 1, \dots, N \quad (1)$$

(we consider only the spatially one-dimensional case), where $\bar{U} = (U^1, \dots, U^N)$.

The original evolution system is usually Lagrangian or Hamiltonian, and this property is then inherited by the equations of slow modulations. A Hamiltonian theory of systems (1) was constructed by B. A. Dubrovin and S. P. Novikov [2] and, then, successfully used by S. P. Tsarev [7] to integrate Hamiltonian systems reducible to the diagonal form

$$U_T^i = V^i(\bar{U})U_X^i, \quad i = 1, \dots, N. \quad (2)$$

In the present paper we give an exposition and justification of some topics in Whitham's averaging theory. Namely, we consider classical single-phase Whitham averaging for a Lagrangian system in which some of the fields (denoted by u^i) have a direct physical meaning, whereas for the other fields (denoted by φ^α) only the x -derivatives $\varphi_x^\alpha(x)$ are physically meaningful. The averaging is carried out on a family of periodic traveling waves of the system rewritten in the coordinates $u^i(x)$, $q^\alpha(x) = \varphi_x^\alpha(x)$. To average such systems directly on the basis of the Lagrangian formalism, Whitham [1] proposed the pseudophase method, which permits one to obtain the averaged equations in the Lagrangian form and then to construct the corresponding Hamiltonian structure [6] (for the case in which pseudophases are lacking, the Hamiltonian formalism for the Whitham equations was constructed by Hayes [11]). However, there is an alternative approach. It is based on the fact that in the variables $(u^i(x), q^\alpha(x))$ the system is Hamiltonian and possesses conservation laws of the form $S_t^i = R_x^i$, which correspond to the energy and momentum conservation laws and to the annihilators of the Poisson bracket. By using these conservation laws, we can rewrite the equations of slow modulations in the form

$$\langle S^i \rangle_T = \langle R^i \rangle_X, \quad (3)$$

where $\langle \dots \rangle$ stands for the averaging on the family of traveling waves [1]. The Hamiltonian structure of Eqs. (3) can be obtained by averaging the original Hamiltonian structure by the Dubrovin–Novikov method [2, 3], but the general proof of the Jacobi identity for the averaged Poisson bracket is lacking [4]. In the present paper we prove that 1) both methods lead to the same equations of slow modulations (this was proved in [4] by the WKB method) and 2) the Poisson bracket averaged by the Dubrovin–Novikov method coincides with that obtained from the averaged Lagrangian formalism (thus, the Jacobi identity for the Dubrovin–Novikov bracket is proved in this case). Furthermore, we prove that if the system has additional conservation laws of the form $S_t = R_x$, then this averaging results in equations consistent with (3).

1. Averaging of conservation laws and the pseudophase method. Let $a^q(x, t)$ be the field functions, and let the action

$$S = \int L[a^q(x, t)] dt = \iint \mathcal{L}(a^q, a_t^q, a_x^q, a_{tt}^q, a_{xx}^q, a_{xt}^q, \dots) dx dt, \quad (4)$$

* Partially supported by the Russian Foundation for Basic Research under grants Nos. 94-01-01478 and 93-011-168.

Landau Institute of Theoretical Physics, Russian Academy of Sciences. Translated from *Funktsional'nyi Analiz i Ego Prilozheniya*, Vol. 29, No. 1, pp. 7–24, January–March, 1995. Original article submitted February 20, 1994.

$q = 1, \dots, N$, be given.

Suppose that an action $\widehat{S} \in T^m: (a^q) \rightarrow (\acute{a}^q)$ of an m -dimensional Abelian group T^m (not necessarily compact) on the manifold of fields is given, the orbits of this action are m -dimensional, and the Lagrangian is invariant, that is,

$$\mathcal{L}(a^q, a_t^q, a_x^q, \dots) = \mathcal{L}(\acute{a}^q, \acute{a}_t^q, \acute{a}_x^q, \dots). \quad (5)$$

By Noether's theorem, each symmetry of the Lagrangian (including translational invariance and independence of time) yields a conservation law. It is easy to verify that these laws have the form

$$\left[\sum_{n \geq 1} n \sum_q a_{nt}^q \frac{\partial \mathcal{L}}{\partial a_{nt}^q} - \mathcal{L} \right]_t + \left[\sum_{n \geq 1} n \sum_q a_{(n-1)x,t}^q \frac{\partial \mathcal{L}}{\partial a_{nx}^q} \right]_x + [\dots]_{tt} + [\dots]_{xx} + [\dots]_{xt} = 0 \quad (6)$$

(conservation of energy) and

$$\left[\sum_{n \geq 1} n \sum_q a_{(n-1)t,x}^q \frac{\partial \mathcal{L}}{\partial a_{nt}^q} \right]_t + \left[\sum_{n \geq 1} n \sum_q a_{nx}^q \frac{\partial \mathcal{L}}{\partial a_{nx}^q} - \mathcal{L} \right]_x + [\dots]_{tt} + [\dots]_{xx} + [\dots]_{xt} = 0 \quad (7)$$

(conservation of momentum); here $a_{nt}^q \equiv \partial^n a^q / \partial t^n$, $a_{nx}^q \equiv \partial^n a^q / \partial x^n$.

Before writing out the other conservation laws, let us pass to new field variables, $(a^q) \rightarrow (u^i, \varphi^\alpha)$, where u^i are constant on the orbits of T^m and φ^α are parameters on these orbits (and coincide with the parameters on the group itself).

In these variables the other conservation laws acquire the form

$$\left[\frac{\partial \mathcal{L}}{\partial \varphi_t^\alpha} \right]_t + \left[\frac{\partial \mathcal{L}}{\partial \varphi_x^\alpha} \right]_x + [\dots]_{tt} + [\dots]_{xx} + [\dots]_{xt} = 0, \quad \alpha = 1, \dots, m. \quad (8)$$

The equations of motion have the form

$$\frac{\partial \mathcal{L}}{\partial a^q} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial a_t^q} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial a_x^q} + \dots = 0, \quad q = 1, \dots, N. \quad (9)$$

Assume that the Lagrangian is nondegenerate. Then we can proceed to the Hamiltonian formalism by setting (the method of Ostrogradskii applied to the case of fields)

$$q_1^\alpha(x) = \varphi^\alpha(x), \quad \dots, \quad q_{n_\alpha}^\alpha(x) = \varphi_{(n_\alpha-1)t}(x), \quad (10)$$

$$r_1^i(x) = u^i(x), \quad \dots, \quad r_{\bar{n}_i}^i(x) = u_{(\bar{n}_i-1)t}^i(x). \quad (11)$$

(Here n_α and \bar{n}_i are the highest orders of time derivatives of φ^α and u^i , respectively, occurring in the Lagrangian. Since we can perform linear transformations of the coordinates φ^α and u^i (separately), it follows that in the generic case all n_α (and all \bar{n}_i) are the same.) Let us introduce the momenta

$$p_1^\alpha(x) = \frac{\delta L}{\delta \varphi_t^\alpha(x)} - \dots + (-1)^{n_\alpha-1} \left[\frac{\delta L}{\delta \varphi_{n_\alpha t}^\alpha(x)} \right]_{(n_\alpha-1)t},$$

.....

$$p_{n_\alpha}^\alpha(x) = \frac{\delta L}{\delta \varphi_{n_\alpha t}^\alpha(x)},$$

$$s_1^i(x) = \frac{\delta L}{\delta u_t^i(x)} - \dots + (-1)^{\bar{n}_i-1} \left[\frac{\delta L}{\delta u_{\bar{n}_i t}^i(x)} \right]_{(\bar{n}_i-1)t},$$

.....

$$s_{\bar{n}_i}^i(x) = \frac{\delta L}{\delta u_{\bar{n}_i t}^i(x)},$$

where

$$\frac{\delta L}{\delta a_{nt}^q(x)} \equiv \frac{\partial \mathcal{L}}{\partial a_{nt}^q}(x) - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial a_{nt,x}^q}(x) + \dots$$

The equations of motion (9) in the variables $q_{\nu_\alpha}^\alpha(x)$, $r_{\mu_i}^i(x)$, $p_{\nu_\alpha}^\alpha(x)$, $s_{\mu_i}^i(x)$ are Hamiltonian. The Hamiltonian is equal to

$$H = \int \left[\sum_{\alpha, \nu_\alpha} p_{\nu_\alpha}^\alpha(x) \dot{q}_{\nu_\alpha}^\alpha(x) + \sum_{i, \mu_i} s_{\mu_i}^i(x) \dot{r}_{\mu_i}^i(x) \right] dx - L, \quad (14)$$

and the Poisson bracket has the form

$$\{q_{\nu_\alpha}^\alpha(x), p_{\nu_\beta}^\beta(y)\} = \delta^{\alpha\beta} \delta_{\nu_\alpha \nu_\beta} \delta(x-y), \quad \{r_{\mu_i}^i(x), s_{\mu_j}^j(y)\} = \delta^{ij} \delta_{\mu_i \mu_j} \delta(x-y) \quad (15)$$

(all other brackets are zero).

