Nonlinear Beltrami equation and τ-function for dispersionless hierarchies

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Abstract

It is proved that the action for nonlinear Beltrami equation (quasi-classical $\bar{\partial}$-problem) evaluated on its solution gives a τ-function for dispersionless KP hierarchy. Infinitesimal transformations of τ-function corresponding to variations of $\bar{\partial}$-data are found. Determinant equations for the function generating these transformations are derived. They represent a dispersionless analogue of singular manifold (Schwarzian) KP equations. Dispersionless 2DTL hierarchy is also considered.

1 Introduction

Dispersionless integrable hierarchies attracted a considerable interest during the last ten years (see e.g. [1]-[13]). Recently it became clear that they play an important role in various problems of hydrodynamics and complex analysis [14]-[21].

Dispersionless integrable hierarchies can be described in different forms within different approaches. In the papers [22, 23, 24] it was shown that such hierarchies can be introduced starting with the nonlinear Beltrami equation (quasi-classical $\bar{\partial}$-problem)

$$S_{\bar{z}} = W(z, \bar{z}, S_z), \tag{1}$$

where $z \in \mathbb{C}$, bar means complex conjugation, $S_z = \frac{\partial S(z,\bar{z})}{\partial z}$, $S_{\bar{z}} = \frac{\partial S(z,\bar{z})}{\partial \bar{z}}$, and W (quasi-classical $\bar{\partial}$-data) is an analytic function of S_z. Applying the quasi-classical $\bar{\partial}$-dressing method based on equation (1), one can get dispersionless integrable hierarchies and the corresponding addition formulae in a very

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regular and simple way. Such an approach reveals also the connection of dispersionless hierarchies with the quasi-conformal mappings on the plane.

In the present paper we demonstrate that the quasi-classical $\bar{\partial}$-dressing method based on equation (1) leads to explicit formula for the τ-function for dispersionless hierarchies, which is connected with the Lagrangian for equation (1) and the corresponding action, evaluated on the solution of the boundary problem for this equation. We will also derive determinant form of the generating equations for the function S_z, defining infinitesimal deformations of the τ-function. In this paper we concentrate on the dispersionless Kadomtsev-Petviashvili (dKP) hierarchy, but we present also the basic formulae for the dispersionless 2DTL hierarchy.

2 τ-function as an action for nonlinear Beltrami equation

It was shown in [22, 23] that the dKP hierarchy is connected with Beltrami equation (1) with the $\bar{\partial}$-data equal to zero outside the unit disc. This problem can be formulated as a boundary problem for equation (1) in the unit disc as follows. Let the function $S_0(z)$ analytic in the unit disc D be given. The problem is to find the function $S = S_0 + \widetilde{S}$, satisfying (1), with $\widetilde{S}$ analytic outside the unit disc and decreasing at infinity (this is in fact a boundary condition on the unit circle, which can be written down using standard projection operator). We suggest that the function W is of the form

$$W(z, \bar{z}, S_z) = \sum_{p=0}^{\infty} w_p(z, \bar{z})(S_z)^p, \tag{2}$$

where $w_p(z, \bar{z})$ are arbitrary smooth functions in the unit disc vanishing on the boundary.

Introducing parameterization of the function $S_0(z)$ in terms of times, $S_0(z) = \sum_{n=1}^{\infty} t_n z^n$, and using the technique of quasi-classical $\bar{\partial}$-dressing method, it is possible to demonstrate that $S(z, \mathbf{t})$ is a solution of Hamilton-Jacobi equations for dKP hierarchy, and the first coefficient of expansion of $\widetilde{S}(z, \mathbf{t})$ as $z \to \infty$ satisfies equations of dKP hierarchy (see [22, 23, 24, 27]).

Here we establish a relation between the action for the problem (1) and the τ-function for dKP hierarchy. This relation illustrates a well-known observation that a transition from dispersionfull to dispersionless hierarchies resembles a transition from quantum mechanics to classical mechanics.

It was noted in [24] that equation (1) is a Lagrangian one. It can be obtained by variation of the action (for the boundary problem in the unit disc)

$$f = -\frac{1}{2\pi \mathrm{i}} \iint_D \left(\frac{1}{2} \widetilde{S}_{\bar{z}} \widetilde{S}_z - W_{-1}(z, \bar{z}, S_z) \right) dz \wedge d\bar{z}, \tag{3}$$

where

$$W_{-1}(z, \bar{z}, S_z) = \sum_{p=0}^{\infty} w_p(z, \bar{z}) \frac{(S_z)^{p+1}}{p+1}, \qquad \partial_\eta W_{-1}(z, \bar{z}, \eta) = W(z, \bar{z}, \eta).$$

One should consider independent variations of $\widetilde{S}$, possessing required analytic properties (analytic outside the unit circle, decreasing at infinity), keeping S_0 fixed.

