

# Large scale inhomogeneity of inertial particles in turbulent flow

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Preferential concentration of inertial particles in turbulent flow is studied by high resolution direct numerical simulations of two-dimensional turbulence. The formation of network-like regions of high particle density, characterized by a length scale which depends on the Stokes number of inertial particles, is observed. At smaller scales, the size of empty regions appears to be distributed according to a scaling law.

The transport of inertial particles in fluids displays properties typical of compressible motion even in incompressible flows. This is a consequence of the difference of density between particles and fluid. The most peculiar effect is the spontaneous generation of inhomogeneity out of an initially homogeneous distribution. The clustering of inertial particles has important physical applications, from rain generation [1], to pollutant distribution and combustion [2]. Starting from the first examples in laminar flow [3], it is now demonstrated both numerically [4, 5, 6] and experimentally [7, 8] that also in fully developed turbulence there is a tendency of inertial particles to form small scale clusters. The parameter characterizing the effect of inertia is the Stokes number  $St$ , defined as the ratio between the particle viscous response time  $\tau_s$  and a characteristic time of the flow  $\tau_v$ . In the limit  $St \rightarrow 0$  inertial particles recover the motion of fluid particles and no clusterization is expected. In the opposite limit  $St \rightarrow \infty$  particles become less and less influenced by the velocity field. The most interesting situation is observed for intermediate values of  $St$  where strong clusterization is observed [6, 7].

In the case of a smooth velocity field the Eulerian characteristic time  $\tau_v$  is a well defined quantity as it can be identified with the inverse Lyapunov exponent  $\tau_v = \lambda_1^{-1}$  of fluid trajectories. In this case some general theoretical predictions are possible [9, 10] such as the exponential growth of high order concentration moments. Detailed numerical simulations in a chaotic random flow have shown maximal clusterization (measured in terms of the dimension of the Lagrangian attractor) for a value  $St \simeq 0.1$  [11].

In the case of turbulent flow, where the velocity field is not smooth, a simple scaling argument suggests that maximal compressibility effects are produced by the smallest, dissipative scales [10]. Nevertheless, for sufficiently large values of  $St$ , the particle response time introduces a characteristic scale in the inertial range which breaks the scale invariance of the velocity field and produces, as we will see, large scales inhomogeneity in particle distribution.

The motion of a spherical particle in an incompressible

flow, when the size of the particle is so small that the surrounding flow can be approximated by a Stokes flow, is governed by the set of equations [12]

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\frac{1}{\tau_s} [\mathbf{v} - \mathbf{u}(\mathbf{x}(t), t)] + \beta \frac{d}{dt} \mathbf{u}(\mathbf{x}(t), t) \end{cases} \quad (1)$$

where  $\mathbf{v}$  represents the Lagrangian velocity of the particle,  $\beta = 3\rho_0/(\rho_0 + 2\rho)$  and  $\rho$  and  $\rho_0$  are the density of particle and fluid respectively. In (1)  $\mathbf{u}(\mathbf{x}, t)$  represents the velocity field whose evolution is given by Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f} \quad (2)$$

In what follows, we will consider the limit heavy particles such that  $\beta \simeq 0$ . In this limit it is easy to show that the Lagrangian velocity possesses a compressible part [10]: expanding (1) to first order in  $\tau_s$  and using  $\nabla \cdot \mathbf{u} = 0$ , one obtains, from (2)

$$\nabla \cdot \mathbf{v} \simeq -\tau_s \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \neq 0 \quad (3)$$

From (3) it is possible to give a dimensional estimation of the relative importance of the compressible part for a turbulent velocity field with scaling exponent  $h$ ,  $\delta_\ell u \sim U(\ell/L)^h$ ,  $L$  and  $U$  being a characteristic large scale and velocity. The scaling exponent for the compressible component of  $\mathbf{v}$  is  $\delta_\ell v \sim U(\ell/L)^{2h-1}$  and thus the relative compressibility scales as  $(\ell/L)^{h-1}$ , i.e. reaches the maximum value at the viscous scale [10]. Nevertheless, we will see that the presence of inertial range of scales in the turbulent flow generates large scale structures in the particle distribution at large  $St$ .