More generally, if the conservation law corresponding to the symmetry of the Lagrangian with respect to a one-parameter transformation family can be put in the form $P_t = Q_x$ (perhaps, ambiguously, because of the presence of mixed t, x -derivatives, as is the case in (6), (7), and (8)), then the functional $\int P dx$ (defined unambiguously) generates this transformation family in the Hamiltonian structure (15).

REMARK. Very frequently, an originally degenerate Lagrangian becomes nondegenerate after a suitable rotation in the plane (x, t) (for example, this is the case for the Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations).

Let us describe the set of functions to be considered in the sequel. Let $\bar{a}(x)$ satisfy the conditions

$$\exists k > 0, \quad \widehat{S} \in T^m \quad \forall x \quad \bar{a}(x + 2\pi/k) = \widehat{S}\bar{a}(x), \quad (16)$$

that is, in the variables (u^i, φ^α) we have

$$u^i(x + 2\pi/k) = u^i(x), \quad \varphi^\alpha(x + 2\pi/k) = \varphi^\alpha(x) + \mu^\alpha, \quad (17)$$

and let the initial data be given in the form

$$\forall x \quad u_{nt}^i(x + 2\pi/k) = u_{nt}^i(x), \quad \varphi_{nt}^\alpha(x + 2\pi/k) = \varphi_{nt}^\alpha(x), \quad n \geq 1. \quad (18)$$

Since \mathcal{L} depends only on the derivatives of φ^α , the evolution equations (9) preserve each of the families (17) and property (18).

In terms of the Hamiltonian variables conditions (17) and (18) acquire the form

$$q_1^\alpha(x + 2\pi/k) = q_1^\alpha(x) + \mu^\alpha, \quad q_{\nu_\alpha}^\alpha(x + 2\pi/k) = q_{\nu_\alpha}^\alpha(x), \quad \nu_\alpha > 1, \quad (19)$$

$$r_{\mu_i}^i(x + 2\pi/k) = r_{\mu_i}^i(x), \quad p_{\nu_\alpha}^\alpha(x + 2\pi/k) = p_{\nu_\alpha}^\alpha(x), \quad s_{\mu_i}^i(x + 2\pi/k) = s_{\mu_i}^i(x). \quad (20)$$

On the family (19), (20) we seek extremals of all possible functionals of the form

$$\lambda_H H + \lambda_P P + \sum_{\alpha} \lambda_{\alpha} I^{\alpha}, \quad \lambda_H, \lambda_P, \lambda_{\alpha} = \text{const}, \quad (21)$$

where H is the Hamiltonian, P is the integral of momentum, and I^α are the functionals that generate the shifts of the corresponding angles φ^α . Thus, we consider the equation

$$\delta \left[\lambda_H H + \lambda_P P + \sum_{\alpha} \lambda_{\alpha} I^{\alpha} \right] = 0. \quad (22)$$

We assume that for any $k, \mu^\alpha, \lambda_H, \lambda_P$, and λ_α the solution to Eq. (22) is unique modulo a translation along the x -axis and the action of a transformation in T^m .

COMMENT. If for any $k > 0$, μ^α , λ_H , λ_P , and λ_α there exists a solution $\bar{a}_{(k, \bar{\mu}, \bar{\lambda})}(x)$, then $\bar{a}_{(k/n, \bar{\mu}, \bar{\lambda})}(x)$ is also a solution for any positive integer n and to each tuple $(k, \bar{\mu}, \bar{\lambda})$ there corresponds countably many such solutions. Accordingly, we assume that our family of solutions has just this very form, and to each $\bar{a}_{(k, \bar{\mu}, \bar{\lambda})}(x)$ we assign the greatest admissible k and the smallest admissible $\bar{\mu}$.

The solutions to Eq. (22) depend on the parameters k , μ^α , λ_P/λ_H , λ_α/λ_H , φ_0^α , and θ_0 , where φ_0^α and θ_0 are the initial phases corresponding to the action of T^m and to the translations along the x -axis. The averaging affects the $2m + 2$ parameters k , μ^α , λ_P/λ_H , and λ_α/λ_H ; the averaged variables do not depend on φ_0^α and θ_0 , so that these last variables do not occur in the equations of slow modulations.

Let us transform system (14), (15) into a new Hamiltonian system by setting

$$q^\alpha(x) = q_{1x}^\alpha(x), \quad p^\alpha(x) = p_1^\alpha(x) \quad (23)$$

(this substitution is well-defined, since the densities of the functionals H , P , and I^α depend only on the derivatives of q_1^α with respect to x).

The new Poisson bracket has the form

$$\{q_{\nu_\alpha}^\alpha(x), p_{\nu_\beta}^\beta(y)\} = \delta^{\alpha\beta} \delta_{\nu_\alpha \nu_\beta} \delta(x - y), \quad \nu_\alpha, \nu_\beta \geq 2, \quad (24)$$

$$\{r_{\mu_i}^i(x), s_{\mu_j}^j(y)\} = \delta^{ij} \delta_{\mu_i \mu_j} \delta(x - y), \quad \{q^\alpha(x), p^\beta(y)\} = \delta^{\alpha\beta} \delta'(x - y) \quad (25)$$

(all other brackets are zero).

Thus, $\int q^\alpha(x) dx$ and $\int p^\beta(x) dx$ are annihilators of the bracket (24), (25), so that every Hamiltonian system with translation-invariant Hamiltonian has $2m + 2$ first integrals.

LEMMA 1. *The family of periodic traveling waves of the Hamiltonian system (14), (23)–(25) can be obtained from the family (19), (20), (22) by factorization with respect to the initial phases φ_0^α .*

PROOF. It is easy to see that conditions (19) and (20) are equivalent to the $2\pi/k$ -periodicity of the functions occurring in (23)–(25).

It follows from conditions (22) that $q_{1t}^\alpha = -(\lambda_P/\lambda_H)q_{1x}^\alpha - \lambda_\alpha/\lambda_H$ and that $\xi_t = -(\lambda_P/\lambda_H)\xi_x$ for the other variables. By differentiating the first equation with respect to x and by substituting $q_{1x}^\alpha = q^\alpha(x)$ we obtain $q_t^\alpha = -(\lambda_P/\lambda_H)q_x^\alpha$; thus, after factorization with respect to the initial phases φ_0^α we obtain exactly the family of periodic traveling waves of system (14), (23)–(25). The lemma is proved.

COROLLARY. *The family of periodic traveling waves of system (14), (23)–(25) depends on $(2m + 2)$ parameters (not including the initial phase θ_0). The averaging of the conservation laws for the energy, momentum, and $2m$ annihilators of the bracket (24), (25), yields the Whitham equations of slow modulations.*

