Exact solutions for one of the extensive chaos model

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Abstract

Sampling equation method is presented to look for exact solutions of nonlinear differential equations. Application of this approach to one of the extensive chaos model is considered. Exact solutions of this model in travelling wave are given. Nonlinear evolution equation for the considered extensive chaos model is shown to have solitary and periodical waves.

Keywords: exact solution, extensive chaos model, nonlinear evolution equation

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1 Introduction

In recent years one can observe a systematical study of a novel type of chaos that is called by "soft-mode turbulence" [\[1,](#page-8-0)[2,](#page-8-1)[3\]](#page-8-2). These chaos types are characterized by a smooth interplay of different spatial scales. Properties of these types are qualitatively different from the well known models that are described by the complex Ginzburg–Landau and the Kuramoto–Sivashinsky equations.

The simplest model exhibiting the soft-mode turbulence can be described by the higher order nonlinear evolution equation with the simplest nonlinearity.

This equation was introduced by N.A. Kudryashov [\[4\]](#page-8-3) and V.N. Nikolaevskiy [\[5\]](#page-8-4) to describe longitudinal seismic waves in viscoelastic media. The simplest case of this equation takes the form

$$
u_t + uu_x + \beta u_{xx} + \delta u_{xxxx} + \varepsilon u_{xxxx} = 0 \tag{1.1}
$$

It is known that the Ginzburg–Landau and the Kuramoto–Sivashinskiy equations are not integrable equations because these ones do not pass the Painleve test $[6, 7]$ $[6, 7]$. However these equations have some list of special solutions $[6, 7, 8, 9, 9]$ $[6, 7, 8, 9, 9]$ $[6, 7, 8, 9, 9]$ $[6, 7, 8, 9, 9]$ $[6, 7, 8, 9, 9]$ [10,](#page-8-9) [11,](#page-8-10) [12\]](#page-8-11).

Eq. [\(1.1\)](#page-0-0) can be normalized. Assuming $\varepsilon \neq 0, \delta \neq 0$ and setting

$$
u = \frac{\delta^2}{\varepsilon} \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}} u', \qquad x = \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{2}} x', \qquad t = \frac{\varepsilon^2}{\delta^3} t', \qquad \sigma = \frac{\beta \varepsilon}{\delta^2}
$$
(1.2)

Then $Eq.(1.1)$ $Eq.(1.1)$ takes the form

$$
u_t + uu_x + \sigma u_{xx} + u_{xxxxx} = 0 \tag{1.3}
$$

(the primes of the variables are omitted).

Equation [\(1.3\)](#page-1-0) is invariant under transformations

$$
u \to -u, \qquad x \to -x \tag{1.4}
$$

which allows us to study this equation for $x \geq 0$.

 $Eq.(1.1)$ $Eq.(1.1)$ does not pass the Painlevé test and this is not integrable equation but one can expect that Eq.[\(1.1\)](#page-0-0) has some special solutions.

The aim of this letter is to present some exact solutions of Eq. (1.1) . The outline of this letter is as follows. The sampling equation method to look for exact solutions of nonlinear differential equations is discussed in Section 2. Application of this approach to search exact solitary solutions of $Eq.(1.1)$ $Eq.(1.1)$ is considered in Section 3. Exact periodic solutions are presented in Section 4.

2 Sampling equation method

It is well known that all nonlinear differential equations can be connectionally divided into three types: exactly solvable, partially solvable and those that have no exact solution. At the present we have a lot of different approaches to look for exact solutions of nonlinear differential equations (see, for a example, refs. [\[6,](#page-8-5)[7,](#page-8-6)[8,](#page-8-7)[9,](#page-8-8)[10,](#page-8-9)[11,](#page-8-10)[12,](#page-8-11)[13,](#page-8-12)[14,](#page-8-13)[15,](#page-8-14)[16,](#page-8-15)[17,](#page-8-16)[18,](#page-8-17)[19,](#page-8-18)[20\]](#page-8-19)). Usually investigators use some sampling functions that are hyperbolic and elliptic functions. However one can note that as a rule partially solvable nonlinear equations have exact solutions that are general solutions of solvable equation of lesser order. In this connection we apply later the sampling equation method to look for exact solutions of Eq.[\(1.1\)](#page-0-0). Our approach takes into consideration the following simple idea.

