Intermittent distribution of tracers advected by a compressible random flow

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Multifractal properties of a tracer density passively advected by a compressible random velocity field are characterized. A relationship is established between the statistical properties of mass on the dynamical fractal attractor towards which the trajectories converge and large deviations of the stretching rates of the flow. In the framework of the compressible Kraichnan model, this result is illustrated by analytical calculations and confirmed by numerical simulations.

We are interested in the passive transport of a scalar density by smooth-in-space compressible random flows in a d-dimensional bounded domain Λ . During transport, the density develops strong inhomogeneities. Our goal is to describe quantitatively their fine structure arising at asymptotically large times. Compressible flows are physically relevant not only at large Mach numbers. For instance, the dynamics of a suspension of finite-size (inertial) particles in an incompressible flow may be approximated in the limit of short Stokes times by that of simple tracers in an effective compressible flow [\[1](#page-3-0), [2\]](#page-3-1). Another application is to the advection of particles floating on the surface of an incompressible fluid [\[3\]](#page-3-2).

A smooth passive density field $\rho(t, x)$ evolves in the velocity field $v(t, x)$ according to the continuity equation

$$
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}
$$

which preserves the total mass that we shall assume for convenience normalized to one. If the initial density at time t_0 is uniform and if the flow is compressible and sufficiently mixing, then the solution $\rho(t)$ of [\(1\)](#page-0-0) will approach, when t_0 tends to $-\infty$, a singular limit $\rho_*(t)$ which is a measure with support on the dynamical attractor towards which the Lagrangian trajectories converge. For random velocities, the measure $\rho_*(t)$ and the dynamical attractor depend on the velocity realization. We shall consider stationary velocity ensembles where the statistics of $\rho_*(t)$ does not depend on t. For convenience, we shall restrict our study to $\rho_* \equiv \rho_*(0)$. One expects the measure ρ_* to have roughly a local product structure with a smooth density along the unstable manifolds of the flow and a fractal-like structure in the transverse directions (such measures are called SRB [\[4](#page-3-3)]).

The Lagrangian average defined by

$$
\langle F \rangle \equiv \overline{\int_{\Lambda} F(\boldsymbol{x} \,|\, \boldsymbol{v}) \,\rho_*(\boldsymbol{x} \,|\, \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x}}\,,
$$

where the overline denotes the expectation with respect to the velocity ensemble, samples points in Λ according to the density ρ_* of the asymptotic tracer distribution on the random attractor. It determines an invariant measure μ_* of the random dynamical system defined on the product of the physical space Λ and the space of the velocity realizations.

We are interested in the small-scale statistics of the mass distribution associated to the measure ρ_* . Denote by $\mathcal{B}_r(x)$ the ball of radius r around the point x and introduce the quantities

$$
m_r(\boldsymbol{x}) \equiv \int_{\mathcal{B}_r(\boldsymbol{x})} \rho_*(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}\,, \quad h_r(\boldsymbol{x}) \equiv \frac{\ln\,m_r(\boldsymbol{x})}{\ln r}\,.
$$

The local dimension at x , defined if the limit exists as $h(\mathbf{x}) \equiv \lim_{r \to 0} h_r(\mathbf{x})$, characterizes the small-scale distribution of mass associated to the limiting density ρ_* . A more global assessment is provided by the Hentschel-Procaccia spectrum for dimensions [\[5,](#page-3-4) [6\]](#page-3-5) of ρ_* :

$$
HP(n | \rho_*) \equiv \lim_{r \to 0} \frac{\ln \int_{\Lambda} m_r(\boldsymbol{x})^{n-1} \rho_*(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}}{(n-1) \ln r} \qquad (2)
$$

for real *n*. In particular, $HP(0 | \rho_*)$ is the fractal dimension of the support of $\rho_*, H^p(1 | \rho_*)$ is known as the information (or capacity) dimension and $HP(2 | \rho_*)$ as the correlation dimension. For random flows, one may define the quenched version of the Hentschel-Procaccia spectrum $HP_{\text{qu}}(n) \equiv HP(n | \rho_*)$ and the **annealed** version $HP_{an}(n)$ where the velocity average is taken under the logarithm in the numerator on the right-hand side of [\(2\)](#page-0-1). One has $(n - 1)HP_{qu}(n) \ge (n - 1)HP_{an}(n)$ and $HP_{qu}(1) = HP_{an}(1)$ (if exists). If the moments of $m_r(x | v)$ with respect to the invariant measure μ_* exhibit the small r scaling

