On bosonic limits of two recent supersymmetric extensions of the Harry Dym hierarchy

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Abstract

Two generalized Harry Dym equations, recently found by Brunelli, Das and Popowicz in the bosonic limit of new supersymmetric extensions of the Harry Dym hierarchy [J. Math. Phys. 44:4756–4767 (2003)], are transformed into previously known integrable systems: one—into a pair of decoupled KdV equations, the other one—into a pair of coupled mKdV equations from a bi-Hamiltonian hierarchy of Kupershmidt.

1 Introduction

Integrable supersymmetric differential equations have been attracting much attention in modern mathematical physics and soliton theory (see, e.g., [1] and references therein). Supersymmetric extensions of known integrable bosonic (or classical) systems are of particular interest, because, if the number N of Grassmann variables is greater than one, those extensions can generate, in their bosonic limits, some new integrable classical systems which generalize the initial ones.

Recently, Brunelli, Das and Popowicz [2] studied supersymmetric extensions of the Harry Dym hierarchy, and found, as bosonic limits of N=2

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supersymmetric extensions, the following two new classical generalizations of the Harry Dym equation:

$$\begin{split} w_{0,t} &= \frac{1}{2} \left(w_0^{-1/2} \right)_{xxx}, \\ w_{1,t} &= \frac{1}{64} \left(-16 w_{1,xxx} w_0^{-3/2} + 96 w_{1,xx} w_{0,x} w_0^{-5/2} \right. \\ &+ 72 w_{1,x} w_{0,xx} w_0^{-5/2} - 258 w_{1,x} w_{0,x}^2 w_0^{-7/2} - 6 w_{1,x} w_1^2 w_0^{-7/2} \right. \\ &+ 9 w_1^3 w_{0,x} w_0^{-9/2} - 108 w_1 w_{0,xx} w_{0,x} w_0^{-7/2} + 219 w_1 w_{0,x}^3 w_0^{-9/2} \right), \end{split}$$

and

$$w_{0,t} = \frac{1}{16} \left(8 \left(w_0^{-1/2} \right)_{xxx} - 6 w_{1,x} w_1 w_0^{-5/2} + 9 w_1^2 w_{0,x} w_0^{-7/2} \right),$$

$$w_{1,t} = \frac{1}{32} \left(-8 w_{1,xxx} w_0^{-3/2} + 48 \left(w_{1,x} w_{0,x} \right)_x w_0^{-5/2} \right)$$

$$- 144 w_{1,x} w_{0,x}^2 w_0^{-7/2} - 6 w_{1,x} w_1^2 w_0^{-7/2}$$

$$+ 9 w_1^3 w_{0,x} w_0^{-9/2} + 12 w_1 w_{0,xxx} w_0^{-5/2}$$

$$- 126 w_1 w_{0,xx} w_{0,x} w_0^{-7/2} + 177 w_{0,x}^3 w_1 w_0^{-9/2} \right),$$
(2)

where w_0 and w_1 are functions of x and t. Note that in the system (1), in the seventh term of the right-hand side of its second equation, we have corrected a misprint made in [2]: the degree of w_0 should be -7/2 there.

In the present paper, we find chains of transformations which relate these new generalized Harry Dym (GHD) equations (1) and (2) with previously known integrable classical systems. In Section 2, the GHD equation (1) is transformed into a pair of decoupled KdV equations. In Section 3, the GHD equation (2) is transformed into a pair of coupled mKdV equations which belongs to the bi-Hamiltonian hierarchy of the modified dispersive water waves equation of Kupershmidt [3] (see also [4], p. 84). Section 4 contains concluding remarks.

2 Transforming the first GHD equation

There are no general methods of transforming a given nonlinear system into another one, less complicated or better studied. The usual way of finding necessary transformations is based on experience, guess and good luck. For this reason, we give no comments on how these transformations were found in the present case.