Since

$$p_1^\alpha(x) = \frac{\partial \mathcal{L}}{\partial \varphi_t^\alpha}(x), \quad \frac{\delta H}{\delta q^\alpha(x)} = \frac{\delta H}{\delta q_{1x}^\alpha(x)} = -\frac{\partial \mathcal{L}}{\partial \varphi_x^\alpha}(x) \quad (26)$$

modulo total derivatives with respect to x and t and the energy and momentum conservation laws have the form (6), (7), we see that the equations of slow modulations have the form

$$\left[\sum_{n \geq 1} n \left(\sum_i \left\langle u_{nt}^i \frac{\partial \mathcal{L}}{\partial u_{nt}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{nt}^\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{nt}^\alpha} \right\rangle \right) - \langle \mathcal{L} \rangle \right]_T + \left[\sum_{n \geq 1} n \left(\sum_i \left\langle u_{(n-1)x,t}^i \frac{\partial \mathcal{L}}{\partial u_{nx}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{(n-1)x,t} \frac{\partial \mathcal{L}}{\partial \varphi_{nx}^\alpha} \right\rangle \right) \right]_X = 0, \quad (27)$$

$$\left[\sum_{n \geq 1} n \left(\sum_i \left\langle u_{(n-1)t,x}^i \frac{\partial \mathcal{L}}{\partial u_{nt}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{(n-1)t,x} \frac{\partial \mathcal{L}}{\partial \varphi_{nt}^\alpha} \right\rangle \right) \right]_T + \left[\sum_{n \geq 1} n \left(\sum_i \left\langle u_{nx}^i \frac{\partial \mathcal{L}}{\partial u_{nx}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{nx}^\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{nx}^\alpha} \right\rangle \right) - \langle \mathcal{L} \rangle \right]_X = 0, \quad (28)$$

$$\left\langle \frac{\partial \mathcal{L}}{\partial \varphi_t^\alpha} \right\rangle_T + \left\langle \frac{\partial \mathcal{L}}{\partial \varphi_x^\alpha} \right\rangle_X = 0, \quad (29)$$

$$\langle q^\alpha \rangle_T = \left\langle \frac{\delta H}{\delta p^\alpha(x)} \right\rangle_X, \quad (30)$$

where $\langle \dots \rangle$ denotes the averaging on the family (19), (20), (22) (or, which is the same, on the family of periodic traveling waves of system (14), (23)–(25)).

It is easy to see that Eqs. (29) correspond to the conservation laws (8).

Let us proceed to the averaging method for the Lagrangian.

Following Whitham, we seek solutions to Eqs. (9) in the form

$$u^i(x, t) = \Phi^i(\theta), \quad \varphi^\alpha(x, t) = \Psi^\alpha(\theta) + \varepsilon^\alpha, \quad (31)$$

where $\theta = kx + \omega t$ is the phase, $\varepsilon^\alpha = \beta^\alpha x + \gamma^\alpha t$ are pseudophases, and $\Phi^i(\theta)$ and $\Psi^\alpha(\theta)$ are 2π -periodic functions.

Next, we set

$$\mathcal{L}(u^i, u_t^i, u_x^i, \dots, \varphi_t^\alpha, \varphi_x^\alpha, \varphi_{tt}^\alpha, \varphi_{xx}^\alpha, \dots) = \mathcal{L}(\Phi^i, \omega \Phi_\theta^i, k \Phi_\theta^i, \dots, \omega \Psi_\theta^\alpha + \gamma^\alpha, k \Psi_\theta^\alpha + \beta^\alpha, \omega^2 \Psi_{\theta\theta}^\alpha, k^2 \Psi_{\theta\theta}^\alpha, \dots),$$

where $\Phi^i = \Phi^i(\theta, k, \omega, \beta^\alpha, \gamma^\alpha)$, $\Psi^\alpha = \Psi^\alpha(\theta, k, \omega, \beta^\alpha, \gamma^\alpha)$, construct the averaged Lagrangian

$$\bar{\mathcal{L}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{L}(\theta, k, \omega, \beta^\alpha, \gamma^\alpha) d\theta = \bar{\mathcal{L}}(k, \omega, \beta^\alpha, \gamma^\alpha) \equiv \bar{\mathcal{L}}(\theta_X, \theta_T, \varepsilon_X^\alpha, \varepsilon_T^\alpha), \quad (32)$$

and obtain the averaged equations

$$[\bar{\mathcal{L}}_{\theta_X}]_X + [\bar{\mathcal{L}}_{\theta_T}]_T = 0, \quad [\bar{\mathcal{L}}_{\varepsilon_X^\alpha}]_X + [\bar{\mathcal{L}}_{\varepsilon_T^\alpha}]_T = 0, \quad (33)$$

or, in the variables $k, \omega, \beta^\alpha, \gamma^\alpha$,

$$[\bar{\mathcal{L}}_\omega]_T + [\bar{\mathcal{L}}_k]_X = 0, \quad k_T = \omega_X, \quad (34)$$

$$[\bar{\mathcal{L}}_{\gamma^\alpha}]_T + [\bar{\mathcal{L}}_{\beta^\alpha}]_X = 0, \quad \beta_T^\alpha = \gamma_X^\alpha. \quad (35)$$

Since the Lagrangian $\bar{\mathcal{L}}$ is independent of time and translation-invariant, Eqs. (34), (35) admit energy and momentum conservation laws. On replacing Eqs. (34) by these laws, we obtain the averaged system in the form

$$\left[\omega \bar{\mathcal{L}} + \sum_\alpha \gamma^\alpha \bar{\mathcal{L}}_{\gamma^\alpha} - \bar{\mathcal{L}} \right]_T + \left[\omega \bar{\mathcal{L}}_k + \sum_\alpha \gamma^\alpha \bar{\mathcal{L}}_{\beta^\alpha} \right]_X = 0, \quad (36)$$

$$\left[k \bar{\mathcal{L}}_\omega + \sum_\alpha \beta^\alpha \bar{\mathcal{L}}_{\gamma^\alpha} \right]_T + \left[k \bar{\mathcal{L}} + \sum_\alpha \beta^\alpha \bar{\mathcal{L}}_{\beta^\alpha} - \bar{\mathcal{L}} \right]_X = 0, \quad (37)$$

$$[\bar{\mathcal{L}}_{\gamma^\alpha}]_T + [\bar{\mathcal{L}}_{\beta^\alpha}]_X = 0, \quad (38)$$

$$\beta_T^\alpha = \gamma_X^\alpha. \quad (39)$$

LEMMA 2. *The solution family (31) coincides with the family (19), (20), (22), and moreover, $\beta^\alpha = \langle q^\alpha \rangle$, $\gamma^\alpha = -(\lambda_P/\lambda_H) \langle q^\alpha \rangle - \lambda_\alpha/\lambda_H$.*

PROOF. The coincidence of these solution families is evident, and (22) implies that

$$\beta^\alpha = (k/2\pi)[\varphi^\alpha(x + 2\pi/k) - \varphi^\alpha(x)] = \langle \varphi_x^\alpha \rangle = \langle q^\alpha \rangle;$$

similarly, we have $\gamma^\alpha = \langle \varphi_t^\alpha \rangle = \langle q_{1t}^\alpha \rangle$, but, by virtue of (22), (22) $q_{1t}^\alpha = -(\lambda_P/\lambda_H) q_{1x}^\alpha - \lambda_\alpha/\lambda_H$, that is,

$$\gamma^\alpha = \langle -(\lambda_P/\lambda_H) q_{1x}^\alpha - \lambda_\alpha/\lambda_H \rangle = -(\lambda_P/\lambda_H) \langle q^\alpha \rangle - \lambda_\alpha/\lambda_H.$$

The lemma is proved.

It is also obvious that the averaging with respect to θ is equivalent to the averaging with respect to x and t .

THEOREM 1. *Equations (27)–(30) coincide with Eqs. (36)–(39).*

PROOF. Let us introduce the functions

$$\tilde{\mathcal{L}}_i = \frac{\partial \mathcal{L}}{\partial u^i}, \quad \hat{\mathcal{L}}_i^n = \frac{\partial \mathcal{L}}{\partial u_{nx}^i}, \quad \mathcal{L}_\alpha^n = \frac{\partial \mathcal{L}}{\partial \varphi_{nx}^\alpha}, \quad \mathcal{L}_i^n = \frac{\partial \mathcal{L}}{\partial u_{nt}^i}, \quad \mathcal{L}_\alpha^n = \frac{\partial \mathcal{L}}{\partial \varphi_{nt}^\alpha}, \quad n \geq 1.$$

For all these functions we set

$$\mathcal{L}_*^* = \mathcal{L}_*^*(\Phi^i, \omega \Phi_\theta^i, k \Phi_\theta^i, \dots, \omega \Psi_\theta^\alpha + \gamma^\alpha, k \Psi_\theta^\alpha + \beta^\alpha, \omega^2 \Psi_{\theta\theta}^\alpha, k^2 \Psi_{\theta\theta}^\alpha, \dots),$$

where $\Phi^i = \Phi^i(\theta, k, \omega, \bar{\beta}, \bar{\gamma})$, $\Psi^\alpha = \Psi^\alpha(\theta, k, \omega, \bar{\beta}, \bar{\gamma})$, and define the action

$$\tilde{S} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{L}(\Phi^i, \omega \Phi_\theta^i, k \Phi_\theta^i, \dots, \omega \Psi_\theta^\alpha + \gamma^\alpha, k \Psi_\theta^\alpha + \beta^\alpha, \omega^2 \Psi_{\theta\theta}^\alpha, k^2 \Psi_{\theta\theta}^\alpha, \dots) d\theta.$$

(Obviously, $\tilde{S} = \langle \mathcal{L} \rangle$.)