Proposition 1 *The function*

$$F(\mathbf{t}) = -\frac{1}{2\pi i} \iint_D \left(\frac{1}{2} \widetilde{S}_{\bar{z}}(\mathbf{t}) \widetilde{S}_z(\mathbf{t}) - W_{-1}(z, \bar{z}, S_z(\mathbf{t})) \right) dz \wedge d\bar{z}, \tag{4}$$

i.e., the action (3) evaluated on the solution of the problem (1), is a τ-function of dKP hierarchy.

Proof In order to prove that $F(\mathbf{t})$ is a τ-function of dKP hierarchy, it is sufficient to demonstrate that (see, e.g., [27])

$$\widetilde{S}(z, \mathbf{t}) = -D(z) F(\mathbf{t}),$$

where $D(z)$ is the quasiclassical vertex operator, $D(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \frac{\partial}{\partial t_n}$, $|z| > 1$. Applying the operator $D(z)$ to the r.h.s. of formula (4), one gets (we change the variable of integration to y)

$$D(z) W_{-1}(y, \bar{y}, S_y(\mathbf{t})) = \widetilde{S}_{\bar{y}} D(z) (\widetilde{S}_y + S_{0y}).$$

Using the formula

$$D(z) S_{0y} = \frac{1}{z - y},$$

we get

$$\frac{1}{2\pi \mathrm{i}} \iint_D \widetilde{S}_{\bar{y}} D(z) S_{0y} dy \wedge d\bar{y} = -\frac{1}{2\pi \mathrm{i}} \oint \frac{1}{z - y} \widetilde{S}(y) dy = -\widetilde{S}(z).$$

Thus we have obtained the crucial term for our proof, and now we should demonstrate that the combination of all other terms, namely,

$$-\frac{1}{4\pi \mathrm{i}}\iint_D (\widetilde{S}_y D(z)\widetilde{S}_{\bar{y}} - \widetilde{S}_{\bar{y}} D(z)\widetilde{S}_y) dy \wedge d\bar{y}$$

is equal to zero. Indeed, using Green's formula and taking into account that $\widetilde{S}_{\bar{y}}$ is equal to zero on the unit circle, and the function $\widetilde{S}(y)$ is analytic outside the unit circle and decreases at infinity, we immediately come to the conclusion that this combination is equal to zero. QED

Remark 1. Note that integral formulae for the τ-function of different type has been derived also in [8, 20].

Remark 2. Let us consider also the integral

$$F_s(\mathbf{t}) = -\frac{1}{2\pi \mathrm{i}}\iint_D \left(\frac{1}{2}S_{\bar{z}}(\mathbf{t})S_z(\mathbf{t}) - W_{-1}(z,\bar{z},S_z(\mathbf{t}))\right) dz \wedge d\bar{z}. \tag{5}$$

Since $\bar{\partial}S_0 = 0$ for $z \in D$, using Green's formula, one obtains

$$F_s = F + \frac{1}{4\pi \mathrm{i}}\oint dz \widetilde{S}(z) S_{0z}.$$

Taking into account the relation $\widetilde{S}(z) = -D(z)F$, one gets

$$F_s = F - \frac{1}{2}\sum_{n=1}^{\infty} t_n \frac{\partial F}{\partial t_n}. \tag{6}$$

The dKP hierarchy and addition formula for F admits scale invariance

$$F(\mathbf{t}) \to F'(\lambda \mathbf{t}) = \lambda^2 F(\mathbf{t}).$$

A full infinitesimal variation of F under this transformation is

$$\delta_s F = \delta_{\mathrm{form}} F + \delta_{\mathrm{t}} F = -2\epsilon F,$$

where $\delta_{\mathrm{form}} F$ denotes a variation of the form of F, while $\delta_{\mathrm{t}} F$ stands for the variation due to the infinitesimal variation of times $\mathbf{t}$. In virtue of (6), one has $\delta_{\mathrm{form}} F = 2\epsilon F_s$. In the particular case $W(z,\bar{z},S_z) = \mu(z,\bar{z})S_z$ we have $F_s = 0$ and thus F is a homogeneous function of times of the second order. Such a class of τ-functions has been considered within different approaches in [14, 7]. Starting with linear Beltrami equation, we obtain only a subclass of τ-functions, which are quadraic with respect to times.

