

# Burgers Equation Revisited

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This paper studies the 1D pressureless turbulence (the Burgers equation). It shows that reliable numerics in this problem is very easy to produce if one properly discretizes the Burgers equation. The numerics it presents confirms the  $7/2$  power law proposed for probability of observing large negative velocity gradients in this problem. It also suggests that the entire probability function for the velocity gradients could be universal, perhaps in some approximate sense. In particular, the probability that the velocity gradient is negative appears to be  $p \approx 0.21 \pm 0.01$  irrespective of the details of the random force. Finally, it speculates that the theory initially proposed by Polyakov, with a particular value of the “anomaly” parameter, may indeed be exact, at least as far as velocity gradients are concerned.

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## I. INTRODUCTION

The problem of the randomly driven Burgers equation, or 1D pressureless turbulence, attracted a lot of attention in the literature in the last decade [1, 2, 3, 4, 5, 6, 7, 8]. Indeed, it is a tantalizing problem. On the one hand, it incorporates many features one would expect from a real 3D turbulent fluid. On the other hand, it is much simpler than the 3D turbulence of the incompressible fluid, usually studied in the framework of the Navier-Stokes equation. Its simplicity prompted some researchers to suggest that it will become the “Ising model” of turbulence. Yet many published papers later, we are still far from a complete solution to this problem. In fact, many of the aspects of the Burgers equation remain the subject of an ongoing debate. Several competing methods exist in the literature, each with its own range of applicability, which provide different, sometimes mutually exclusive answers to various aspects of the 1D turbulence.

The situation could be improved if a way existed to check various statements made about the Burgers equation. Such a check could be provided by numerical simulations. But the numerical simulations in turbulence are hard to conduct, they require large expenditure of resources, and the answers they provide are often too ambiguous to either confirm or disprove a particular theory. The situation in Burgers turbulence seems to be no different. Despite several influential papers on the numerical aspects of Burgers turbulence, those studies failed to rule out completely or confirm any of the competing theories currently on the market. The exception to this seems to be the paper of J. Bec [8], where he confirmed the asymptotics of a certain probability distribution function (discussed below) predicted earlier in Ref. [5].

In this paper, I would like to point out that there exists an alternative way to study the Burgers equation numerically. The method involves minimizing the Hopf-Cole functional instead of trying to solve a complicated nonlinear differential equation. That turned out to be extremely easy to do numerically. A ten line program,

which can be written in half an hour, produces, after several minutes of computer time, clean graphs of probability distribution functions. With the help of this program, I report confirmation of some of the existing theories. I also hope that the method, being so simple, can be used by other researchers to do a quick check if their proposed theory indeed does not contradict the numerical experiment.

The rest of this paper is organized as follows. In section II I formulate the problem of Burgers turbulence and summarize many of the known facts about it. In section III I show how Burgers equation must be discretized in order to facilitate doing numerics. In section IV I discuss the statistics of the velocity gradients of the Burgers equation using the methods of Lagrangian trajectories. I also discuss the “anomaly” hypothesis of Polyakov and explain its meaning within the Lagrangian approach. In section V I discuss the numerical simulations, show that they are compatible with the asymptotic results of Refs. [5, 8] and show that they are also compatible with the Polyakov hypothesis if the “anomaly” parameter  $\beta$  is chosen to be equal to  $3/2$ . Finally in the last section VI I also show that the compatibility with the known results on the behavior of pinned charge density waves [12, 13, 16, 17] also forces the value  $\beta = 3/2$ . I speculate that since the problem of pinned charge density waves and the asymptotics of Refs. [5, 8] relate to different properties of Burgers turbulence, there is a chance that the Polyakov’s theory with  $\beta = 3/2$  is indeed exact, at least as far as the calculation of probabilities of velocity gradients is concerned, unless a remarkable and unlikely coincidence is at play here.

## II. FORMULATION OF THE PROBLEM

The Burgers turbulence problem can be formulated in just a few lines. Take a pressureless fluid in 1D, introduce its velocity field  $u(x, t)$  where  $x$  is space and  $t$  is time and write down the Navier-Stokes equation

$$\partial_t u + u \partial_x u - \nu \partial_x^2 u = f(x, t). \quad (1)$$

Here  $\nu$  is the viscosity of the fluid which is taken to be very small.  $f(x, t)$  is a force acting on the fluid, which is taken to be random, white noise in time, and a smooth function in space

$$\langle f(x, t)f(x, t') \rangle = \delta(t - t')F(x - x'), \quad (2)$$

where  $F(x)$  is a function which is positive at  $x = 0$  and smoothly goes to zero when  $x \gg L$ , where  $L$  sets the force length scale. As far as boundary conditions are concerned, it is usually assumed that  $u$  is either periodic in space (in which case  $F$  has to be periodic as well), or the space is infinite and  $u \rightarrow 0$  at infinity. Note that the center of mass motion decouples in both cases, to give