Let us assume we have two differential equations

$$
E[y] = 0 \tag{2.1}
$$

and

$$
D[u] = 0 \tag{2.2}
$$

and let us also assume that $Eq.(2.1)$ $Eq.(2.1)$ is not integrable equation but Eq. (2.2) is solvable equation of lesser order then $Eq.(2.1)$ $Eq.(2.1)$. If we find the transformation for solution y of Eq.[\(2.1\)](#page-1-1) that allows us to connect y with the general solution of Eq. (2.2) we have the following relation between Eq. (2.1) and (2.2)

$$
E[y] = \hat{A}D[u] \tag{2.3}
$$

where \hat{A} is a operator and y is a transformation that is determined by the formula

$$
y = F(u) \tag{2.4}
$$

This raises the question as to whether finding transformation [\(2.4\)](#page-2-0) and exactly solvable equation [\(2.2\)](#page-1-2) as the sampling equation.

One of the impressive method to look for the transformation like [\(2.4\)](#page-2-0) is the singular manifold method by J.Weiss, M.Tabor and G.Carnevalle [\[21\]](#page-8-20) that is used to study both integrable and nonintegrable differential equations. Success of this approach for nonintegrable differential equations is explained by so-called truncated expansions that are transformations similar to formula [\(2.4\)](#page-2-0). In this case for the polynomial class of nonintegrable equations [\(2.1\)](#page-1-1) one can suggest corresponding exactly solvable equation [\(2.2\)](#page-1-2) as the Riccati equation, the elliptic equation or other solvable ordinary differential equation [\[6,](#page-8-5) [7,](#page-8-6) [10,](#page-8-9) [22\]](#page-8-21).

As a example let us consider the ordinary differential equation in the form

$$
E[y] = y_{xxxx} + yy_{xxx} - 6yy_{xx} - 6y_x^2 - 6y^2y_x - \beta y = 0
$$
\n(2.5)

This equation is not integrable equation but this one has some exact solutions.

Taking into consideration leading members of Eq.[\(2.5\)](#page-2-1) one can find that solution of $Eq. (2.5)$ $Eq. (2.5)$ have the second degree singularity. In this connection we can find solution of Eq.[\(2.5\)](#page-2-1) using the truncated expansion

$$
y(z) = A_0 + A_1 Y + A_2 Y^2
$$
\n(2.6)

where A_0 , A_1 and A_2 are unknown parameters and $Y(z)$ satisfies to the Riccati equation

$$
D[Y] = Y_z + Y^2 - \alpha = 0 \tag{2.7}
$$

Here α is a parameter that will be found too. Substituting transformation (2.6) info Eq. (2.5) and taking into account Eq. (2.7) and its consequences

$$
Y_{zz} = 2Y^3 - 2\alpha Y \tag{2.8}
$$

$$
Y_{zzz} = -6Y^4 + 8\alpha Y^2 - 2\alpha^2\tag{2.9}
$$

$$
Y_{zzzz} = 24Y^5 - 40\alpha Y^3 + 16\alpha^2 Y \tag{2.10}
$$

we have solution of Eq.[\(2.5\)](#page-2-1) at $\beta = 0$ that is expressed by Eq.[\(2.6\)](#page-2-2) where $Y(z)$ is the solution of $Eq.(2.7)$ $Eq.(2.7)$

One can use another transformation

$$
y(z) = B_0 + B_1 R \tag{2.11}
$$

Where B_0 and B_1 are constant that are found. As this takes place we take into consideration that R has second degree singularity and $R = R(z)$ is a solution of the elliptic function equation

$$
R_z^2 = -2R^3 + aR^2 + 2bR + d \tag{2.12}
$$

In this case we obtain the elliptic solution at $\beta = 0$ again.