It is well known that dKP τ-function F obeys the dispersionless addition formula [7, 11]

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} = 0, \quad (7)$$
$$z_1, z_2, z_3 \in \mathbb{C} \setminus D.$$

The formula (4) gives a solution to this equation in terms of solution of nonlinear Beltrami equation (1).

3 Variations of the τ-function

The function W is the $\bar\partial$ data for the dKP hierarchy. Its variations provide us with a wide class of variations of the function F. For the functions W of the form (2), varying $w_n(z, \bar z)$, one has

$$\delta W = \sum_{n=1}^{\infty} \delta w_n(z, \bar z)\, (S_z)^n\,, \qquad \delta W_{-1} = \sum_{n=1}^{\infty} \frac{\delta w_n}{n+1}\, (S_z)^{n+1}\,,$$

and

$$\delta F = \frac{1}{2\pi \mathrm{i}} \iint_D (\delta W_{-1})(z, \bar z, S_z) dz \wedge d\bar z. \tag{8}$$

Considering elementary variation $\delta w_{n_0} = \epsilon \alpha_{n_0} \delta(z - z_0)$, $\delta w_n = 0, n \neq n_0$, one gets

$$\delta F = \frac{\epsilon}{2\pi \mathrm{i}} \frac{\alpha_{n_0}}{(n_0 + 1)} (S_z)^{n_0+1} |_{z=z_0}, \tag{9}$$

and, respectively,

$$\delta \widetilde{S} = -\frac{\epsilon}{2\pi \mathrm{i}} \frac{\alpha_{n_0}}{(n_0 + 1)} D(z) \left(S_z(z_0)\right)^{n_0+1}. \tag{10}$$

Taking superposition of elementary variations (9), we obtain a general variation of the form

$$\delta F = \epsilon f(S_z(z_0)), \tag{11}$$

where f is an arbitrary analytic function (summation over different points and integration over z_0 are also possible).

The formulae (10), (9) can be also derived considering the deformations of nonlinear Beltrami equation (1).

Remark 3. Since a variation of the $\bar{\partial}$-data W transforms solution of the dKP hierarchy into another solution, then the formula (11) defines an infinitesimal symmetry transformation for the function F satisfying equation (7). It is possible to prove this statement directly starting with the formula

$$p(z_0) - p(z) + z\exp(-D(z)S(z_0)) = 0, \quad z_0 \in \mathbb{C}, \quad z \in \mathbb{C} \setminus D. \tag{12}$$

Derivation of this formula can be found in, e.g., [27].

3.1 Determinant form of equation for ϕ

Existence of symmetry transformation of the form (11) leads us directly to equation for the function $\phi = S_z(z_0)$. Indeed, let us consider a special symmetry transformation (11) of the form

$$F' = F + \epsilon \exp(\Theta\phi),$$

where Θ is an arbitrary parameter, and substitute it to (7). Then we get a system of linear equations

$$\begin{cases} x + y + z = 0, \\ (D_2D_3\phi)x + (D_1D_3\phi)y + (D_1D_2\phi)z = 0, \\ (D_2\phi)(D_3\phi)x + (D_1\phi)(D_3\phi)y + (D_1\phi)(D_2\phi)z = 0, \end{cases}$$

where we use notations $D_i = D(z_i)$ and

$$x = (z_2 - z_3)e^{D_2D_3F}, \quad y = (z_3 - z_1)e^{D_3D_1F}, \quad z = (z_1 - z_2)e^{D_1D_2F}.$$

The condition that determinant of this system is equal to zero gives the equation for the function ϕ, ($\phi_i = D_i\phi$)

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \phi_2\phi_3 & \phi_1\phi_3 & \phi_1\phi_2 \\ \phi_{23} & \phi_{13} & \phi_{12} \end{pmatrix} = 0. \tag{13}$$

Expanding the l.h.s. of this equation into powers of parameters z_1^{-1}, z_2^{-1}, z_3^{-1}, in the order $z_1^{-1}z_2^{-2}z_3^{-3}$ one gets the equation

$$\partial_x \left(\frac{\phi_t}{\phi_x} - \frac{3}{8}\left(\frac{\phi_y}{\phi_x}\right)^2 \right) = \frac{3}{4}\partial_y \left(\frac{\phi_y}{\phi_x}\right), \tag{14}$$

which is the KP singular manifold equation in dispersionless limit.

It is interesting to note that equation (13) written in the form

$$D_1 \log\left(\frac{D_2\phi}{D_3\phi}\right) + D_2 \log\left(\frac{D_3\phi}{D_1\phi}\right) + D_3 \log\left(\frac{D_2\phi}{D_3\phi}\right) = 0 \tag{15}$$

has been derived in [28] as a naive continuous limit of the discrete Möbius-invariant KP equation [29] (see equation (20)) in connection with the Menelaus theorem. It is easy to check that equation (13) (or, equivalently, (15)) is invariant under conformal transformation of dependent variable $\phi \to f(\phi)$, where f is an analytic function.