It is easy to verify that

$$\frac{\delta \tilde{S}}{\delta \Phi^i(\theta)} = 0, \quad \frac{\delta \tilde{S}}{\delta \Psi^\alpha(\theta)} = 0 \quad (40)$$

on the solution family (31). Furthermore, it is obvious that

$$\begin{aligned} \bar{\mathcal{L}}_\omega &= \sum_{n \geq 1} n \left(\sum_i \omega^{n-1} \langle \Phi_{n\theta}^i \mathcal{L}_i^n \rangle + \sum_\alpha \omega^{n-1} \langle \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right) \\ &\quad + \int_0^{2\pi} \sum_i \frac{\delta \tilde{S}}{\delta \Phi^i(\theta)} \Phi_\omega^i d\theta + \int_0^{2\pi} \sum_\alpha \frac{\delta \tilde{S}}{\delta \Psi^\alpha(\theta)} \Psi_\omega^\alpha d\theta, \\ \bar{\mathcal{L}}_k &= \sum_{n \geq 1} n \left(\sum_i k^{n-1} \langle \Phi_{n\theta}^i \hat{\mathcal{L}}_i^n \rangle + \sum_\alpha k^{n-1} \langle \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right) \\ &\quad + \int_0^{2\pi} \sum_i \frac{\delta \tilde{S}}{\delta \Phi^i(\theta)} \Phi_k^i d\theta + \int_0^{2\pi} \sum_\alpha \frac{\delta \tilde{S}}{\delta \Psi^\alpha(\theta)} \Phi_k^\alpha d\theta, \\ \bar{\mathcal{L}}_{\beta^\nu} &= \langle \mathcal{L}_\nu^1 \rangle + \int_0^{2\pi} \sum_i \frac{\delta \tilde{S}}{\delta \Phi^i(\theta)} \Phi_{\beta^\nu}^i d\theta + \int_0^{2\pi} \sum_\alpha \frac{\delta \tilde{S}}{\delta \Psi^\alpha(\theta)} \Psi_{\beta^\nu}^\alpha d\theta, \\ \bar{\mathcal{L}}_{\gamma^\nu} &= \langle \mathcal{L}_\nu^1 \rangle + \int_0^{2\pi} \sum_i \frac{\delta \tilde{S}}{\delta \Phi^i(\theta)} \Phi_{\gamma^\nu}^i d\theta + \int_0^{2\pi} \sum_\alpha \frac{\delta \tilde{S}}{\delta \Psi^\alpha(\theta)} \Psi_{\gamma^\nu}^\alpha d\theta \end{aligned}$$

(here notation such as $\bar{\mathcal{L}}$ and $\langle \mathcal{L} \rangle$ is used for the averaged variables).

By (40), we have

$$\begin{aligned} \bar{\mathcal{L}}_\omega &= \sum_{n \geq 1} n \left(\sum_i \omega^{n-1} \langle \Phi_{n\theta}^i \mathcal{L}_i^n \rangle + \sum_\alpha \omega^{n-1} \langle \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right), \\ \bar{\mathcal{L}}_k &= \sum_{n \geq 1} n \left(\sum_i k^{n-1} \langle \Phi_{n\theta}^i \hat{\mathcal{L}}_i^n \rangle + \sum_\alpha k^{n-1} \langle \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right), \\ \bar{\mathcal{L}}_{\beta^\alpha} &= \langle \mathcal{L}_\alpha^1 \rangle, \quad \bar{\mathcal{L}}_{\gamma^\alpha} = \langle \mathcal{L}_\alpha^1 \rangle. \end{aligned}$$

It follows from Eqs. (27)–(30) and from Lemma 2 that

$$\begin{aligned}
& \sum_{n \geq 1} n \left(\sum_i \left\langle u_{nt}^i \frac{\partial \mathcal{L}}{\partial u_{nt}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{nt}^\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{nt}^\alpha} \right\rangle \right) - \langle \mathcal{L} \rangle \\
&= \sum_{n \geq 1} n \left(\sum_i \langle \omega^n \Phi_{n\theta}^i \mathcal{L}_i^n \rangle \right) + \sum_{n \geq 1} n \left(\sum_\alpha \langle \omega^n \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right) + \sum_\alpha \langle \gamma^\alpha \mathcal{L}_\alpha^1 \rangle - \langle \mathcal{L} \rangle \\
&= \omega \bar{\mathcal{L}}_\omega + \sum_\alpha \gamma^\alpha \bar{\mathcal{L}}_{\gamma^\alpha} - \bar{\mathcal{L}}, \\
& \sum_{n \geq 1} n \left(\sum_i \left\langle u_{(n-1)x,t}^i \frac{\partial \mathcal{L}}{\partial u_{nx}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{(n-1)x,t}^\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{nx}^\alpha} \right\rangle \right) \\
&= \sum_{n \geq 1} n \left(\sum_i \langle k^{n-1} \omega \Phi_{n\theta}^i \widehat{\mathcal{L}}_i^n \rangle + \sum_\alpha \langle k^{n-1} \omega \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right) + \sum_\alpha \langle \gamma^\alpha \mathcal{L}_\alpha^1 \rangle \\
&= \omega \bar{\mathcal{L}}_k + \sum_\alpha \gamma^\alpha \bar{\mathcal{L}}_{\beta^\alpha};
\end{aligned}$$

thus, (27) coincides with (36), and

$$\begin{aligned}
& \sum_{n \geq 1} n \left(\sum_i \left\langle u_{(n-1)t,x}^i \frac{\partial \mathcal{L}}{\partial u_{nt}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{(n-1)t,x}^\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{nt}^\alpha} \right\rangle \right) \\
&= \sum_{n \geq 1} n \left(\sum_i \langle \omega^{n-1} k \Phi_{n\theta}^i \mathcal{L}_i^n \rangle + \sum_\alpha \langle \omega^{n-1} k \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right) + \sum_\alpha \langle \beta^\alpha \mathcal{L}_\alpha^1 \rangle \\
&= k \bar{\mathcal{L}}_\omega + \sum_\alpha \beta^\alpha \bar{\mathcal{L}}_{\gamma^\alpha}, \\
& \sum_{n \geq 1} n \left(\sum_i \left\langle u_{nx}^i \frac{\partial \mathcal{L}}{\partial u_{nx}^i} \right\rangle + \sum_\alpha \left\langle \varphi_{nx}^\alpha \frac{\partial \mathcal{L}}{\partial \varphi_{nx}^\alpha} \right\rangle \right) - \langle \mathcal{L} \rangle \\
&= \sum_{n \geq 1} n \left(\sum_i \langle k^n \Phi_{n\theta}^i \widehat{\mathcal{L}}_i^n \rangle \right) + \sum_{n \geq 1} n \left(\sum_\alpha \langle k^n \Psi_{n\theta}^\alpha \mathcal{L}_\alpha^n \rangle \right) + \sum_\alpha \langle \beta^\alpha \mathcal{L}_\alpha^1 \rangle - \langle \mathcal{L} \rangle \\
&= k \bar{\mathcal{L}}_k + \sum_\alpha \beta^\alpha \bar{\mathcal{L}}_{\beta^\alpha} - \bar{\mathcal{L}},
\end{aligned}$$

which means that (28) coincides with (37).

Furthermore,

$$\left\langle \frac{\partial \mathcal{L}}{\partial \varphi_t^\alpha} \right\rangle = \langle \mathcal{L}_\alpha^1 \rangle = \bar{\mathcal{L}}_{\gamma^\alpha}, \quad \left\langle \frac{\partial \mathcal{L}}{\partial \varphi_x^\alpha} \right\rangle = \langle \mathcal{L}_\alpha^1 \rangle = \bar{\mathcal{L}}_{\beta^\alpha};$$

that is, (29) coincides with (38).

Now, by (22),

$$\left\langle \frac{\delta H}{\delta p_1^\alpha(x)} \right\rangle = \left\langle -\frac{\lambda_P}{\lambda_H} \{q_1^\alpha(x), P\} - \sum_\beta \frac{\lambda_\beta}{\lambda_H} \{q_1^\alpha(x), I^\beta\} \right\rangle = \left\langle -\frac{\lambda_P}{\lambda_H} q_{1x}^\alpha - \frac{\lambda_\alpha}{\lambda_H} \right\rangle = -\frac{\lambda_P}{\lambda_H} \langle q^\alpha \rangle - \frac{\lambda_\alpha}{\lambda_H},$$

and (30) coincides with (39) by virtue of Lemma 2.