However if we use formula (2.11) and take the first Painlevé equation except Eq.[\(2.12\)](#page-3-1)

$$
R_{zz} = 3R^2 + \beta x \tag{2.13}
$$

we find that Eq.[\(2.5\)](#page-2-1) has exact solution [\(2.11\)](#page-3-0) at $B_0 = 0, B_1 = 1$ and $\beta \neq 0$ where $R(z)$ is the Painlevé transcendent. We can see that we have obtained much more interesting solution of Eq. (2.5) than (2.7) and (2.11) . This solution can not be found using sampling functions.

3 Exact solitary solutions of Eq.(1.1).

Let us look for exact solutions of $Eq.(1.1)$ $Eq.(1.1)$ in the form of travelling waves using variables

$$
u(x,t) = y(z), \t z = x - C_0 t \t (3.1)
$$

Eq.[\(2.1\)](#page-1-1) takes the form after integration over z

$$
C_1 - C_0 y + \frac{1}{2} y^2 + \beta y_z + \delta y_{zzz} + \varepsilon y_{zzzzz} = 0
$$
\n(3.2)

Assuming

$$
y = a_0 z^p \tag{3.3}
$$

and substituting into leading members of Eq.[\(3.2\)](#page-3-2) we have $a_0 = 30240\varepsilon$ and $p = -5$. Solution of Eq.[\(3.2\)](#page-3-2) has fifth degree singularity and following to the sampling equation method we can look for the exact solution of Eq.[\(3.2\)](#page-3-2) in the form

$$
y(z) = A_0 + A_1 Y + A_2 Y^2 + A_3 Y^3 + A_4 Y^4 + A_5 Y^5
$$
\n(3.4)

Where $Y(z)$ satisfies to the Riccati equation

$$
D[Y] = Y_z + Y^2 - \alpha = 0 \tag{3.5}
$$

Constants A_0 , A_1 , A_2 , A_3 , A_4 , A_5 and α are found after substitution of the truncated expansion (3.4) into Eq. (2.1) . We need also to take into account the following formulas

$$
Y_{zz} = 2Y^3 - 2\alpha Y
$$

\n
$$
Y_{zzz} = -6Y^4 + 8\alpha Y^2 - 2\alpha^2
$$

\n
$$
Y_{zzzz} = 24Y^5 - 40\alpha Y^3 + 16\alpha^2 Y
$$
\n(3.6)

 $Y_{zzzzz} = -120Y^6 + 240\alpha Y^4 - 136\alpha^2 Y^2 + 16\alpha^3$

As a result of calculations we have

$$
A_5 = 30240 \,\varepsilon, \quad A_4 = 0, \quad A_3 = \frac{2520\delta}{11} - 50400 \varepsilon \alpha, \quad A_2 = 0,
$$

$$
A_1 = -\frac{2520}{11} \,\delta \,\alpha + 20160 \,\varepsilon \,\alpha^2 + \frac{1260}{251} \,\beta - \frac{12600}{30371} \frac{\delta^2}{\varepsilon}, \quad A_0 = C_0
$$
 (3.7)

Where β is determined by the formula

$$
\beta = -\frac{213811840 \varepsilon^3 \alpha^3 - 10204656 \delta \varepsilon^2 \alpha^2 - 2045 \delta^3 - 92400 \delta^2 \varepsilon \alpha}{121 \varepsilon (9240 \varepsilon \alpha + 79 \delta)} \tag{3.8}
$$

Denoting

$$
\alpha = \frac{\delta w}{\varepsilon} \tag{3.9}
$$

we obtain for w the following six values

$$
w_1 = -\frac{1}{220}, \quad w_2 = -\frac{5}{176}, \quad w_3 = -\frac{1}{440}
$$
\n
$$
(3.10)
$$

$$
w_4 = \frac{1}{52800} \left(557 - \frac{46031}{m} + m \right) \tag{3.11}
$$

$$
m = (113816753 + 1260\sqrt{8221079733})^{\frac{1}{3}} \approx 610,966
$$
\n(3.12)

$$
w_{5,6} = \frac{1}{52800} \left(\frac{46031}{2m} - \frac{m}{2} + 557 \pm \frac{i\sqrt{3}}{2} \left(m + \frac{46031}{m} \right) \right)
$$
(3.13)