The determinant form (13) gives some hint for the geometric interpretation of this equation. Indeed, as it is known (see, e.g., [30]), the area of the plane triangle with the coordinates of vertices given by the pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), can be written as

$$A = \frac{1}{2}\left|\det\mathbf{A}\right|, \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Thus equation (13) means that area (may be complex) of corresponding triangle vanish, that is, obviously, the only possibility for the geometry admitting arbitrary transformation $\phi \to f(\phi)$.

More general generating equation for the gauge-invariant function $\mathcal{S} = S(z_1) - S(z_0)$

$$\sum \epsilon_{ijk} D_j \log(\exp(D_i\mathcal{S}) - 1) = 0 \tag{16}$$

derived in [24] (from which equation (15) can be easily obtained by the limit $z_1 \to z_0$) can be also written in the determinant form,

$$\det\begin{pmatrix} 1 & 1 & 1 \\ (e^{\mathcal{S}_2} - 1)(e^{\mathcal{S}_3} - 1) & (e^{\mathcal{S}_3} - 1)(e^{\mathcal{S}_1} - 1) & (e^{\mathcal{S}_1} - 1)(e^{\mathcal{S}_2} - 1) \\ \mathcal{S}_{23} & \mathcal{S}_{13} & \mathcal{S}_{12} \end{pmatrix} = 0,$$

where the subscript i refers to the derivative D_i. We believe it is possible to obtain this equation in a manner similar to the derivation of (13).

3.2 Determinant form of discrete SKP equation

The basic idea of derivation of the determinant form (13) is applicable to the dispersionfull case too (in fact, originally we used it in the dispersionfull case first). This idea gives another way to obtain discrete Swartzian KP equation, which was introduced in [29]. Instead of equation (7), we start from the well-known addition formula for the KP τ-function

$$c_1(T_1\tau)(T_2T_3\tau) + c_2(T_2\tau)(T_1T_3\tau) + c_3(T_3\tau)(T_1T_2\tau) = 0, \tag{17}$$

where T_i denotes a Sato shift, c_i are certain coefficients.

Proposition 2 *If for some function Φ the function*

$$\widetilde{\tau} = \tau(1 + \Theta\Phi), \tag{18}$$

satisfies equation (17) for arbitrary Θ (i.e., formula (18) defines a Bäcklund transformation for the τ-function), then the function Φ satisfies the equation

$$\det \begin{pmatrix} 1 & 1 & 1 \\ (T_1\Phi + T_2T_3\Phi) & (T_2\Phi + T_1T_3\Phi) & (T_3\Phi + T_1T_2\Phi) \\ (T_1\Phi)(T_2T_3\Phi) & (T_2\Phi)(T_1T_3\Phi) & (T_3\Phi)(T_1T_2\Phi) \end{pmatrix} = 0. \tag{19}$$

Proof Substituting (18) into (17), we obtain a system of linear equations

$$\begin{cases} x + y + z = 0, \\ (T_1\Phi + T_2T_3\Phi)x + (T_2\Phi + T_1T_3\Phi)y + (T_3\Phi + T_1T_2\Phi)z = 0, \\ (T_1\Phi)(T_2T_3\Phi)x + (T_2\Phi)(T_1T_3\Phi)y + (T_3\Phi)(T_1T_2\Phi)z = 0, \end{cases}$$

where

$$x = c_1(T_1\tau)(T_2T_3\tau), \quad y = c_2(T_2\tau)(T_1T_3\tau), \quad z = c_3(T_3\tau)(T_1T_2\tau).$$

Then, from the requirement that determinant of this linear system should be equal to zero, we get equation (19), QED.

It is easy to check that equation (19) coincides with discrete SKP equation, introduced in [29] in multiplicative form

$$(T_2\Delta_1\Phi)(T_3\Delta_2\Phi)(T_1\Delta_3\Phi) = (T_2\Delta_3\Phi)(T_3\Delta_1\Phi)(T_1\Delta_2\Phi), \tag{20}$$

where $\Delta_i = T_i - 1$ is a difference operator.

4 Dispersionless 2DTL hierarchy

Most of the results presented above for dispesionless KP hierarchy are valid after some modification for dispersionless 2DTL hierarchy. First we will outline the basic notations, following the work [27], and then we will discuss how the main formulae are modified.