$$\frac{d}{dt} \int dx u = \int dx f, \quad (3)$$

which is but the standard Brownian motion. For this reason one usually selects the force  $f$  to be a derivative of another function  $h(x, t)$ ,  $f = \partial_x h$ . With this restriction, the average velocity vanishes  $\langle u \rangle = 0$  and it makes sense to define the *root mean square velocity* as  $U_{\text{rms}} = \sqrt{\langle u^2 \rangle}$ . It measures the average amplitude of the fluctuation of the velocity field. A basic assumption in the theory of turbulence is that this quantity does not depend on the viscosity  $\nu$  as it is taken to zero. That allows to determine  $U_{\text{rms}}$  via simple hydrodynamic scaling to be  $U_{\text{rms}} \propto (LF(0))^{1/3}$ .

Once the equation (1) is written, one would like to calculate various averages and probabilities. A particular popular question focuses on the probability distribution function of the velocity at two different points. Define  $P(\sigma)$  as the probability of observing a velocity gradient  $\sigma = \partial_x u(x)$  in a particular point in space. Similarly  $P_d(v, r)$  is the probability of observing a difference in velocities  $v = u(x + r) - u(x)$  at two points separated by a distance  $r$  from each other. These quantities received a lot of attention in the literature, possibly because they are related to the turbulent advection of particles suspended in the fluid, which is a very important topic in real 3D turbulence.

The following is a highly subjective historical overview of the progress made in understanding (1). As always, the choice of highlights is very personal one, and I apologize to the authors whose contributions are not mentioned here.

The first to address the behavior of  $P_d(v)$  were V. Yakhot and A. Chekhlov [1], who argued that it should not be symmetric under  $v \rightarrow -v$ . They were also the first to study this function numerically.

Soon thereafter, A. Polyakov [2] suggested an approach to calculation of these functions. He suggested an equation that  $P_d(v)$  should satisfy. The drawback of his approach was that it was not just one equation but a family of equations parametrized by a parameter (referred to as “ $b$ -anomaly”) whose value remained undetermined. At about the same time, Bouchaud, Mezard and Parisi [3] attacked the problem using replica trick and the Gaussian variational ansatz. The problem with their approach

was that it was not well suited to computing quantities such as  $P(\sigma)$  and  $P_d(v)$ .

In subsequent work [4] of the author with A. Migdal, it was shown that at least the asymptotics of  $P(\sigma)$  and  $P_d(v)$  at  $\sigma, v \rightarrow +\infty$  can be derived carefully in the saddle point approximation. It was shown that

$$P(\sigma) \propto \exp\left(-\frac{2\sigma^3}{3G}\right), \quad \sigma \gg G^{1/3} \approx \frac{U_{\text{rms}}}{L} \quad (4)$$

$$P_d(v) \propto \exp\left(-\frac{2v^3}{3r^3G}\right), \quad v \gg U_{\text{rms}}$$

where  $G = -\partial_x^2 F(x)|_{x=0}$ . The drawback of the saddle point approach was that it allowed to determine the right hand side asymptotics only. But that asymptotics is expected to depend on whether the force  $f$  of (1) is Gaussian. The asymptotics (4) is true only if the random force of the Burgers equation is Gaussian. But it does not have to be. For example, one could take the random force whose strength is always less than a certain value,  $|f| < f_{\text{max}}$ . Then the asymptotics (4) would not be expected to hold. But if it is so, then one could legitimately ask if at least some part of the functions  $P(v)$  and  $P_d$  are universal, that is, they are independent on the details of the random force  $f$ , except via the force length scale  $L$ . On top of it, since any random Gaussian force leads to Eq. (4), one could ask if at least for the Gaussian force, the entire functions  $P(\sigma)$  and  $P_d(v)$  are universal (that is, independent of the force correlation function  $F(x)$ ).

Some time later, an important paper by W. E. K. Khanin, A. Mazel and Y. Sinai [5] came out. It was argued in that paper that the asymptotics of  $P(\sigma)$  at large negative  $\sigma$  is

$$P(\sigma) \propto \frac{1}{|\sigma|^{5/2}}, \quad \sigma \ll -\frac{U_{\text{rms}}}{L}. \quad (5)$$

The arguments leading to Eq. (5) seemed to be well defined and valid independent of the details of the random force setup. Those arguments were later elaborated in a later paper by U. Frisch, J. Bec and B. Villone [6]. In all of these papers, the asymptotics Eq. (4) was not disputed.

At the same time, these authors argued that the only universal feature of the probability distribution function  $P(\sigma)$  is its “left tail” Eq. (5). The rest of the function was proposed to be nonuniversal, depending on the details of the random force.