Exact solutions of $Eq.(1.1)$ $Eq.(1.1)$ can be written in the form

$$
y(z) = 30240 \,\varepsilon \, Y^5 + \left(\frac{2520}{11} \,\delta - 50400 \,\varepsilon \,\alpha\right) Y^3 +
$$

+
$$
\left(-\frac{2520}{11} \,\delta \,\alpha + 20160 \,\varepsilon \,\alpha^2 + \frac{1260}{251} \,\beta - \frac{12600}{30371} \,\frac{\delta^2}{\varepsilon}\right) Y + C_0
$$
 (3.14)

where $Y = Y(z)$ is a solution of Eq.[\(3.5\)](#page-3-4)

$$
Y(z) = \sqrt{\alpha} \tanh\left(\sqrt{\alpha}z + \varphi_0\right) \tag{3.15}
$$

Constant C_1 is determined by formula

$$
C_1 = \frac{4112640}{11} \frac{\delta^5 w^4}{\varepsilon^3} - 9999360 \frac{\delta^5 w^5}{\varepsilon^3} - \frac{5080320}{251} \frac{\delta^3 w^3 \beta}{\varepsilon^2} - \frac{55460160}{30371} \frac{\delta^5 w^3}{\varepsilon^3} +
$$

+
$$
\frac{1}{2} C_0^2 + \frac{660240}{2761} \frac{\delta^3 w^2 \beta}{\varepsilon^2} - \frac{25200}{30371} \frac{\delta^5 w^2}{\varepsilon^3} - \frac{1260}{251} \frac{\beta^2 \delta w}{\varepsilon} + \frac{12600}{30371} \frac{\beta \delta^3 w}{\varepsilon^2}
$$
(3.16)

Substituting solution [\(3.15\)](#page-5-0) into [\(3.14\)](#page-5-1) and taking into account that $\alpha =$ $\alpha_i = w_i \delta / \varepsilon$ $(i = 1, ..., 6)$ we have different solutions of Eq.[\(1.1\)](#page-0-0) in the form of solitary waves.

4 Exact periodic solutions of Eq.(1.1).

We can see that solutions of $Eq(1.1)$ $Eq(1.1)$ have fivth degree singularity and one can also look for exact solution of $Eq.(1.1)$ $Eq.(1.1)$ in the form

$$
y(z) = B_1 + B_2 R(z) + B_3 R_z + B_4 R^2 + B_5 R R_z
$$
\n(4.1)

where B_k $(k = 1, ..., 5)$ are constants and $R = R(z)$ is a second degree singularity solution of the elliptic function equation

$$
R_z^2 = -2R^3 + aR^2 + 2bR + d \tag{4.2}
$$

From Eq.[\(4.2\)](#page-5-2) we get that $R(z)$ satisfies also to equations

$$
R_{zz} = -3R^2 + aR + b
$$

\n
$$
R_{zzz} = -6RR_z + aR_z
$$

\n
$$
R_{zzzz} = 30R^3 - 15aR^2 - 18bR - 6d + a^2R + ab
$$

\n
$$
R_{zzzzz} = 90R^2R_z - 30aRR_z - 18bR_z + a^2R_z
$$

\n
$$
R_{zzzzzz} = -630R^4 + 420aR^3 + 504bR^2 + 180Rd -
$$
\n(4.3)

$$
-63 a2R2 - 108 abR - 30 ab - 18 b2 + a3R + a2b
$$

Substituting (4.1) into Eq. (1.1) and taking into account formulas (4.2) and (4.3) we find