$$
\langle m_r^n \rangle \sim r^{\xi_n},\tag{3}
$$

then $HP_{an}(n+1) = \xi_n/n$. There are various aspects of behavior [\(3\)](#page-0-2). First, a non-linear dependence of the scaling exponents ξ_n implies intermittency in the mass distribution and, in particular, the presence of non-Gaussian tails in the probability density function (PDF) of m_r . Second, this behavior suggests that the PDFs of h_r take, for small radii, the large deviations form $e^{(\ln r) S(h)}$ with the rate function $S(h)$ and the scaling exponents ξ_n related by the Legendre transform: $S(h) = \max_n [\xi_n - nh].$ Third, the (annealed) fractal dimensions $f(h)$ of the random sets on which the local dimension $h(x)$ is equal to h (the multifractal spectrum), is expected to be equal to $h - S(h)$.

The aim of this letter is to relate quantitatively the dimensional spectrum $HP_{an}(n)$ to the large deviations of the stretching rates of the flow (sometimes also called finite-time Lyapunov exponents). We note here that in [\[2\]](#page-3-1) a similar relation was discussed for $n = 1$ and, in a qualitative way, for large n.

Let $\mathbf{X}(t, x)$ denote the Lagrangian trajectory passing $(\partial_j X^i(t, x))$ describes the flow linearized around the traat time zero through point x. The matrix $W(t, x) \equiv$ jectory \mathbf{X} . The eigenvalues of the positive matrix W^TW may be written, arranged in non-increasing order, as $e^{2t\sigma_1}, \ldots, e^{2t\sigma_d}$ with $\sigma_i \equiv \sigma_i(\tau, x|v)$ called the stretching rates of the flow. The limits

$$
\lambda_i \equiv \lim_{t \to \infty} \sigma_i(t)
$$

exist almost surely with respect to the invariant measure μ_* and are constant if the latter is ergodic. They define the Lyapunov exponents of the flow. This is the content of the Multiplicative Ergodic Theorem [\[7\]](#page-3-6). As argued in [\[8](#page-3-7)], if the exponents λ_i are all different then the PDF of the stretching rates takes the large deviation form

$$
e^{-t H(\sigma_1,\ldots,\sigma_d)} \theta(\sigma_1-\sigma_2)\ldots\theta(\sigma_{d-1}-\sigma_d) \qquad (4)
$$

for large t with convex rate function H attaining its minimum, equal to zero, at $\sigma_i = \lambda_i$.

If the top Lyapunov exponent λ_1 is negative, then the attractor measure ρ_* is expected to be atomic with trivial mass statistics. For $\lambda_1 > 0$, Ledrappier and Young [\[9](#page-3-8)] showed under rather general assumptions that the local dimension h is almost surely equal to the Lyapunov dimension $d_L = j + \delta$ where $0 < \delta \leq 1$ is such that $\lambda_1 + \ldots + \lambda_j + \delta \lambda_{j+1} = 0$. The above statement implies that

$$
HP_{\text{qu}}(1) = HP_{\text{an}}(1) = \frac{d\xi_n}{dn}|_{n=0} = d_L. \tag{5}
$$

The Lyapunov dimension d_L was introduced by Kaplan and Yorke [\[10](#page-3-9)] and may be heuristically interpreted as the dimension of objects keeping a constant volume during the time evolution. Eq. [\(5\)](#page-1-0) gives only partial information about the mass scaling as compared to the full set of exponents ξ_n .