First, the transformation

$$w_0 = u(x,t)^{-2}, w_1 = v(x,t), t \mapsto -4t$$
 (3)

brings the GHD equation (1) into the following simpler form:

$$u_{t} = u^{3}u_{xxx},$$

$$v_{t} = u^{3}v_{xxx} + 9u^{2}v_{x}u_{xx} + 12u^{2}u_{x}v_{xx} + 27uvu_{x}u_{xx}$$

$$+ \frac{57}{2}vu_{x}^{3} + \frac{75}{2}uu_{x}^{2}v_{x} + \frac{9}{8}u^{6}v^{3}u_{x} + \frac{3}{8}u^{7}v^{2}v_{x}.$$

$$(4)$$

Second, we try to transform x, u and v in (4) as follows:

$$x = p(y, t),$$
 $u(x, t) = p_y(y, t),$ $v(x, t) = q(y, t).$ (5)

This is an extension of the transformation used by Ibragimov [5] to relate the original Harry Dym equation with the Schwarzian-modified KdV equation. In the case of scalar evolution equations, the Ibragimov transformation (i.e. (5) with v = q = 0) is an essential link in chains of transformations between constant separant equations and non-constant separant ones [6, 7]. The transformation (5) really works and relates the system (4) with the system

$$p_{t} = p_{yyy} - \frac{3}{2}p_{y}^{-1}p_{yy}^{2},$$

$$q_{t} = q_{yyy} + 9p_{y}^{-1}p_{yyy}q_{y} + 27p_{y}^{-2}p_{yy}p_{yyy}q + 18p_{y}^{-2}p_{yy}^{2}q_{y}$$

$$+ 9p_{y}^{-1}p_{yy}q_{yy} + \frac{3}{2}p_{y}^{-3}p_{yy}^{3}q + \frac{3}{8}p_{y}^{6}q^{2}q_{y} + \frac{9}{8}p_{y}^{5}p_{yy}q^{3}.$$
(6)

To verify this, one may use the following identities:

$$u\partial_x = \partial_y, \qquad u_t = p_{yt} - p_y^{-1} p_{yy} p_t, \qquad v_t = q_t - p_y^{-1} q_y p_t.$$
 (7)

Note that (5) is not an invertible transformation: it maps the system (6) into the system (4), whereas its application in the opposite direction, from (4) to (6), requires one integration by y. We have omitted the terms $\alpha(t)p_y$ and $\alpha(t)q_y$ in the right-hand sides of the first and second equations of (6), respectively, where this arbitrary function $\alpha(t)$ appeared as a 'constant' of that integration.

Third, we make the transformation

$$f(y,t) = p_y^{-1} p_{yy}, \qquad g(y,t) = p_y^3 q,$$
 (8)

admitted by the system (6) owing to the form of its equations, and obtain the pair of decoupled mKdV equations

$$f_t = (f_{yy} - \frac{1}{2}f^3)_y, \qquad g_t = (g_{yy} + \frac{1}{8}g^3)_y.$$
 (9)

Needless to say that the pair of Miura transformations

$$a(y,t) = \pm f_y - \frac{1}{2}f^2, \qquad b(y,t) = \pm \frac{1}{2}ig_y + \frac{1}{8}g^2,$$
 (10)

with independent choice of the \pm signs, relates (9) with the two copies of the KdV equation

$$a_t = a_{yyy} + 3aa_y, b_t = b_{yyy} + 3bb_y.$$
 (11)

3 Transforming the second GHD equation

We follow the same three-step transformation as used in Section 2. First, the transformation (3) brings the GHD equation (2) into the form

$$u_{t} = u^{3}u_{xxx} - \frac{9}{4}u^{7}v^{2}u_{x} - \frac{3}{4}u^{8}vv_{x},$$

$$v_{t} = 3u^{2}vu_{xxx} + u^{3}v_{xxx} + 36uvu_{x}u_{xx} + 12u^{2}v_{x}u_{xx}$$

$$+ 12u^{2}u_{x}v_{xx} + 24vu_{x}^{3} + 36uu_{x}^{2}v_{x} + \frac{9}{4}u^{6}v^{3}u_{x} + \frac{3}{4}u^{7}v^{2}v_{x}.$$
(12)

Second, we apply the transformation (5) to the system (12) and obtain

$$p_{t} = p_{yyy} - \frac{3}{2}p_{y}^{-1}p_{yy}^{2} - \frac{3}{8}p_{y}^{7}q^{2},$$

$$q_{t} = 3p_{y}^{-1}p_{yyyy}q + 24p_{y}^{-2}p_{yy}p_{yyy}q + 12p_{y}^{-1}p_{yyy}q_{y} + q_{yyy}$$

$$-3p_{y}^{-3}p_{yy}^{3}q + \frac{27}{2}p_{y}^{-2}p_{yy}^{2}q_{y} + 9p_{y}^{-1}p_{yy}q_{yy} + \frac{9}{4}p_{y}^{5}p_{yy}q^{3} + \frac{3}{8}p_{y}^{6}q^{2}q_{y},$$

$$(13)$$

where the terms $\alpha(t)p_y$ and $\alpha(t)q_y$, with arbitrary $\alpha(t)$, have been omitted in the right-hand sides of the first and second equations, respectively.