The theorem is proved.

2. Additional conservation laws. Let us now give an independent proof of the fact that the averaging of the conservation laws corresponding to energy, momentum, and I^α on the family (19), (20), (22) and the equations $\beta_T^\alpha = \gamma_X^\alpha$ (which correspond to the averaging of the annihilators $\int q^\alpha(x) dx$ of the Poisson bracket (24), (25)) imply the ‘‘conservation of waves’’ $k_T = \omega_X$. Furthermore, we shall prove that

if the Hamiltonian system (14), (15) has additional conservation laws $S_t = R_x$ corresponding to some fluxes on the space of $a^q(x)$ which preserve the Lagrangian and commute with the fluxes generated by the integral of momentum and by I^α (in this case their densities and the fluxes themselves are independent of φ^α and, in the variables used in (24), (25), correspond to conservation laws for system (14), (24), (25)), then the averaging of these additional conservation laws yields equations consistent with (27)–(30).

THEOREM 2. *The averaging of the conservation laws for the energy, momentum, and $2m$ annihilators of the Hamiltonian system (14), (24), (25) on the family (19), (20), (22) implies the conservation of waves $k_T = \omega_X$.*

PROOF. In accordance with the preceding, we consider the equations

$$\delta \left[\lambda_H H + \lambda_P P + \sum_{\alpha} \lambda_{\alpha} I^{\alpha} \right] = 0$$

on the phase space of functions that satisfy the conditions

$$u^i(x + 2\pi/k) = u^i(x), \quad \varphi^\alpha(x + 2\pi/k) = \varphi^\alpha(x) + 2\pi\beta^\alpha/k.$$

We assume that the variations $\delta u^i(x)$, $\delta \varphi^\alpha(x)$ are uniformly bounded, that is, $u^i(x) + \delta u^i(x)$, $\varphi^\alpha(x) + \delta \varphi^\alpha(x)$ belong to the same family (15) as $u^i(x)$, $\varphi^\alpha(x)$. Thus,

$$\frac{\delta}{\delta a^q(x)} \int \mathcal{P} dx = \frac{\partial \mathcal{P}}{\partial a^q} (x) - \frac{\partial}{\partial x} \frac{\partial \mathcal{P}}{\partial a_x^q} (x) + \dots$$

Any functional that commutes with the Hamiltonian, with the momentum, and with I^α leaves each of the families (19), (20) invariant.

Let $\bar{U} = (U^1, \dots, U^{2m+2})$ be a collection of parameters (excluding the phases φ_0^α and θ_0) on the solution family (19), (20), (22). Let $\vec{\xi}$ be a tangent vector to the level surface $k = \text{const}$, $\beta^\alpha = \text{const}$ in the space with coordinates (U^1, \dots, U^{2m+2}) . Then the function variations corresponding to the translation along $\vec{\xi}$ are uniformly bounded. Set

$$H = \int \mathcal{P}_H dx, \quad P = \int \mathcal{P}_P dx, \quad I^\alpha = \int \mathcal{P}_\alpha dx$$

(the corresponding conservation laws have the form $(\mathcal{P}_H)_t = (J_H)_x$, $(\mathcal{P}_P)_t = (J_P)_x$, $(\mathcal{P}_\alpha)_t = (J_\alpha)_x$). By (22), we have

$$\lambda_H(\bar{U}) \partial_{\vec{\xi}} \langle \mathcal{P}_H \rangle + \lambda_P(\bar{U}) \partial_{\vec{\xi}} \langle \mathcal{P}_P \rangle + \sum_{\alpha} \lambda_{\alpha}(\bar{U}) \partial_{\vec{\xi}} \langle \mathcal{P}_{\alpha} \rangle = 0$$

for any $\vec{\xi}$ such that $\partial_{\vec{\xi}} k = 0$ and $\partial_{\vec{\xi}} \beta^\alpha = 0$. This equation is equivalent to the relation

$$\lambda_H(\bar{U}) d \langle \mathcal{P}_H \rangle + \lambda_P(\bar{U}) d \langle \mathcal{P}_P \rangle + \sum_{\alpha} \lambda_{\alpha}(\bar{U}) d \langle \mathcal{P}_{\alpha} \rangle = \mu(\bar{U}) dk + \sum_{\alpha} \mu_{\alpha}(\bar{U}) d\beta^{\alpha}$$

for some functions $\mu(\bar{U})$ and $\mu_{\alpha}(\bar{U})$.

As was already shown, under conditions (19) and (20) the solutions to (22) have the form (31), that is,

$$u^i(x, t) = \Phi^i(\theta), \quad \varphi^\alpha(x, t) = \Psi^\alpha(\theta) + \varepsilon^\alpha, \quad \theta = kx + \omega t, \quad \varepsilon^\alpha = \beta^\alpha x + \gamma^\alpha t,$$

where $\Phi^i(\theta)$, $\Psi^\alpha(\theta)$ are 2π -periodic functions.

In the original Lagrangian formalism let us make the rotation by an angle χ in the plane (x, t) :

$$\hat{x} = x \cos \chi - t \sin \chi, \quad \hat{t} = x \sin \chi + t \cos \chi.$$

The passage to the Hamiltonian structure will now give a new Poisson bracket, in which the translations along the “old” t -axis will be generated by the functional $\int (\mathcal{P}_H \cos \chi - J_H \sin \chi) d\hat{x}$, the translations along the old coordinate x by the functional $\int (\mathcal{P}_P \cos \chi - J_P \sin \chi) d\hat{x}$, and the action of T^m by the functionals

$\int (\mathcal{P}_\alpha \cos \chi - J_\alpha \sin \chi) d\dot{x}$, where all densities and fluxes of the conservation laws are expressed via the new coordinates $a^q(\dot{x})$. (Indeed, $\mathcal{P}_t = J_x \iff (\mathcal{P} \cos \chi - J \sin \chi)_t = (\mathcal{P} \sin \chi + J \cos \chi)_{\dot{x}}$.)

However, the solution family (31) remains the same, and moreover, $\dot{k} = k \cos \chi - \omega \sin \chi$, $\dot{\beta}^\alpha = \beta^\alpha \cos \chi - \gamma^\alpha \sin \chi$, $\dot{\omega} = k \sin \chi + \omega \cos \chi$, $\dot{\gamma}^\alpha = \beta^\alpha \sin \chi + \gamma^\alpha \cos \chi$.

By writing out the conditions

$$\widehat{A} \delta \left[\lambda_H \int \mathcal{P}_H dx + \lambda_P \int \mathcal{P}_P dx + \sum_\alpha \lambda_\alpha \int \mathcal{P}_\alpha dx \right] = 0$$

in the new variables for some function in the family (31), we obtain

$$\widehat{A}(\chi) \delta \left[\lambda_H \int (\mathcal{P}_H \cos \chi - J_H \sin \chi) d\dot{x} + \lambda_P \int (\mathcal{P}_P \cos \chi - J_P \sin \chi) d\dot{x} + \sum_\alpha \lambda_\alpha \int (\mathcal{P}_\alpha \cos \chi - J_\alpha \sin \chi) d\dot{x} \right] = 0,$$

where \widehat{A} and $\widehat{A}(\chi)$ are the old and the new Hamiltonian operators, respectively.