These claims were later checked numerically by R. Kraichnan and T. Gotoh [7], but they were not able to unambiguously confirm or reject the power law (5). Some time later J. Bec [8] did a very thorough numerical analysis of the “left tail” and concluded that the power law Eq. (5) is indeed correct.

Here I would like to report a numerical confirmation of the tail Eq. (5). However, I would also like to suggest that the left tail is not the only universal feature of the function  $P(\sigma)$ .

### III. DISCRETE BURGERS EQUATION

To study the Burgers equation numerically, its space and time have to be discretized. One way to do that would be to write derivatives as differences. That is a very crude scheme, however. Burgers equation allows a much more gentle way of discretizing its time, which is based on the Hopf-Cole transformation. In this section I am going to present the scheme as it is, and show how it produces the Burgers equation in the continuum limit.

Consider the energy function  $E_i(x)$  where  $x$  changes from 0 to  $2\pi$  and  $i$  is the integer index (which will later become time). Now write down the following definition

$$E_i(x) = \min_{0 \leq y \leq 2\pi} [1 - \cos(x - y) + E_{i-1}(y)] + h_i(x). \quad (6)$$

Here  $h_i$  is a random function, which I take as

$$h_i(x) = A_i \cos(x - \phi_i), \quad (7)$$

where  $A_i$  is a random variable (for example, random Gaussian with or taking values uniformly distributed from 0 to some maximum  $A_{\max}$ ), and  $\phi_i$  is a random phase uniformly distributed from 0 to  $2\pi$ . The sign min in Eq. (6) should be understood as a minimization over  $y$  which varies from 0 to  $2\pi$  of the expression in the square brackets. Eq. (6) plays the role of a definition of  $E_i$  in terms of  $E_{i-1}$ . As a starting point, one can take  $E_0(x) = 0$ . Finally, the functions  $E_i(x)$  will automatically be periodic in  $x$ .

The equation Eq. (6) is the discrete Burgers equation. To see that, let us take the continuum limit. In the notations of (6), the continuum limit is achieved when  $h_i \ll 1$ ,

$E_i \ll 1$ . The minimization condition reads

$$\sin(x - y) - \partial_y E_{i-1}(y) = 0. \quad (8)$$

Since  $E_i$  is small, one can expand

$$x = y + \partial_y E_{i-1}. \quad (9)$$

Substituting it back to (6) and expanding  $E_{i-1}(x + y - x)$  in powers of  $y - x$ , one finds

$$E_i(x) - E_{i-1}(x) + \frac{1}{2} (\partial_x E_{i-1}(x))^2 = h_i(x). \quad (10)$$

The final step is replacing the discrete index  $i$  with the continuous time  $t$ , to find

$$\partial_t E + \frac{1}{2} (\partial_x E)^2 = h, \quad (11)$$

which is the so-called KPZ equation. Introducing the velocity  $u = \partial_x E$ , one gets the Burgers equation

$$\partial_t u + u \partial_x u = \partial_x h, \quad (12)$$

with  $\partial_x h$  identified as the force  $f$ . In this equation, the viscosity term  $\nu u_{xx}$  is absent. But it is easy to see that find a global minimum in (6) is equivalent to the infinitesimal viscosity term in (12), following, for example, Feigelman [9]. The derivation below is somewhat technical and is given for completeness only, so it is possible to skip the equations Eqs. (13)-(15) without losing any essential information.

Introduce an alternative way of writing down (6)

$$\exp(-E_i(x_i)) = \lim_{\nu \rightarrow 0} \int \prod_{j < i} dx_j \exp \left\{ -\frac{2}{\nu} \sum_{j \leq i} [1 - \cos(x_j - x_{j-1}) + h_j(x)] \right\}. \quad (13)$$

It is clear that the integral, in the limit of vanishing  $\nu$  effectively minimizes the expression in the exponential, doing the work which is accomplished by the min sign

in (6). Passing to the continuum limit, one finds the functional integral

$$\exp(-E(y, T)) = \lim_{\nu \rightarrow 0} \int \mathcal{D}x(t) \exp \left\{ -\frac{2}{\nu} \int_0^T dt \left[ \frac{1}{2} \dot{x}^2 + h(x, t) \right] \right\}, \quad x(T) = y. \quad (14)$$

But the right hand side is none other than the propagator of the Schrödinger equation, which enables us to conclude that  $\exp(-E)$  satisfies the Schrödinger equation and as

a consequence,

$$\partial_t E + \frac{1}{2} (\partial_x E)^2 - \nu \partial_x^2 E = h, \quad \partial_t u + u \partial_x u - \nu \partial_x^2 u = \partial_x h, \quad (15)$$

which concludes the derivation. (In the last line of (15) I replaced the arguments of the function  $E(y, T)$  by  $x$  and  $t$ .)