$$
B_4 = 0
$$
, $B_2 = 0$, $B_1 = C_0$, $B_5 = -3780 \varepsilon$, $B_3 = 630 \varepsilon a + \frac{630}{11} \delta$ (4.4)

$$
\beta = \frac{10}{121} \frac{\delta^2}{\varepsilon} \tag{4.5}
$$

As this takes place parameters b and d in Eq.[\(4.1\)](#page-5-3) take two values

$$
b_{1,2} = -\frac{1}{12}a^2 + \frac{1}{1452} \frac{\delta^2}{\varepsilon^2} \pm \frac{1}{5082} \frac{\sqrt{21}\delta^2}{\varepsilon^2}
$$
 (4.6)

$$
d_{1,2} = \frac{1}{108}a^3 + \frac{13}{359370} \frac{\delta^3}{\varepsilon^3} \pm \frac{1}{119790} \frac{\sqrt{21}\delta^3}{\varepsilon^3} - \frac{1}{4356} \frac{a\delta^2}{\varepsilon^2} \mp \frac{1}{15246} \frac{a\sqrt{21}\delta^2}{\varepsilon^2}
$$
(4.7)

Constant C_1 in Eq.[\(2.1\)](#page-1-1) in this case has two values too

$$
C_1^{(1,2)} = -\frac{10854}{161051} \frac{\delta^5}{\varepsilon^3} + \frac{1}{2} C_0^2 \mp \frac{2484}{161051} \frac{\sqrt{21} \delta^5}{\varepsilon^3}
$$
(4.8)

Using [\(4.4\)](#page-6-1) and [\(4.5\)](#page-6-2) we obtain as resultant expression for the solution $y(z)$ in the form of periodic waves.

$$
y(z) = C_0 + 630 \left(\varepsilon a + \frac{\delta}{11} - 6\varepsilon R\right) R_z \tag{4.9}
$$

where $R = R(z)$ is a solution of the following equations

$$
R_z^2 = -2R^3 + aR^2 - \frac{1}{6}a^2R + \frac{1}{726}\frac{R\delta^2}{\varepsilon^2} \pm \frac{1}{2541}\frac{R\sqrt{21}\delta^2}{\varepsilon^2} +
$$

$$
\frac{1}{108}a^3 + \frac{13}{359370}\frac{\delta^3}{\varepsilon^3} \pm \frac{1}{119790}\frac{\sqrt{21}\delta^3}{\varepsilon^3} - \frac{1}{4356}\frac{a\delta^2}{\varepsilon^2} \mp \frac{1}{15246}\frac{a\sqrt{21}\delta^2}{\varepsilon^2}
$$
(4.10)

Assuming that R_1, R_2 and R_3 with $R_1 \ge R_2 \ge R_3$ real roots of equations

$$
2R^3 - aR^2 + \left(\frac{1}{6}a^2 - \frac{\delta^2}{726\varepsilon^2} \mp \frac{\delta^2 \sqrt{21}}{2541\varepsilon^2}\right)R - \frac{1}{108}a^3 - \frac{3}{359379\varepsilon^3} \mp \frac{\delta^3 \sqrt{21}}{119790\varepsilon^3} + \frac{a\delta^2}{4356\varepsilon^2} \pm \frac{a\delta^2 \sqrt{21}}{15246\varepsilon^2} = 0
$$
\n(4.11)

We have solutions of Eq.[\(4.10\)](#page-7-0) in the form

$$
R(z) = R_2 + (R_1 - R_2) \text{cn}^2 (z \sqrt{R_1 - R_2}, S), \quad S^2 = \frac{R_1 - R_2}{R_1 - R_3} \tag{4.12}
$$

Thus Eq.[\(1.1\)](#page-0-0) have a few exact solutions at different values of equation parameters. These solution are solitary and periodic waves and they are determined by the formulas [\(3.14\)](#page-5-1) and [\(4.9\)](#page-6-3). We hope these solutions can be useful for test of the numerical simulations of soft-mode turbulence.

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