Since $\widehat{A}(\chi)$ is nondegenerate, we obtain

$$\delta \left[\lambda_H \int (\mathcal{P}_H \cos \chi - J_H \sin \chi) d\dot{x} + \lambda_P \int (\mathcal{P}_P \cos \chi - J_P \sin \chi) d\dot{x} + \sum_\alpha \lambda_\alpha \int (\mathcal{P}_\alpha \cos \chi - J_\alpha \sin \chi) d\dot{x} \right] = 0.$$

Since the averaging on the family (31) in any direction in the plane (x, t) (except for $\theta = \text{const}$, $\varepsilon^\alpha = \text{const}$) gives the same result, we can conclude, as above, that

$$\begin{aligned} & \lambda_H(\bar{U})(d\langle \mathcal{P}_H \rangle \cos \chi - d\langle J_H \rangle \sin \chi) + \lambda_P(\bar{U})(d\langle \mathcal{P}_P \rangle \cos \chi - d\langle J_P \rangle \sin \chi) \\ & + \sum_\alpha \lambda_\alpha(\bar{U})(d\langle \mathcal{P}_\alpha \rangle \cos \chi - d\langle J_\alpha \rangle \sin \chi) \\ & = \mu(\chi, \bar{U}) dk + \sum_\alpha \mu_\alpha(\chi, \bar{U}) d\beta^\alpha, \end{aligned}$$

that is,

$$\begin{aligned} & \lambda_H(\bar{U})(d\langle \mathcal{P}_H \rangle \cos \chi - d\langle J_H \rangle \sin \chi) + \lambda_P(\bar{U})(d\langle \mathcal{P}_P \rangle \cos \chi - d\langle J_P \rangle \sin \chi) \\ & + \sum_\alpha \lambda_\alpha(\bar{U})(d\langle \mathcal{P}_\alpha \rangle \cos \chi - d\langle J_\alpha \rangle \sin \chi) \\ & = \mu(\chi, \bar{U})(dk \cos \chi - d\omega \sin \chi) + \sum_\alpha \mu_\alpha(\chi, \bar{U})(d\beta^\alpha \cos \chi - d\gamma^\alpha \sin \chi). \end{aligned} \quad (41)$$

For $\chi = 0$ and $\chi = \pi/2$ we obtain, respectively,

$$\lambda_H(\bar{U}) d\langle \mathcal{P}_H \rangle + \lambda_P(\bar{U}) d\langle \mathcal{P}_P \rangle + \sum_\alpha \lambda_\alpha(\bar{U}) d\langle \mathcal{P}_\alpha \rangle = \mu(0, \bar{U}) dk + \sum_\alpha \mu_\alpha(0, \bar{U}) d\beta^\alpha, \quad (42)$$

$$\lambda_H(\bar{U}) d\langle \mathcal{P}_H \rangle + \lambda_P(\bar{U}) d\langle J_P \rangle + \sum_\alpha \lambda_\alpha(\bar{U}) d\langle J_\alpha \rangle = \mu(\pi/2, \bar{U}) d\omega + \sum_\alpha \mu_\alpha(\pi/2, \bar{U}) d\gamma^\alpha. \quad (43)$$

Let us multiply (42) by $\cos \chi$ and (43) by $\sin \chi$ and subtract the results from (41). We obtain

$$\begin{aligned} & [\mu(\chi, \bar{U}) - \mu(0, \bar{U})] \cos \chi dk - [\mu(\chi, \bar{U}) - \mu(\pi/2, \bar{U})] \sin \chi d\omega \\ & + \sum_\alpha [\mu_\alpha(\chi, \bar{U}) - \mu_\alpha(0, \bar{U})] \cos \chi d\beta^\alpha - \sum_\alpha [\mu_\alpha(\chi, \bar{U}) - \mu_\alpha(\pi/2, \bar{U})] \sin \chi d\gamma^\alpha = 0. \end{aligned}$$

Assuming that the differentials dk , $d\omega$, $d\beta^\alpha$, and $d\gamma^\alpha$ are linearly independent (on the space (\bar{U})), we obtain $\mu(\chi, \bar{U}) \equiv \mu(0, \bar{U}) = \mu(\pi/2, \bar{U}) = \mu(\bar{U})$, $\mu_\alpha(\chi, \bar{U}) \equiv \mu_\alpha(0, \bar{U}) = \mu_\alpha(\pi/2, \bar{U}) = \mu(\bar{U})$. Consequently, on the space (\bar{U}) we have

$$\begin{aligned}\lambda_H(\bar{U}) d\langle \mathcal{P}_H \rangle + \lambda_P(\bar{U}) d\langle \mathcal{P}_P \rangle + \sum_\alpha \lambda_\alpha(\bar{U}) d\langle \mathcal{P}_\alpha \rangle &= \mu(\bar{U}) dk + \sum_\alpha \mu_\alpha(\bar{U}) d\beta^\alpha, \\ \lambda_H(\bar{U}) d\langle J_H \rangle + \lambda_P(\bar{U}) d\langle J_P \rangle + \sum_\alpha \lambda_\alpha(\bar{U}) d\langle J_\alpha \rangle &= \mu(\bar{U}) d\omega + \sum_\alpha \mu_\alpha(\bar{U}) d\gamma^\alpha;\end{aligned}$$

these relations readily imply the statement of the theorem. The theorem is proved.

THEOREM 3. *Suppose that for any tuple λ_H , λ_P , λ_α , k , β^α system (22) has a unique solution in the class (19), (20) modulo translations along the x -axis and the action of I^α . Let the Hamiltonian system (14), (15) have an additional first integral $S_t = R_x$, corresponding to some flux on the space of $a^q(x)$, and suppose that this flux commutes with the fluxes generated by H , P , and I^α and preserves the action. Then Eqs. (27)–(30) imply that $\langle S \rangle_T = \langle R \rangle_X$.*

PROOF. The flux generated by the functional $\int S dx$ preserves the solution family (19), (20), (22) and can at most generate a linear dependence of the phases φ_0^α , θ_0 on time on this family. Since the Poisson bracket (15) is nondegenerate, it follows that

$$\delta \left[\int S dx + \zeta(\bar{U}) \int \mathcal{P}_P dx + \sum_\alpha \zeta_\alpha(\bar{U}) \int \mathcal{P}_\alpha dx \right] = 0$$

on the family (19), (20), (22) for some function $\zeta(\bar{U})$ and $\zeta_\alpha(\bar{U})$. Literally repeating the argument in the proof of Theorem 2, we see that

$$\begin{aligned}d\langle S \rangle + \zeta(\bar{U}) d\langle \mathcal{P}_P \rangle + \sum_\alpha \zeta_\alpha(\bar{U}) d\langle \mathcal{P}_\alpha \rangle &= \eta(\bar{U}) dk + \sum_\alpha \eta_\alpha(\bar{U}) d\beta^\alpha, \\ d\langle R \rangle + \zeta(\bar{U}) d\langle J_P \rangle + \sum_\alpha \zeta_\alpha(\bar{U}) d\langle J_\alpha \rangle &= \eta(\bar{U}) d\omega + \sum_\alpha \eta_\alpha(\bar{U}) d\gamma^\alpha\end{aligned}$$

on the manifold (\bar{U}) for some functions $\eta(\bar{U})$, $\eta_\alpha(\bar{U})$. The desired assertion now follows readily from Theorem 2. The theorem is proved.

3. Hamiltonian formalism. Equations (34), (35) are Hamiltonian with respect to the Poisson bracket

$$\{\beta^\alpha(X), \bar{\mathcal{L}}_{\gamma^\beta}(Y)\} = \delta^{\alpha\beta} \delta'(X - Y), \quad \{k(X), \bar{\mathcal{L}}_\omega(Y)\} = \delta'(X - Y) \quad (44)$$

(all other brackets are zero); the Hamiltonian is equal to

$$H = \int \left(\omega \bar{\mathcal{L}}_\omega + \sum_\alpha \gamma^\alpha \bar{\mathcal{L}}_{\gamma^\alpha} - \bar{\mathcal{L}} \right) dX, \quad (45)$$

the integral of momentum

$$P = \int \left(k \bar{\mathcal{L}}_\omega + \sum_\alpha \beta^\alpha \bar{\mathcal{L}}_{\gamma^\alpha} \right) dX$$

generates the translation along the X -axis, and the functionals $\int \beta^\alpha(X) dX$, $\int \bar{\mathcal{L}}_{\gamma^\alpha}(X) dX$, $\int k(X) dX$, $\int \bar{\mathcal{L}}_\omega(X) dX$ are annihilators of the bracket (44) (see [6]).

Here we shall prove that the bracket (44) coincides with the averaged Dubrovin–Novikov bracket [2, 3] for Eqs. (27)–(30).