All this suggests that the following steps must be taken to simulate the Burgers equation numerically. First, generate the random “force”  $h_i(x)$  using (7).  $A_i$  can be taken, for example, uniformly distributed on the interval from 0 to  $A_{\max}$ . Then solve the relationship (6) recursively to find  $E_i(x)$ . The actual value of  $E_i$  is not important for the numerics of the Burgers equation. What is important is the value of  $y$  which minimizes the expression in the square brackets of (6) for each value of  $x$ . Given the relationship between  $x$  and  $y$ , one finds for the Burgers velocity  $u_i = \partial_x E_i(x)$

$$u_i(x) = u_{i-1}(y) + \partial_x h_i(x). \quad (16)$$

This is the discrete analog of the obvious equation  $\frac{d}{dt}u(x(t), t) = f(x, t)$  where  $\frac{d}{dt}x(t) = u(x(t), t)$ , the so-called Lagrangian trajectory (see below). Finally, for the velocity gradients  $\sigma_i(x) = \partial_x^2 E_i(x)$  one finds

$$\sigma_i(x) = \cos(x - y) \frac{\sigma_{i-1}(y)}{\cos(x - y) + \sigma_{i-1}(y)} + \partial_x^2 h_i(x), \quad (17)$$

which is the discrete version of the Eq. (21) considered below. The relations Eq. (6), Eq. (16), and Eq. (17) are those one has to iterate to solve the Burgers equation numerically. To stay close to the continuum limit in time, one has to take  $A_{\max} \ll 1$ . And finally, to do the minimization in (6) in practice one has to discretize space as well. With 2000 spacial discretization point, reliable data can be accumulated over about 10000 steps in time, which takes about 10 minutes computer time on a Sun workstation. The result of one such calculation, with  $A_{\max} = 0.01$  is shown on Fig 1.

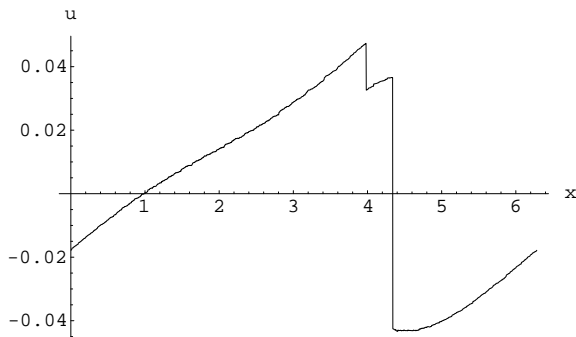


FIG. 1: A typical plot of  $u(x)$  at a fixed time  $t$  taken from an actual computer simulation

The random force chosen according to (7) fixes the function  $F$  to be  $F(x) \propto \cos(x)$  and the characteristic length scale  $L$  is then of the order of the size of the system. If one wants to use a shorter range force, one could take

$$h_i(x) = A_i \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{(x - 2\pi n + \phi_i)^2}{L^2} \right\}, \quad (18)$$

or any other similar function, as long as it is periodic in space  $x$ .

#### IV. STATISTICS OF THE VELOCITY GRADIENTS

The properties of the solution to Burgers equation has been discussed in many publication (see for example [6]). Here I list those which are relevant for the present discussion.

In the Burgers equation, the viscosity  $\nu$  is supposed to be infinitesimally small. Yet it cannot be set to zero. The reason for that is, the solution to this equation develop shock waves, or sharp drops in the value of the velocity field. The width of this drops is of the order of  $1/\nu$  and they become discontinuities in  $u$  when  $\nu$  is taken to zero. In the middle of the drop  $\nu \partial_x^2 u$  has a finite limit as  $\nu \rightarrow 0$ .

To study the gradients of the velocity it is advantageous to introduce the notion of the lagrangian coordinate  $x(t)$ . It is the point which moves together with the fluid

$$\frac{dx(t)}{dt} = u(x(t)). \quad (19)$$

A discrete version of the lagrangian trajectory was given by Eq. (9). Once  $x(t)$  reaches a shock wave, it becomes “trapped” in it. For the purposes of this paper, it is natural to assume that  $x(t)$  ends as soon as it reaches the shock wave (this is precisely what Eq. (9) does). Introduce the notion of the gradient of the velocity on a trajectory defined by  $x(t)$ ,

$$\sigma(t) = \partial_x u(x(t), t). \quad (20)$$

As a consequence of the Burgers equation,

$$\dot{\sigma} + \sigma^2 = \partial_x f(x(t), t). \quad (21)$$

Notice that the  $\nu$  term can be neglected since, by construction,  $x(t)$  avoids shock waves and terminates once it reaches them.

The equation Eq. (21) can be interpreted as a Langevin equation with the random force  $g(t) = \partial_x f(x(t), t)$ . Before proceeding further, one needs to establish the statistics of  $g(t)$ .  $x(t)$  is correlated with  $f$  in a complicated way, and it is not *a priori clear* if those correlations can be neglected.