In this letter we address the problem of transport of heavy inertial particles in fully developed two-dimensional turbulence in the inverse cascade regime. High resolution direct numerical simulations of two-dimensional Navier-Stokes equation with white in time, random forcing  $\mathbf{f}$  localized at small scales  $\ell_f$  have been performed. As customary, a friction term  $-\alpha \mathbf{u}$  is added

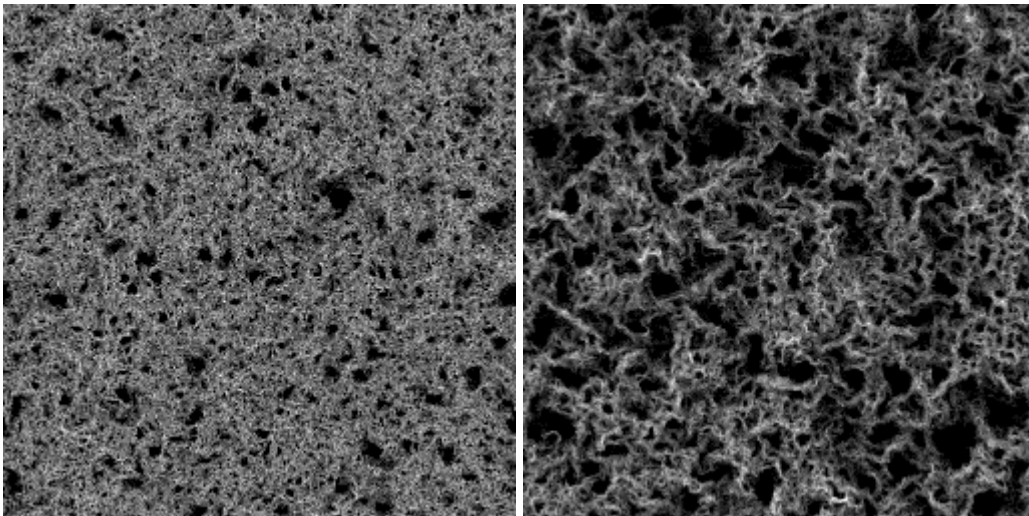


FIG. 1: Snapshots of particle concentrations taken at the same time in stationary condition for two realizations with different Stokes numbers  $St = 0.12$  (left) and  $St = 1.2$  (right) started from identical initial conditions and advected by the same turbulent flow. The number of particles in both cases is  $1024^2$ . Gray levels correspond to particle concentration. Turbulent velocity is obtained by integration of two-dimensional Navier–Stokes equation by a standard pseudo-spectral code on doubly periodic unit square domain at resolution  $N^2 = 1024^2$ .

to (2) in order to extract energy from the system at the friction scale  $\ell_{fr} \sim \varepsilon^{1/2} \alpha^{-3/2}$ . The intermediate scales  $\ell_f \ll \ell \ll \ell_{fr}$  define the inertial range in which Kolmogorov scaling  $\delta_{\ell} u \sim U(\ell/L)^{1/3}$  is observed [13]. Lagrangian tracers are placed at random with initial zero velocity and integrated according to (1) with a given  $\tau_s$ . After a scratch run long several  $\tau_s$ , Lagrangian statistics is accumulated for typically some tens of  $\tau_s$ . Stokes time is made dimensionless by rescaling with the Lagrangian Lyapunov exponent of fluid particles,  $St \equiv \lambda_1 \tau_s$ . Fig. 1 shows typical distributions of inertial tracers in stationary conditions at different values of  $St$ , obtained starting from the same initial random distribution. One observes in both cases strong inhomogeneous distributions with empty “holes”, in the second case on much larger scales.

Maximum compressibility effects are expected at small scales and can be described by the Lyapunov spectrum for inertial particles. We recall that for a generic dynamical system the sum of the Lyapunov exponents gives the exponential rate of expansion (or contraction) of the hypervolume in phase space. In our case, from (1) we have  $\sum_{i=1,4} \lambda_i = -2/\tau_s$ , thus volumes are contracted at a constant rate. Let us observe that when  $\tau_s \rightarrow \infty$ , phase space contraction rate vanishes, and we thus expect less clusterization. As a consequence of the structure of (1) we find that two Lyapunov exponents are close to  $-1/\tau_s$ , representing the rate of adjustment of Lagrangian velocity to the Eulerian one. The first Lyapunov exponent is found positive, as the trajectories are chaotic and the second, negative, determines the dimension of the attractor according to the definition of Lyapunov dimension [14]

$$d_L = K + \frac{\sum_{i=1}^K \lambda_i}{|\lambda_{K+1}|} \quad (4)$$

where  $K$  is defined as the largest integer such that  $\sum_{i=1}^K \lambda_i \geq 0$ . In the inset of Fig. 2 we show the dependence of the first Lyapunov exponent on Stokes number. As  $St$  increases,  $\lambda_1(St)$  decreases monotonically from the neutral value  $\lambda_1(0) \simeq 0.72$ . The behavior of Lyapunov dimension is also shown in Fig. 2. In the limit  $St \rightarrow 0$ , particles become neutral and thus one recovers the homogeneous distribution with  $d_L = 2$ . The presence of a minimum around  $St \simeq 0.1$  was already discussed in the case of smooth flow [11] and indicates a value for which compressibility effects are maximum. We remark that the curve  $d_L(St)$  of Fig. 2 is almost identical to the one obtained in synthetic smooth flow [11], and is thus insensitive to the presence of the hierarchy of scales typical of a turbulent flow.