To relate the mass distribution in small balls to the fluctuations of the stretching rates, we focus on the twodimensional case. The expected smoothness of ρ_* in the

FIG. 1: Two-dimensional sketch of the backward-in-time evolution between times t and 0 of a small parallelogram of length r_1 in the direction of the unstable manifold and r_2 in the direction perpendicular to it.

unstable direction leads us to consider small parallelograms with one side of size r_1 parallel to the unstable manifold and with extension r_2 in the direction perpendicular to it. We expect that the moments of the mass m_{r_1,r_2} in such parallelograms obey

$$
\langle m_{r_1,r_2}^n \rangle \sim r_1^n r_2^{\xi_n - n} \tag{6}
$$

for small r_1, r_2 as long as $\xi_n - n \geq 0$. When $\xi_n < n$, we expect [\(6\)](#page-1-1) to be replaced for $r_2 > r_1$ by

$$
\langle m_{r_1,r_2}^n \rangle \sim r_1^{\xi_n} r_2^0. \tag{7}
$$

This may be viewed as analogous to the stretching in one direction of a fractal set of dimension D, while contracting it in the other one. In the expanding direction, the set will behave (down to a scale depending on the stretching) as if projected on a line, so it will have dimension $min(1, D)$. In the other direction it will have the complementary dimension $D - \min(1, D) = \max(D - 1, 0)$.

Let us consider such a parallelogram at $t > 0$ and let us look at its pre-image at time zero. While the direction parallel to the unstable manifold is exponentially contracted backward-in-time with a rate given by the largest stretching rate σ_1 , the other direction typically expands with an exponential rate σ_2 . Hence the time-zero preimage of the original parallelogram is (approximately) another parallelogram as sketched in Fig. [1.](#page-1-2) Conservation of mass and stationarity of the statistics lead to the relation

$$
\langle \, m^n_{r_1,r_2} \rangle \approx \langle \, m^n_{r_1\mathrm{e}^{-t\sigma_1}} \, ,_{r_2\mathrm{e}^{-t\sigma_2}} \rangle.
$$

If there is a sufficiently rapid loss of memory in the Lagrangian dynamics (i.e. if the invariant measure is sufficiently mixing) then the expectation on the right-hand side should factorize for large t (such a factorization holds for all t in the Kraichnan model discussed below). In such situation, using the large deviation form [\(4\)](#page-1-3) of the PDF of the stretching rates, we infer that

$$
\langle m_{r_1,r_2}^n \rangle \sim \int_{\sigma_1 \ge \sigma_2} \langle m_{r_1 e^{-t\sigma_1}, r_2 e^{-t\sigma_2}}^n \rangle e^{-tH(\sigma_1, ..., \sigma_2)} d\sigma_1 d\sigma_2.
$$

Consistency of the above relation with the scaling [\(6\)](#page-1-1) requires that

$$
1 \sim \int_{\sigma_1 \geq \sigma_2} e^{-t [n\sigma_1 + (\xi_n - n)\sigma_2 + H(\sigma_1, \sigma_2)]} d\sigma_1 d\sigma_2.
$$

Since t is assumed large, a saddle-point argument implies then the following relation between the scaling exponents ξ_n and the rate function H of the stretching rates:

$$
\min_{\sigma_1 \ge \sigma_2} [n\sigma_1 + (\xi_n - n)\sigma_2 + H(\sigma_1, \sigma_2)] = 0. \tag{8}
$$

Alternative formulations are

$$
\xi_n = n - \max_{\substack{\sigma_1 \ge \sigma_2 \\ \sigma_2 < 0}} \frac{1}{\sigma_2} \left[n\sigma_1 + H(\sigma_1, \sigma_2) \right] = \min_{h \ge 0} \left[h n + S(h) \right]
$$

for $S(h) = \min_{\sigma > 0} \sigma^{-1} H((h-1)\sigma, -\sigma)$. These formulae are valid for $\xi_n \geq n$. Similarly, from [\(7\)](#page-1-4) we may get the formula valid for $\xi_n \leq n$

$$
\xi_n = - \max_{\substack{\sigma_1 \ge \sigma_2 \\ \sigma_1 < 0}} \frac{1}{\sigma_1} H(\sigma_1, \sigma_2). \tag{9}
$$

It is easily checked from [\(8\)](#page-1-5) that $(d\xi_n/dn)|_{n=0} = 1 \lambda_1/\lambda_2$ which coincides, in our settings, with formula [\(5\)](#page-1-0). Also from [\(8\)](#page-1-5) one may deduce that

$$
\lim_{t \to \infty} \frac{1}{t} \left\langle |\mathbf{R}(t)|^{-\xi_1} \right\rangle = 0,
$$

for $\mathbf{R}(t) = W(t, x) \mathbf{R}_0$, meaning that the generalized Lyapunov exponent of order $-\xi_1$ vanishes. This is a known result for stochastic flows of diffeomorphisms, see [\[11\]](#page-3-10). Relation [\(8\)](#page-1-5) can be extended to dimensions higher than two using similar arguments.