Fortunately, the discrete Burgers equation discussed in the previous section provides an answer to this question.  $g(t)$  is none other than  $\partial_x^2 h_i(x)$ . But  $x$  is related to  $y$  via Eq. (9), and therefore,  $x$  knows nothing about the random phase  $\phi_i$  included in the definition of  $h_i$  in Eq. (7). Therefore,  $g(t)$  is indeed just a white noise random force

$$\langle g(t)g(t') \rangle = G \delta(t - t') \quad (22)$$

where, as before,  $G = -\partial_x^2 F(x)|_{x=0} \approx (U_{\text{rms}}/L)^3$ . Given a Langevin equation Eq. (21), a Fokker-Planck equation

can be written down for the probability  $P(\sigma)$

$$\frac{\partial}{\partial \sigma} \left( \frac{G}{2} \frac{\partial}{\partial \sigma} + \sigma^2 \right) P = \frac{\partial P}{\partial t}. \quad (23)$$

In the present context this equation was first written down in [10], but in that paper it was given an interpretation different from the one here. In fact, a slightly more general form of this equation will be useful below. It has the form

$$\frac{\partial}{\partial \sigma} \left( \frac{G}{2} \frac{\partial}{\partial \sigma} + \sigma^2 \right) P + \beta P = \frac{\partial P}{\partial t} \quad (24)$$

where  $\beta$  is an arbitrary parameter. At this stage,  $\beta = 0$ . However, later a version of Eq. (24) with nonzero  $\beta$  will be useful. Notice that the large  $\sigma$  asymptotics of  $P$  obtained from the Eq. (24) is Eq. (4), irrespective of the value of  $\beta$ .

Eq. (21) is unstable. As soon as  $\sigma$  becomes large negative, it quickly reaches infinity. This process is none other than the formation of a shock wave. A quick estimate of the probability  $P(\sigma)$  in this regime is easy to do. Once  $\sigma$  is large enough and negative, one can neglect  $g(t)$  in Eq. (21) completely and solve the equation to find

$$\sigma(t) = \frac{1}{t - t_0}, \quad t < t_0 \quad (25)$$

The probability can be estimated to be

$$P(\sigma) \propto \int dt \delta(\sigma - \sigma(t)) \propto \frac{1}{\sigma^2}, \quad \sigma \ll -\frac{U_{\text{rms}}}{L}. \quad (26)$$

This estimate was used in some publications to conclude that this should be the correct power law in the probability distribution function of the Burgers equation Eq. (5). However, to say so would be premature.

It is instructive to reproduce the power law Eq. (26) with the help of the Fokker-Planck equation. To do that, observe that one has to solve this equation with an absorbing boundary conditions at  $\sigma \rightarrow -\infty$ , corresponding to the trajectories disappearing when shock waves are formed. The standard substitution

$$P = \exp \left( -\frac{\sigma^3}{3G} \right) \Psi(\sigma). \quad (27)$$

leads to the Schrödinger equation of a quantum mechanical particle moving in a potential  $U(\sigma) = \sigma^4/(2G) - (\beta + 1)\sigma$

$$-\frac{G}{2} \frac{\partial^2 \Psi}{\partial \sigma^2} + \left( \frac{\sigma^4}{2G} - (\beta + 1)\sigma \right) \Psi = -\frac{\partial \Psi}{\partial t}. \quad (28)$$

The solution to this equation reads

$$\Psi(\sigma) = \sum_{n=0}^{\infty} C_n \exp(-E_n t) \Psi_n(\sigma), \quad (29)$$

where  $C_n$  are arbitrary coefficients,  $E_n$  are the eigenstates, and  $\Psi_n$  are the eigenfunctions of the Schrödinger equation satisfying

$$-\frac{G}{2} \frac{\partial^2 \Psi_n}{\partial \sigma^2} + \left( \frac{\sigma^4}{2G} - (\beta + 1)\sigma \right) \Psi_n = E_n \Psi_n. \quad (30)$$

At large times  $t \gg (E_1 - E_0)^{-1}$ , only the ground state survives

$$P(\sigma, t) \propto \exp(-E_0 t) P_0(\sigma), \quad t \gg (E_1 - E_0)^{-1}, \\ P_0(\sigma) = \exp \left( -\frac{\sigma^3}{3G} \right) \Psi_0(\sigma). \quad (31)$$

A detailed analysis of Eqs. (28)-(31) at arbitrary  $\beta$  can be found in the paper by S. Boldyrev [11]. In the case of interest,  $\beta = 0$ . Ref. [11] shows that in this case  $E_0 > 0$ , and it confirms the asymptotics Eq. (26) for  $P_0(\sigma)$ . The asymptotics at large positive  $\sigma$  coincides with Eq. (4). The fact that  $E_0 > 0$  is not surprising. It means that the total probability  $\int d\sigma P(\sigma)$  decreases with time. This is due to elimination of trajectories which ended on shock waves.