In the turbulent scenario, instead of becoming more homogeneous, for increasing  $St$  the inertial particle distribution develops structures on larger scales, as is evident in Fig. 1. The most evident dishomogeneities are related to the presence of empty regions (“holes”) on different scales. We have thus studied the hole statistics at varying Stokes number.

We have performed a coarse graining of the system by dividing it into small boxes forming the sites of a square lattice and counting the number of particles contained in each small box. From this coarse grained density we have computed the probability density function of holes, defined as connected regions of empty boxes.

Probability density functions (pdf) of hole areas are shown in Fig. 3 for different Stokes numbers. The hole distributions follow a power law with an exponent  $-1.8 \pm 0.2$  up to an exponential cutoff at a scale  $A_{St}$  which moves to larger sizes with  $St$ , as shown in the inset of Fig. 3. At variance with the smooth flow case, in

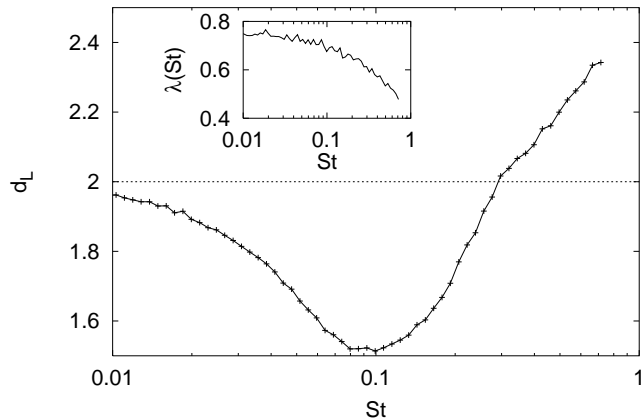


FIG. 2: Lyapunov dimension for heavy particles in two-dimensional turbulence as a function of the Stokes number. Inset: the first Lyapunov exponent  $\lambda_1$ .

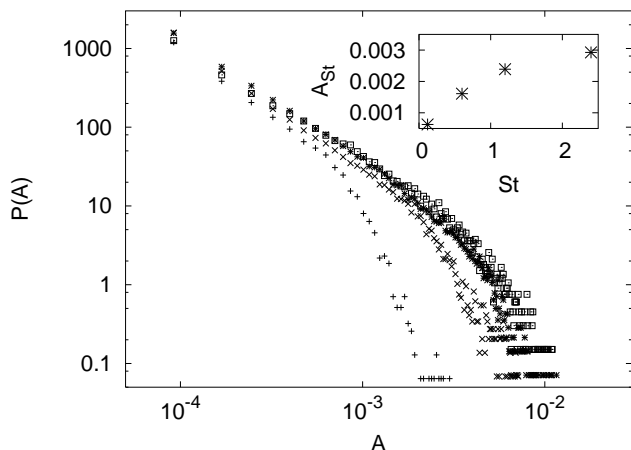


FIG. 3: Probability density functions (pdf) of hole areas, normalized with the area of the box, for  $St = 0.12$  (+),  $St = 0.6$  ( $\times$ ),  $St = 1.2$  (\*) and  $St = 2.4$  ( $\square$ ). Holes are defined as connected regions of coarse grained distribution with zero density. Probability density functions are computed over 100 independent realizations of  $1024^2$  particles each. In the inset we show the dependence of the cutoff area  $A_{St}$  (defined by the condition that 1% of holes has larger area) on  $St$ . The robustness of the hole area census with respect to particle statistics have been checked by increasing the number of tracers up to  $4 \times 10^6$ .

which particle density recovers homogeneity for  $St > 0.1$  [11], in the turbulent case inhomogeneities are pushed to larger scales when increasing  $St$ .

The hole area pdf is independent on the number of particles used in the simulations. This is a non trivial property, reflecting the fact that inertial particles cluster on network-like structures where a clear-cut distinction between empty regions and particle-rich regions can be observed. We have also verified that the choice of the small coarse graining scale does not modify the hole area

pdf at larger scales.

These results support the following physical picture. According to (1), heavy particles filter out velocity fluctuations on timescales shorter than  $\tau_S$ . We may thus expect that inhomogeneity does appear at a scale  $\ell_S$  whose characteristic time is of order of  $\tau_S$ . For a velocity field with a roughness exponent  $h \leq 1$ , this would lead to  $A_{St} \sim St^{2/(1-h)}$ , i.e. to the increase of hole areas with  $St$ , in qualitative agreement with Fig. 3. Clusterization of inertial particles in turbulence appears to be a self-similar process up the cutoff scale  $A_{St}$ . Indeed the pdfs of hole areas collapse when areas are rescaled with the cutoff area, as shown in Fig. 4.