For the dispersionless 2DTL hierarchy the $\bar{\partial}$-data are localized on the domain G which is an annulus $a < |z| < b$, where a, b ($a, b \in \mathbb{R}$, $a, b > 0$; $b > a$) are arbitrary (instead of the unit disc in KP case). To set the quasi-classical $\bar{\partial}$-problem (1) correctly, in general we do not need to require analyticity of the function S_0 in G, it is enough to have analyticity of its derivative S_{0z}. A generic function S_0 with S_{0z} analytic in G can be represented as

$$S_0(t, \mathbf{x}, \mathbf{y}) = t \log z + \sum_{n=1}^{\infty} z^n x_n + \sum_{n=1}^{\infty} z^{-n} y_n,$$

where t, x_n, y_n are free parameters [24]. We assume that $\widetilde{S}(z) \sim \sum_{n=1}^{\infty} \frac{S_n}{z^n}$ as $z \to \infty$ and denote $\widetilde{S}(0) = \phi$, $G_+ = \{z, |z| > b\}$, $G_- = \{z, |z| < a\}$. Relations, characterizing the τ-function F, are $\phi = DF$, $\widetilde{S}(z_1) = -D_+(z_1)F$ ($z_1 \in G_+$), $\widetilde{S}(z_2) = \phi - D_-(z_2)F$ ($z_2 \in G_-$), where $D_+(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \frac{\partial}{\partial x_n}$, $D_-(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n \frac{\partial}{\partial y_n}$, $D = \frac{\partial}{\partial t}$.

4.1 τ-function for dispersionless 2DTL hierarchy

Boundary problem for nonlinear Beltrami equation (1) in this case is formulated on the boundary of the annulus G, and integration in the formula (4) goes over the annulus.

Proposition 3 *The function*

$$F(t, \mathbf{x}, \mathbf{y}) = \frac{-1}{2\pi i} \iint_G \left(\frac{1}{2} \widetilde{S}_{\bar{z}}(t, \mathbf{x}, \mathbf{y}) \widetilde{S}_z(t, \mathbf{x}, \mathbf{y}) - W_{-1}(z, \bar{z}, S_z(t, \mathbf{x}, \mathbf{y})) \right) dz \wedge d\bar{z}$$

is a τ-function of dispersionless 2DTL hierarchy.

The proof is completely analogous to the KP case.

The formula for the τ-function may be also considered in a more general context, for arbitrary domain G. In this case it defines F as a functional on the space of functions $S_0(z)$, having the derivative S_{0z} analytic in G. The form of corresponding hierarchy depends on parameterization of this space in terms of 'times'.

4.2 Symmetries and singular manifold equations

Variations of the function F, preserving the hierarchy, have the same form (11) as in KP case.

Dispersionless 2DTL equations can be obtained from KP case (equations (16), (15), (13)) by the transformations $D_1 \to D_+$, $D_2 \to D_- + \partial_t$, $D_3 \to \partial_t$

or, equivalently, $D_1 \to D_-$, $D_2 \to D_+ - \partial_t$, $D_3 \to -\partial_t$, connected by the the symmetry $D_+ \to D_-$, $D_- \to D_+$, $\partial_t \to -\partial_t$.

Equation (16) for $\mathcal{S}$ takes the form

$$\left(e^{D_+\mathcal{S}}(e^{-\partial_t\mathcal{S}}-1)+e^{D_-\mathcal{S}}(e^{\partial_t\mathcal{S}}-1)\right)D_+D_-\mathcal{S}$$
$$-(e^{D_+\mathcal{S}}-1)(e^{D_-\mathcal{S}}-1)\partial_t(D_- - D_+ + \partial_t)\mathcal{S} = 0. \quad (21)$$

In the order $z_+(z_-)^{-1}$ of expansion of equation (21) one gets

$$\mathcal{S}_{xy} - \mathcal{S}_x\mathcal{S}_y\frac{\mathcal{S}_{tt}}{(e^{\mathcal{S}_t}-1)(1-e^{-\mathcal{S}_t})} = 0, \quad (22)$$

that is the dispersionless 2DTL singular manifold equation.

Generating equation for conformally invariant case reads

$$(\partial_t\phi)(D_+D_-\phi)(\partial_t + D_- - D_+)\phi = (D_+\phi)(D_-\phi)\partial_t(\partial_t + D_- - D_+)\phi. \quad (23)$$

The order of expansion $z_+(z_-)^{-1}$ gives the conformally invariant dispersionless 2DTL equation

$$\frac{\phi_{xy}}{\phi_x\phi_y} = \frac{\phi_{tt}}{\phi_t\phi_t}. \quad (24)$$

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