However, this cannot be the correct probability of observing a velocity gradient  $\sigma$  at a given point in space. Indeed, introduce the density of trajectories  $\rho(x, t)$ . It satisfies the continuity equation

$$\partial_t \rho + \partial_x (u\rho) = 0. \quad (32)$$

In particular, the density around the given Lagrangian trajectory  $x(t)$  is given by  $\rho_L(t) = \rho(x(t), t)$  and it satisfies

$$\frac{d}{dt} \rho_L + \sigma \rho_L = 0 \quad (33)$$

It is clear that the probability that a particular trajectory happens to go through an observation point where a velocity gradient is measured is inversely proportional to the density of trajectories at this point. Thus a more physically relevant probability distribution function is defined as

$$P(\sigma) = \left\langle \frac{1}{\rho_L} \delta(\sigma - \sigma(t)) \right\rangle. \quad (34)$$

This function satisfies a Fokker-Planck equation Eq. (24), but with  $\beta = 1$ . In this case, it is possible to show that  $E_0 < 0$  and

$$P(\sigma) \propto \frac{1}{|\sigma|^3}, \quad \sigma \ll -\frac{U_{\text{rms}}}{L}. \quad (35)$$

This asymptotics was proposed in the paper [7]. But it cannot be the correct asymptotics either. Indeed, while the above analysis correctly removes the trajectories which end at the points where a new shock wave is formed, it fails to take into account those which fall into ‘‘mature’’ shock waves, the ones which formed in the past. That is why  $E_0 < 0$  and the probability grows with time. That comes from overcounting the trajectories.

Unfortunately, the Eq. (21) does not seem to contain enough information to determine if there is a shock wave nearby. If that is indeed the case, then the full Burgers equation Eq. (1) has to be solved. That is the core of the problem of Burgers turbulence, and the reason the solution to it seems so close and yet so hard to get. The full Burgers equation Eq. (1) leads to a full fledged 2D quantum field theory (via Martin-Siggia-Rose formalism, for example). That theory is presumably very hard, if not impossible, to solve. The “reduced” equation Eq. (21), on the other hand, leads to a quantum mechanics encoded in Eq. (28). The solution to quantum mechanics problems are clearly within reach. The question is, therefore, if it is possible to extract all the relevant information about the Burgers equation from the Lagrangian equation Eq. (21). At this stage of the present analysis, it does not seem possible.

To move further, I will make the following leap of faith. The equation Eq. (24) was studied in Ref. [11] for arbitrary  $\beta$ . It was found that while the large positive  $\sigma$  asymptotics of  $P$  is always given by Eq. (4), the other asymptotics is given by

$$P(\sigma) \propto \frac{1}{\sigma^{\beta+2}}, \quad \sigma \ll -\frac{U_{\text{rms}}}{L}. \quad (36)$$

On the other hand, it is known from the analysis of Ref. [5] and [6] that the correct power of this asymptotics is  $7/2$ . To match this asymptotics, I choose  $\beta = 3/2$ . Then I propose to solve the equation Eq. (24) and compare what it gives with the numerics. The results of the comparison is shown in next section.

The equation Eq. (24) with an arbitrary  $\beta$  was proposed in [2] (but for  $P_d(v)$  as opposed to  $P(\sigma)$ ) where  $\beta$  was referred to as  $b$ -anomaly. But whether the present discussion has anything to do with the methods of Ref. [2] is unclear. To derive Eq. (24) with  $\beta = 3/2$ , one needs to show that the rate at which the trajectories fall into mature shock waves is proportional to the extra  $\frac{1}{2}\sigma P$  term generated in the Fokker-Planck equation Eq. (24). How to do that is not known to me. Next, the Eq. (24) cannot be exact. After all, it clearly breaks down even if the force is not Gaussian. However, it may be correct in some mean field sense, for moderate values of  $\sigma$ . Finally, I'd like to note that for this value of  $\beta$ ,  $E_0 < 0$  and the probability still grows with time. That must be because this description gives the relative probability of observing one value of  $\sigma$  over another, not the absolute probability, and the overall normalization must be enforced by  $P \rightarrow P/\Pi$  where  $\Pi = \int d\sigma P(\sigma)$ .

## V. NUMERICAL SIMULATIONS

With the setup described in the preceding sections, it is very easy to calculate  $P(\sigma)$  numerically. Unlike some of the previous work on this subject, I did not try to compute  $P(\sigma)$  from the data. Instead, I computed the

integrated probability density

$$N(\sigma) = \int_{-\infty}^{\sigma} d\mu P(\mu). \quad (37)$$

Obviously its left tail is expected to go as  $N(\sigma) \propto 1/|\sigma|^{5/2}$ . To plot it numerically, it is enough to collect the data into an array, to sort it in the increasing order, and plot the position of an array entry as a function of its value. Fig 2. depicts  $N(\sigma)$  in the case when the force was chosen to be as in Eq. (7),  $A_{\text{max}} = 0.01$ , and  $U_{\text{rms}} \approx 0.02$ .