Third, the transformation (8) relates the system (13) with the following system of coupled mKdV equations:

$$f_t = \left(f_{yy} - \frac{3}{4} g g_y - \frac{1}{2} f^3 - \frac{3}{8} f g^2 \right)_y,$$

$$g_t = \left(g_{yy} + 3g f_y - \frac{3}{2} f^2 g - \frac{5}{8} g^3 \right)_y.$$
(14)

The system (14) does not admit any further transformation into a system of coupled KdV equations. It is possible to transform (14) into a system of

a KdV-mKdV type, but we will not follow this way. Instead, we notice that the system (14) is invariant under the change of variables $f \mapsto f$, $g \mapsto -g$. Therefore the transformation

$$f = c_1(a+b), g = c_2(a-b),$$
 (15)

with any nonzero constants c_1 and c_2 , relates the system (14) with a system of symmetrically coupled mKdV equations for a(y,t) and b(y,t), which is invariant under $a \mapsto b$, $b \mapsto a$. Systems of symmetrically coupled mKdV equations possessing higher-order generalized symmetries were classified by Foursov [8]. The choice of

$$c_1 = 1, \qquad c_2 = \pm i \tag{16}$$

in the transformation (15) brings the system (14) into the form

$$a_{t} = (a_{yy} + 3aa_{y} - 3ba_{y} + a^{3} - 6a^{2}b + 3ab^{2})_{y},$$

$$b_{t} = (b_{yy} + 3bb_{y} - 3ab_{y} + b^{3} - 6b^{2}a + 3ba^{2})_{y},$$
(17)

which is exactly the case (K) in the Foursov classification [8].

Foursov [8] proved that the system (17) represents the third-order generalized symmetry of the system of coupled Burgers equations

$$a_t = (a_y + a^2 - 2ab)_y, b_t = (-b_y + 2ab - b^2)_y,$$
 (18)

and found the bi-Hamiltonian structure of this hierarchy with the Hamiltonian operators

$$P = \begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} -2a\partial_y - a_y & \partial_y^2 + (a-b)\partial_y + a_y \\ -\partial_y^2 + (a-b)\partial_y - b_y & 2b\partial_y + b_y \end{pmatrix}.$$
(19)

In its turn, the system of coupled Burgers equations (18) has a long history. As a system of coupled second-order evolution equations possessing higher-order symmetries, it appeared in the classifications of Mikhailov, Shabat and Yamilov [9] and Olver and Sokolov [10]. Moreover, the bi-Hamiltonian structure (19) turns out to be not new. Indeed, the transformation

$$a = -r, b = s - r, t \mapsto -\frac{1}{2}t (20)$$

relates the system (18) with the modified dispersive water waves equation

$$r_{t} = \frac{1}{2} \left(-r_{y} + 2rs - r^{2} \right)_{y},$$

$$s_{t} = \frac{1}{2} \left(s_{y} - 2r_{y} - 2r^{2} + 2rs + s^{2} \right)_{y}$$
(21)

which was introduced, together with its bi-Hamiltonian structure, by Kupershmidt [3] (see also [4], p. 84). The bi-Hamiltonian structures of (18) and (21) are related by the transformation (20) as well. For this reason, the system (17) is equivalent to a third-order member of the bi-Hamiltonian hierarchy of the modified dispersive water waves equation (21).

4 Conclusion

In this paper, we found chains of transformations which relate the new GHD equations (1) and (2) of Brunelli, Das and Popowicz with previously known integrable systems. The transformations (3), (5), (8) and (10) relate the GHD equation (1) with the pair of decoupled KdV equations (11). The transformations (3), (5), (8), (15) with the choice of (16), and (20) relate the GHD equation (2) with a third-order member of the bi-Hamiltonian hierarchy of the modified dispersive water waves equation (21).

It can be observed in the literature (see, e.g., [5, 6, 7, 9] and references therein) that quite often a newly-found remarkable equation turns out to be related to a well-studied old equation through an explicit chain of transformations. In such a situation, one gets a possibility not to study the new equation directly but to derive its properties from the well-known properties of the corresponding old equation, using the transformations obtained. Now this applies to the new generalized Harry Dym equations of Brunelli, Das and Popowicz as well.

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