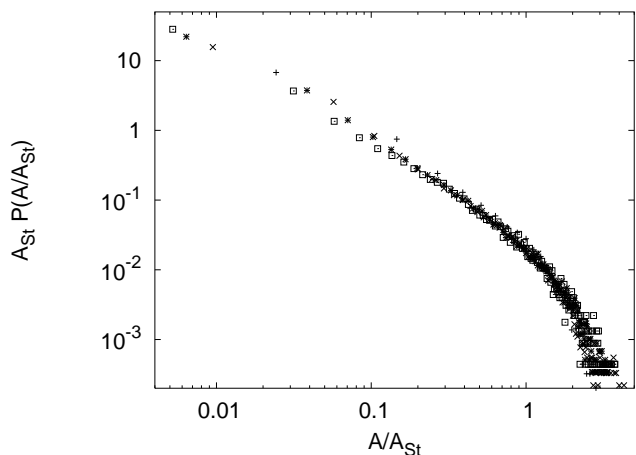


FIG. 4: Probability density functions of hole areas rescaled with the cutoff areas  $A_{St}$  computed from Fig.3. Symbols as in Fig.3.

A natural measure of the degree of clusterization of inertial particles which has been considered in the literature [4, 6, 7] is the deviation of the particle density distribution from Poisson law,  $D(R) = (\sigma_R - \sigma_R^{(P)})/\lambda_R$ , where  $\lambda_R$  and  $\sigma_R$  represent the mean and standard deviation of particle distribution coarse grained at scale  $R$  and  $\sigma_R^{(P)}$  is the corresponding standard deviation for a Poisson distribution. The maximum of the deviation  $D(R)$ , which depends on  $St$  (see Fig. 5) is then used for defining the typical scale of clusters,  $R_{St}$ . We have found that this definition of  $D(R)$  is strongly sensitive to particle statistics, nevertheless, the increase of  $R_{St}$  on  $St$  (inset of Fig. 5) seems to be a robust result.

The presence of structures in the inertial particle distribution is often attributed [4] to the fact that heavy particles are expelled from vortical regions. Although structures are related to the presence of many active scales in the turbulent flow, one should recall that in 2D turbulence, as a consequence of the direct vorticity cascade, vorticity is concentrated at the small scales [13] and no large scale coherent structures appear. Holes emerge as a result of the delayed dynamics (1), which filters the scales of the underlying turbulent flow characterized by times of the order of the Stokes time  $\tau_S$ .

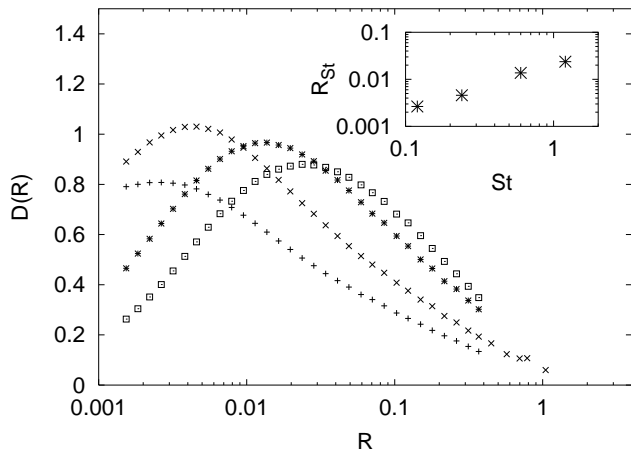


FIG. 5: Deviation of the inertial particle distribution from Poisson law as a function of the coarse grain scale for different  $St$ . The maximum of  $D(R)$  defines the clusterization scale  $R_{St}$ , whose dependence on  $St$  is shown in the inset.

As discussed before, clustering occurs also in synthetic flows where a hierarchy of time scales is absent, just as

a consequence of the dissipative character of the motion [11]. However, it appears from our simulations that to fully understand the geometry of inertial particle distribution in a turbulent flow the presence of structures characterized by a large set of time scales cannot be ignored.

We conclude that the geometry of inertial particle clusters in developed turbulence is controlled both by the dissipative effective dynamics of the particle motion at small scales, and by the tendency of inertial particles to filter the active scales characterized by times of the order of the characteristic relaxation time of the particles. A full understanding of the geometry of particle clusters in developed turbulence is particularly relevant for several applications, such as coalescence processes or chemical reactions. The reaction rate of two chemical species is a function of their concentration. When particles of different species are transported by the same turbulent flow, their local concentrations are not independent and the presence of large scale correlations in the particle distribution can in principle influence the reaction velocity.

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