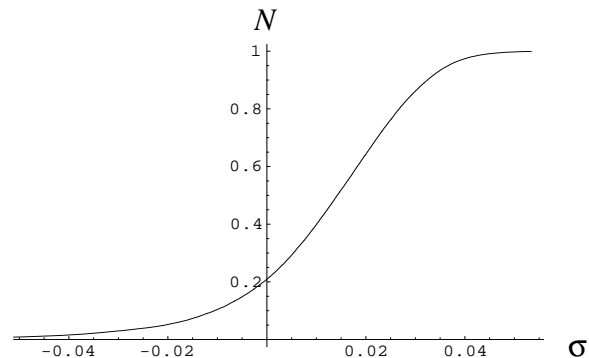


FIG. 2: The integrated probability density  $N(\sigma)$  as a function of  $\sigma$

Fig 3. shows the left tail in the log-log format. For comparison, a straight line with the slope  $-5/2$  is plotted. The agreement is striking. One could object that the

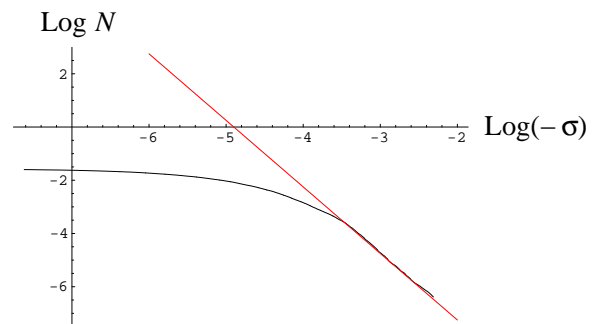


FIG. 3: The plot of  $\log N$  as a function of  $\log(-\sigma)$  for negative  $\sigma$ . The straight line has a slope of  $-2.5$

power law in this graph does not extend over a sufficiently long range of  $\sigma$ . To extend it further, one needs to do numerics at weaker force for longer periods of time (to keep close to the continuum limit even at large velocity fluctuations). It was not the task of the present work to do large scale numerical simulations, and I believe the agreement I have here is good enough (especially in view of the work reported in Ref. [8]).

Fig 4. shows the same probability function, but plotted together with the (numerical) solution to the equation Eq. (24) with  $\beta$  chosen to be  $\beta = 3/2$ . To arrive at this picture, one parameter - the units of measure of  $\sigma$  -

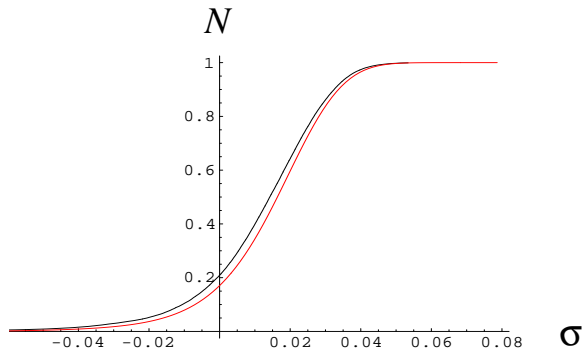


FIG. 4:  $N(\sigma)$  taken from the numerics (upper curve) and extracted from the solution to the Eq. (24) with  $\beta = 3/2$ .

had to be adjusted to make the graphs overlap as much as possible. The agreement is not perfect. Yet the curves are close enough to suggest that perhaps it is not coincidental either. One interesting number is the probability  $p$  that the velocity gradient is negative. It is especially interesting, because it is independent of the adjustments of the units of  $\sigma$ . Numerically  $p \approx 0.21 \pm 0.01$ . The solution to the Eq. (24) gives about 0.18. According to some claims in the literature, this number is not universal. However, the value of  $p$  is reproducible whether I do the numerics with the long ranged force Eq. (7) or shorter range force Eq. (18) with various values of  $L$ . So numerically  $p$  appears to be universal, or perhaps approximately universal.

To conclude, the numerical data does not confirm that the Eq. (24) with the parameter  $\beta = 3/2$  is the *exact* solution to the problem. But it may be an *approximate*, perhaps in some mean field sense, solution to the problem, which reproduces the actual probability density rather close. I believe this question deserves further investigations.