Let us describe the construction of the Dubrovin–Novikov bracket. Let the original evolution system

$$a_t(x) = K(a, a_x, \dots) = \{a(x), H[a]\}, \quad a = (a^q), \quad (46)$$

be Hamiltonian with respect to the local translation-invariant field-theoretic Poisson bracket

$$\{a^q(x), a^p(y)\} = \sum_{k=0}^M B_k^{qp}(a(x), a_x(x), \dots) \delta^{(k)}(x-y), \quad (47)$$

and let the Hamiltonian $H[a]$ be a local field functional,

$$H[a] = \int h(a(x), a_x(x), \dots) dx. \quad (48)$$

Suppose that system (46) has N pairwise commuting local integrals

$$I^i[a] = \int \mathcal{P}^i(a(x), a_x(x), \dots) dx, \quad i = 1, \dots, N, \quad \{I^i, I^j\} = 0. \quad (49)$$

(In our case, $N = 2m+2$ and the integrals have the form $\int q^\alpha(x) dx$, $\int p^\alpha(x) dx$, $\int \mathcal{P}_P dx$, and $\int \mathcal{P}_H dx$.) Consider the pairwise brackets of the densities of the integrals (49)

$$\begin{aligned} & \{\mathcal{P}^i(a(x), a_x(x), \dots), \mathcal{P}^j(a(y), a_y(y), \dots)\} \\ &= \sum_{k \geq 0} A_k^{ij}(a(x), a_x(x), \dots) \delta^{(k)}(x-y), \quad i, j = 1, \dots, 2m+2. \end{aligned} \quad (50)$$

By (49), $A_0^{ij} = \partial_x Q^{ij}$. The averaged Dubrovin–Novikov bracket has the form (we denote $\langle \mathcal{P}^i \rangle = U^i$)

$$\begin{aligned} \{U^i(X), U^j(Y)\} &= \langle A_1^{ij} \rangle(\bar{U}(X)) \delta'(X-Y) + \frac{\partial \langle Q^{ij} \rangle(\bar{U})}{\partial U^k} U_X^k \delta(X-Y) \\ &\equiv g^{ij}(\bar{U}(X)) \delta'(X-Y) + b_k^{ij}(\bar{U}(X)) U_X^k \delta(X-Y), \end{aligned} \quad (51)$$

where $\langle \dots \rangle$ stands for the averaging on the family of periodic traveling waves of the Hamiltonian system (46), determined by the conditions

$$\delta \left[k \int \mathcal{P}_H dx - \omega \int \mathcal{P}_P dx + \sum_{\alpha} \left(\mu_{\alpha} \int q^{\alpha}(x) dx + \lambda_{\alpha} \int p_{\alpha}(x) dx \right) \right] = 0, \quad (52)$$

where k is the wave number, ω is the frequency, and μ_{α} and λ_{α} are arbitrary constants. Just as in the proof of Theorem 2, it follows that

$$k(\bar{U}) d \langle \mathcal{P}_H \rangle - \omega(\bar{U}) d \langle \mathcal{P}_P \rangle + \sum_{\alpha} (\mu_{\alpha}(\bar{U}) d \langle q^{\alpha} \rangle + \lambda_{\alpha}(\bar{U}) d \langle p^{\alpha} \rangle) = \mu(\bar{U}) dk \quad (53)$$

for some function $\mu(\bar{U})$.

The theory of the brackets (51) is closely related to Riemannian geometry; in particular, their skew symmetry implies that

$$g^{ij} = g^{ji}, \quad b_k^{ij} + b_k^{ji} = \partial g^{ij} / \partial U^k, \quad (54)$$

and it follows from the Leibniz identity that under invertible smooth changes $U^i \rightarrow \tilde{U}^i(\bar{U})$ of the field variables the functions g^{ij} behave as the contravariant components of a metric tensor, and the functions $\Gamma_{jk}^i = -g_{js} b_k^{si}$ behave as the components of the differential-geometric connection consistent (by (54)) with the metric.

If $g^{i,j}$ is nondegenerate, then the Jacobi identity for the bracket (51) is equivalent to the symmetry of the connection Γ_{jk}^i and to the vanishing of the curvature tensor of the metric, $R_{ijkl}^i \equiv 0$ (see [2, 3]).

However, the survey [3] does not contain the proof of the Jacobi identity for the bracket (51) obtained by averaging (see [4]). In other words, it is not proved that the connection Γ_{jk}^i is symmetric and that the curvature tensor of the metric $g^{ij}(\bar{U})$ is zero. By proving that the Dubrovin–Novikov bracket coincides with the bracket (44) rewritten in the variables $(\langle q^{\alpha} \rangle, \langle p^{\alpha} \rangle, \langle \mathcal{P}_P \rangle, \langle \mathcal{P}_H \rangle)$ (as was shown in the proof of Theorem 1, we have $\langle q^{\alpha} \rangle = \beta^{\alpha}$ and $\langle p^{\alpha} \rangle = \bar{\mathcal{L}}_{\gamma^{\alpha}}$), we at the same time prove that for the case in question the Dubrovin–Novikov bracket satisfies the Jacobi identity.

THEOREM 4. *The Dubrovin–Novikov bracket (51) coincides with the bracket (44) rewritten in the variables $(\langle q^\alpha \rangle, \langle p^\alpha \rangle, \langle \mathcal{P}_P \rangle, \langle \mathcal{P}_H \rangle)$.*

PROOF. Let the bracket (44) in the variables $(\langle q^\alpha \rangle, \langle p^\alpha \rangle, \langle \mathcal{P}_P \rangle, \langle \mathcal{P}_H \rangle)$ have the form

$$\{U^i(X), U^j(Y)\} = \tilde{g}^{ij}(\bar{U}(X))\delta'(X - Y) + \tilde{b}_k^{ij}(\bar{U}(X))U_X^k\delta(X - Y). \quad (55)$$

It is easy to verify [3] that the Dubrovin–Novikov bracket can be obtained as follows. With the bracket (47) we associate a bracket on the space of fields $a^q(x, X)$ by setting

$$\{a^q(x, X), a^p(y, Y)\}_0 = \sum_{k=0}^M B_k^{qp}(a(x, X), a_x(x, X), \dots)\delta^{(k)}(x - y)\delta(X - Y), \quad (56)$$

and then in the densities of the integrals $I_i^j(X) = \int \mathcal{P}_i(a(x, X), a_x(x, X), \dots) dx$ and in the bracket (56) we replace the operator ∂_x by $\partial_x + \varepsilon\partial_X$, where $\varepsilon \ll 1$ is the ratio of the “rapid” scale to the “slow” scale (for example, the ratio of the wavelength to the typical length of variation of the parameters \bar{U}). Then we evaluate the Poisson bracket of the resultant integrals $I_\varepsilon^i(X)$ in the new Hamiltonian structure:

$$\begin{aligned} & \{a^q(x, X), a^p(y, Y)\}_\varepsilon \\ &= \sum_{k=0}^M B_k^{qp}(a(x, X), a_x(x, X) + \varepsilon a_X(x, X), \dots) \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X} \right)^k \delta(x - y)\delta(X - Y). \end{aligned} \quad (57)$$

Set

$$h_\varepsilon^{ij}(X, Y) = \{I_\varepsilon^i(X), I_\varepsilon^j(Y)\}_\varepsilon = \varepsilon h_1^{ij}(X, Y) + O(\varepsilon^2)$$

and consider the restriction of the function $h_1^{ij}(X, Y) = h_1^{ij}([a], X, Y)$ to the submanifold of functions $a(x, X)$ such that for any fixed X these functions, regarded as a functions of x , belong to the family of traveling wave solutions to system (46). The function $\langle \mathcal{P}^i \rangle(X)$, $\theta_0(X)$ can serve as coordinates on this manifold, and the Dubrovin–Novikov bracket has the form

$$\{\langle \mathcal{P}^i \rangle(X), \langle \mathcal{P}^j \rangle(Y)\} = \tilde{h}_1^{ij}(X, Y),$$

where $\tilde{h}_1^{ij}(X, Y)$ are the functions $h_1^{ij}(X, Y)$ restricted to the cited submanifold (it is easy to see that they are independent of $\theta_0(X)$). It readily follows that in this Hamiltonian structure the functionals

$$\int \langle \mathcal{P}_H \rangle dX, \quad \int \langle \mathcal{P}_P \rangle dX, \quad \int \langle q^\alpha \rangle dX, \quad \int \langle p^\alpha \rangle dX \quad (58)$$

generate the averaged equations (27)–(30), the translation along the X -axis, and the “zero” fluxes, respectively on the space of fields $U^i(X)$.