To illustrate the small-scale properties of the mass distribution, we focus in the sequel on transport by a random compressible velocity field chosen in the framework of the Kraichnan model [\[12\]](#page-3-11). For the domain Λ we take a periodic box. The velocity \boldsymbol{v} is taken centered Gaussian, with covariance

$$
\overline{v^{i}(\boldsymbol{x}+\boldsymbol{\ell},t+\tau)\,v^{j}(\boldsymbol{x},t)}=2(D_{0}\delta^{ij}-d^{ij}(\boldsymbol{\ell}))\,\,\delta(\tau),
$$

where, for small separations ℓ , the function d^{ij} satisfies

$$
d^{ij}(\ell) = \frac{D_1}{2} \left[(d + 1 - 2\wp) \, \delta^{ij} \ell^2 + 2(\wp d - 1) \, \ell^i \ell^j + o(\ell^2) \right]
$$

assuring local isotropy. The parameter \wp , called the compressibility degree, is chosen in the interval [0, 1]. Its extreme values correspond, respectively, to incompressible and to potential velocity fields. For such velocity ensembles, the distribution of the stretching rates is explicitly known [\[8,](#page-3-7) [13\]](#page-3-12). The corresponding rate function takes the simple form

$$
H = \frac{1}{C_1} \Big[\sum_{i=1}^d (\sigma_i - \lambda_i)^2 + C_2 \Big(\sum_{i=1}^d (\sigma_i - \lambda_i) \Big)^2 \Big].
$$

 $C_1 \equiv 4D_1 (d+\wp(d-2)), C_2 \equiv (1-\wp d)/(\wp(d-1) (d+2)).$ The Lyapunov exponents are $\lambda_i = D_1[d(d-2j+1) 2\varphi(d+(d-2)j)$. In the two-dimensional case, the largest Lyapunov exponent is negative when $\wp > 1/2$. For $\wp <$ 1/2, the exponents for the mass distribution obtained from (8) and (9) read

$$
\xi_n = \begin{cases} \frac{2n + \sqrt{(1+2\wp)^2 - 8\wp n}}{1+2\wp} - 1 & \text{if } n \le n_{\text{cr}},\\ \xi_{\infty} & \text{if } n \ge n_{\text{cr}} \end{cases} \tag{10}
$$

where the critical moment n_{cr} and the saturation exponent ξ_{∞} are given by

$$
\begin{cases}\nn_{\text{cr}} = \frac{1}{2}\sqrt{1 + \frac{1}{2\wp}} & \text{if } 0 < \wp \le 1/6, \\
\xi_{\infty} = 2n_{\text{cr}} - 1 & \\
n_{\text{cr}} = \xi_{\infty} = \xi_1 = \frac{2 - 4\wp}{1 + 2\wp} & \text{if } 1/6 \le \wp < \frac{1}{2}.\n\end{cases}
$$

The two different behaviors are illustrated in Fig. 2. In both cases, the events contributing to the saturation of the exponents are those for which a mass of order unity is concentrated inside the small ball.

FIG. 2: (a) Ellipses in the (n, ξ_n) plane corresponding to the unconstrained minimum in [\(8\)](#page-1-5). Dotted parts refer to situations when either the minimum is reached for $\sigma_1 < \sigma_2$ or $\xi_n < n$. Two values of \wp are represented to illustrate both the case \wp < 1/6 and \wp > 1/6. (b) Location where the minimum is reached in the (σ_1, σ_2) plane.