## VI. PINNED CHARGE DENSITY WAVES

In the papers of the author with J.T. Chalker [12, 13] a remarkable correspondence was established between the randomly driven Burgers equation and the physics of pinned charge density waves. It would go beyond this paper to discuss charge density waves, but mathematically the correspondence can be formulated in the following way. Consider the functional

$$\mathcal{E}[x(t)] = \int_0^T dt \left[ \frac{1}{2} \dot{x}^2 + h(x, t) \right], \quad (38)$$

where  $h(x, t)$  is the same random function as the one in Eq. (13). Let us find a function  $x_0(t)$  which, when substituted for  $x(t)$  in Eq. (38), gives an absolute minimum for the functional  $\mathcal{E}$ . Obviously it satisfies the minimization

equation

$$-\frac{d^2 x_0}{dt^2} + \frac{\partial h(x, t)}{\partial x} \Big|_{x=x_0(t)} = 0. \quad (39)$$

Now consider the energy cost of  $x(t)$  deviating from  $x_0(t)$ . Writing  $x(t) \approx x_0(t) + \psi(t)$  we find the equation for the normal modes of oscillations about the absolute minimum, given by

$$\left[ -\frac{d^2}{dt^2} + \frac{\partial^2 h(x, t)}{\partial x^2} \Big|_{x=x_0(t)} \right] \psi = \omega^2 \psi. \quad (40)$$

With the problem thus set up, one needs to calculate the average number of modes with frequencies less than a given frequency  $\omega$ . Such a function is denoted as  $N(\omega)$ .

This problem, which was first formulated in the context of charge density waves in Ref. [14], received a substantial amount of attention in the literature (see Refs. [9, 12, 13, 15, 16, 17]). After initial disagreements, a consensus was built which gave for the function  $N(\omega)$  the value

$$N(\omega) \propto \omega^5, \quad (41)$$

for sufficiently small  $\omega$ . This was first proposed in Ref. [16] and then the derivation was improved in subsequent publications until Ref. [12] derived Eq. (41) in a systematic way.

One of the results of Refs. [12, 13] relates the calculation of  $N(\omega)$  to the following specific question in Burgers turbulence. Consider a Burgers fluid Eq. (1) driven by a random force  $f(x, t) = \partial_x h(x, t)$ . Consider a Lagrangian trajectory  $x_0(t)$  which moves with a fluid and never gets absorbed by a shock wave for times  $t < T$ . Consider a velocity gradient  $\sigma(t)$  defined in Eq. (20). Let us find time intervals  $t_1 < t < t_2$  such that  $\sigma(t) < 0$  within those intervals. Now let us calculate the probability that

$$\int_{t_1}^{t_2} dt \sigma(t) < \log(\omega), \quad (42)$$

for some small value of  $\omega$ . Then the function  $N(\omega)$  of the charge density wave problem coincides with this probability.

Notice that knowing the tails of the probability distribution  $P(\sigma)$  would not help calculating  $N(\omega)$ . This is because the typical functions  $\sigma(t)$  which satisfy Eq. (42) are not those for which  $\sigma$  is very large negative, but rather moderate negative  $\sigma(t)$  which however persist over long time intervals. Therefore, the results of W. E *et al* [5] are useless if we were to calculate  $N(\omega)$  with the help of Eq. (42).

Let us however use the methods of this paper, in particular Eq. (28), to calculate  $N(\omega)$ . To do so, we need to write down a Feynman path integral formalism which corresponds to Eq. (28) and then use the Lagrange multiplier method described in Ref. [13], section VI C, subsection 1, corrected however for the presence of a parameter

$\beta$  in Eq. (28). Then we find

$$N(\omega) \propto \omega^{2\beta+2}. \quad (43)$$

Now we know the exact answer to the problem, Eq. (41). If the methods described in this paper were to reproduce the exact answer, we have to choose  $\beta = 3/2$ . However, *this is the same value of  $\beta$  required to reproduce the asymptotics of W. E et al!*

Two possible explanations of this are possible. First is, perhaps the Eq. (28) with  $\beta = 3/2$  indeed reproduces the right asymptotics of W. E et al and for some reason also reproduces the function  $N(\omega)$  correctly, but this equation is still generally not correct and the probability distribution derived with its help and shown on Fig. 4 is not correct either. Second is, the Eq. (28) is indeed correct and gives the correct solution to the Burgers problem.

The first explanation seems far fetched, as a remarkable coincidence must be at play to give rise to it. The second explanation sounds much more likely. And yet, no derivation of Eq. (28) is known at this point.

## VII. CONCLUSION

In this paper, I demonstrate that it is easy to obtain reliable numerical data on the behavior of the 1D Burgers

equation. I show that the data confirms the 7/2 tail of the probability distribution suggested in the literature. The data also suggests that the entire probability distribution function could be universal, determined by the solution to the equation Eq. (24) with the ‘‘anomaly’’ parameter  $\beta = 3/2$ . I also discuss that it is possible to reproduce the known solution to the pinned charge density wave problem by applying the techniques discussed here and choosing  $\beta = 3/2$ , which gives extra weight to these methods.

## VIII. ACKNOWLEDGEMENTS

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