As was shown in the proof of Theorem 1, $\langle \mathcal{P}_H \rangle$, $\langle \mathcal{P}_P \rangle$, and $\langle q^\alpha \rangle$, $\langle p^\alpha \rangle$ coincide, respectively, with the densities of the Hamiltonian, of the momentum, and of the annihilators of the Hamiltonian system (44), (45), and the fluxes generated by the functionals (58) by virtue of the Dubrovin–Novikov bracket coincide with the corresponding fluxes for the bracket (44). Since the related equations in the coordinates $\langle \mathcal{P}^i \rangle(X) = U^I(X)$ have the form $U_{t_j}^i = b_k^{ij}(\bar{U})U_X^k$ and $U_{t_j}^i = \tilde{b}_k^{ij}(\bar{U})U_X^k$, respectively, it readily follows that $b_k^{ij} = \tilde{b}_k^{ij}$ and consequently, by (54), $g^{ij} = \tilde{g}^{ij} + f^{ij}$, where $f^{ij} = \text{const}$.

Furthermore, let

$$\begin{aligned} I_\varepsilon^i(X) &= I_0^i(X) + \varepsilon I_1^i(X) + O(\varepsilon^2), \\ \{\dots, \dots\}_\varepsilon &= \{\dots, \dots\}_0 + \varepsilon \{\dots, \dots\}_1 + O(\varepsilon^2); \end{aligned}$$

then

$$h_1^{ij}(X, Y) = \{I_0^i(X), I_1^j(Y)\}_0 + \{I_1^i(X), I_0^j(Y)\}_0 + \{I_0^i(X), I_0^j(Y)\}_1,$$

and, by (52) (we set $P = I^{2m+1}$, $H = I^{2m+2}$) on the cited submanifold we have

$$\sum_{i,j} \tau_i(X)\tau_j(Y)\tilde{h}_1^{ij}(X, Y) \equiv 0,$$

where $(\tau_i) = (\mu_1, \dots, \mu_m, \lambda_1, \dots, \lambda_m, -\omega, k)$. Since $\tilde{h}_1^{ij}(X, Y)$ has the form (51), we see that this relation is equivalent to the relation

$$\sum_{i,j} \tau_i(\bar{U}) \tau_j(\bar{U}) g^{ij}(\bar{U}) \equiv 0;$$

thus, $(\tau_i(\bar{U})) = (\mu_1(\bar{U}), \dots, \mu_m(\bar{U}), \lambda_1(\bar{U}), \dots, \lambda_m(\bar{U}), -\omega(\bar{U}), k(\bar{U}))$ is an isotropic covector with respect to the metric $g^{ij}(\bar{U})$. In view of (53), we see that in the variables $(\beta^\alpha, \bar{\mathcal{L}}_{\gamma^\alpha}, k, \bar{\mathcal{L}}_\omega)$ this covector has the coordinates $(0, \dots, 0, 0, \dots, 0, \mu, 0)$ and is isotropic with respect to the metric (44). It follows that in the variables (U^i) the covector $(\mu_1(\bar{U}), \dots, \mu_m(\bar{U}), \lambda_1(\bar{U}), \dots, \lambda_m(\bar{U}), -\omega(\bar{U}), k(\bar{U}))$ is isotropic with respect to both the metric $g^{ij}(\bar{U})$ and the metric $\tilde{g}^{ij}(\bar{U})$, that is, it is isotropic with respect to f^{ij} . However, $f^{ij} = \text{const}$, and $\mu_\alpha, \lambda_\alpha, \omega, k$ can be arbitrary by assumption. Thus, any covector is isotropic with respect to f^{ij} ; that is, $f^{ij} \equiv 0$. The theorem is proved.

EXAMPLE. The multicomponent NLS-type equation

$$iu_t^i + u_{xx}^i + V'(h)u^i = 0, \quad h = \sum_i |u^i|^2, \quad i = 1, \dots, n,$$

after the substitution $u^i = \sqrt{w^i} \exp(i \int v^i dx)$ takes the form

$$v_t^i = \left[\partial_x \left(\frac{w_x^i}{2w^i} \right) + \left(\frac{w_x^i}{2w^i} \right)^2 - (v^i)^2 + V' \left(\sum_i w^i \right) \right]_x, \quad (59)$$

$$w_t^i = -2(w^i v^i)_x. \quad (60)$$

The system has the Hamiltonian structure

$$\{v^i(x), w^k(y)\} = \delta^{ik} \delta'(x-y),$$

$$H = \int \left[\sum_i \left(-\frac{(w_x^i)^2}{4w^i} - w^i (v^i)^2 \right) + V \left(\sum_i w^i \right) \right] dx,$$

and the integral of momentum has the form $P = \int \sum_i v^i w^i dx$.

The bracket has the form described in the preceding, and we can proceed to a nondegenerate Lagrangian formalism in the system after the substitution $w^i = q_x^i$ with allowance for the fact that $v^i = -q_t^i / (2q_x^i)$ (thus, in accordance with the general procedure, we regard v^i as generalized momenta and express them via q_t^i). The system takes the Lagrangian form with the Lagrangian

$$\mathcal{L} = \sum_i \left[-\frac{(q_t^i)^2}{4q_x^i} + \frac{(q_{xx}^i)^2}{4q_x^i} \right] + V \left(\sum_i q_x^i \right).$$

We find the family of traveling waves of system (59), (60) from the conditions

$$cv_x^i = \left[\frac{w_x^i}{2w^i} \right]_{xx} + \left[\left(\frac{w_x^i}{2w^i} \right)^2 \right]_x - [(v^i)^2]_x + \left[V' \left(\sum_k w^k \right) \right]_x, \quad cw_x^i = -2(w^i v^i)_x.$$

It follows that

$$cv^i = \left[\frac{w_x^i}{2w^i} \right]_x + \left(\frac{w_x^i}{2w^i} \right)^2 - (v^i)^2 + V' \left(\sum_k w^k \right) + A^i, \quad cw^i + 2w^i v^i = B^i,$$

that is, $v^i = (B^i - cw^i) / (2w^i)$ and

$$\frac{c^2}{4} + \left[\frac{w_x^i}{2w^i} \right]_x + \left(\frac{w_x^i}{2w^i} \right)^2 - \left(\frac{B^i}{2w^i} \right)^2 + V' \left(\sum_k w^k \right) + A^i = 0. \quad (61)$$

System (61) has the first integral

$$\sum_i \left(\frac{c^2}{4} w^i + \frac{(w_x^i)^2}{4w^i} + \frac{(B^i)^2}{4w^i} + A^i w^i \right) + V \left(\sum_i w^i \right) = E$$

and is Hamiltonian with generalized momenta $p^i = w_x^i / (2w^i)$, Hamiltonian

$$H = \sum_i \left(\frac{c^2}{4} w^i + w^i (p^i)^2 + \frac{(B^i)^2}{4w^i} + A^i w^i \right) + V \left(\sum_i w^i \right)$$

and bracket $\{w^i, p^j\} = \delta^{ij}$.

By applying the canonical transformation $\bar{q}^i = 2\sqrt{w^i}$, $\bar{p}^i = \sqrt{w^i} p^i$, we obtain

$$H = \sum_i \left((\bar{p}^i)^2 + \left(\frac{c\bar{q}^i}{4} \right)^2 + \left(\frac{B^i}{\bar{q}^i} \right)^2 + \frac{A^i (\bar{q}^i)^2}{4} \right) + V \left(\sum_i \frac{(\bar{q}^i)^2}{4} \right), \quad \{\bar{q}^i, \bar{p}^j\} = \delta^{ij}.$$

Thus, the periodic traveling waves of system (59), (60) correspond to closed trajectories of a particle that moves in the n -dimensional space with the potential

$$\Pi(\bar{q}) = \sum_i \left(\left(\frac{c\bar{q}^i}{4} \right)^2 + \left(\frac{B^i}{\bar{q}^i} \right)^2 + \left(\frac{A^i \bar{q}^i}{2} \right)^2 \right) + V \left(\sum_i \frac{(\bar{q}^i)^2}{4} \right)$$

with all possible c , A^i , B^i . Assuming that (as usual) closed trajectories are isolated on each energy level, we obtain a $(2n+2)$ -parameter (excluding the initial phase shift θ_0) family of traveling waves (or several such families) with the parameters c , A^i , B^i , E , on which we can average the $2n+2$ first integrals P , H , $\int v^i dx$, and $\int w^i dx$; all statements proved above are valid for this family.

In closing, the authors express their gratitude to S. P. Novikov for his attention to the work, and also to I. M. Krichever, E. V. Ferapontov, and O. I. Mokhov for useful remarks and fruitful discussion in the course of the present study.

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Translated by V. E. Nazaikinskii