Numerical simulations confirm the values of the scaling exponents obtained explicitly for the compressible Kraichnan model. To distinguish the two cases, two different values of the compressibility degree are investigated ($\wp = 1/10 < 1/6$ and $\wp = 3/10 > 1/6$). The velocity field \boldsymbol{v} is generated by the superposition of nine independent Gaussian modes and the density is approximated by considering a large number of Lagrangian tracers. The exponents obtained numerically after averaging over 10^5 turnover times and for $N = 10^5$ tracers are shown in Fig. [3.](#page-2-1) Although statistical convergence of the average is quite slow, these expensive simulations are in a rather good agreement with the theory, in particular with the saturation of the exponents after the critical order.

FIG. 3: Scaling exponents ξ_n for the mass distribution associated to the advection of a density field by a Kraichnan velocity field. For two different degrees of compressibility, the exponents obtained match those predicted by theory which are represented as dashed lines.

The determination of the exponents may be improved for positive integer orders n by considering only $n+1$ particles. This method allows to perform very long time averages (here of the order of 10^8 turnover times) required for good convergence of the statistics at small scales. As shown in Fig. [4,](#page-3-13) for $\wp = 3/10$, the moments clearly scale

over several decades. For the lower value of compressibility $\wp = 1/10$, convergence is slower because of the smaller probability for the tracked particles to come close together.

FIG. 4: For $n+1$ particles $(n=1, 2 \text{ and } 3)$ the distribution of the maximum of the distances of the reference particle to the other n particles is represented. It can be shown that this PDF is proportional to $d\langle m_r^n \rangle / d \ln r$. The insets show the difference with small- r asymptotes with slopes given by the theoretical exponents.

Finally, let us mention that, for integer orders, the exponent ξ_n for the Kraichnan model can be linked to the homogeneity degree of the stationary single-time density correlation function

$$
\mathcal{F}_{n+1} \equiv \overline{\rho(\mathbf{0})\rho(\boldsymbol{x}_1)\cdots\rho(\boldsymbol{x}_n)}\,.
$$

The nth order moment of m_r may be written as

$$
\langle m_r^n \rangle = \int_{\mathcal{B}_r(\mathbf{0}) \times \cdots \times \mathcal{B}_r(\mathbf{0})} \mathcal{F}_{n+1}(\mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_n) \, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_n \,. \tag{11}
$$

In the stationary regime of the Kraichnan model \mathcal{F}_{n+1} is a zero mode of the operator

$$
M_{n+1}^{\dagger} \equiv \sum_{0 \leq k,\ell \leq n} \partial_{x_k^i} \partial_{x_\ell^j} \left[\left(D_0 \delta^{ij} - d^{ij} (\boldsymbol{x}_k - \boldsymbol{x}_\ell) \right) \cdot \right]
$$

We need not write these zero modes explicitly. The homogeneity degree of the isotropic solution of lowest degree can be found simply by requiring its positivity and imposing on it certain continuity and integrability conditions. The branch $n < n_{cr}$ in [\(10\)](#page-2-2) is obtained by requiring the solution to be continuous at $x_1 = \ldots = x_n$. On this subspace, M_{n+1}^{\dagger} is degenerate, owing to the fact that collinear points remain collinear when transported by the (linearized) flow. If in addition we restrict M_{n+1}^{\dagger} to the rotationally invariant sector, it becomes an ordinary second-order homogenous differential operator. It has two scaling solutions. The one with exponent $\xi_n - nd$ gives, through [\(11\)](#page-3-14), the non-saturated $(n < n_{cr})$ branch of [\(10\)](#page-2-2). Such a solution breaks down when the corresponding zero mode ceases to be integrable at small x_i which may occur if the restriction of the zero mode to the collinear sector is not integrable near 0, i.e. $\xi_n - nd \leq -1$. This gives n_{cr} and ξ_{∞} in the case $\wp < 1/6$. Another scenario is if the zero mode has a non-integrable singularity around the collinear geometry. This gives n_{cr} and ξ_{∞} in the case $\wp > 1/6$.

An important open question not touched upon by this paper concerns the case of rough-in-space velocity fields appearing in the limit of very high Reynolds numbers. That problem cannot be formulated in terms of stretching rates but the relationship with density correlations and zero modes still holds.

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When this work was essentially finished, we learned from A. Fouxon that he has also derived the relationship [\(8\)](#page-1-5) between the scaling exponents of mass and the large deviations of the stretching rates.

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