

Integrable systems on $so(4)$ related with XXX spin chains with boundaries.

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Abstract

We consider two-site XXX Heisenberg magnets with different boundary conditions, which are integrable systems on $so(4)$ possessing additional cubic and quartic integrals of motion. The separated variables for these models are constructed using the Sklyanin method.

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1 Introduction

In classical mechanics a lot of integrable systems on $so(4)$ and $e(3)$ algebras are related with the integrable spin chains. For instance [1], different Gaudin magnets coincide with the Euler, Lagrange, Neumann and Clebsh systems on $e(3)$ and with the Manakov, Steklov systems on $so(4)$. The separated variables for all these models may be derived from the separation of variables for the XYZ Gaudin model [2].

Some degenerate cases of the Heisenberg spin chain are connected with the Goryachev-Chaplygin top [3], auxiliary symmetric Neumann system and Kowalevski-Goryachev-Chaplygin top [4]. The separated variables for all these models may be derived from the separation of variables for the XYZ spin chain [5].

In this note we consider XXX Heisenberg magnet with boundaries [6, 7] using the Lax matrix for the standard two-site XXX Heisenberg magnet

$$T(\lambda) = \begin{pmatrix} \lambda - s_3 + i\delta_1 & s_1 + is_2 \\ s_1 - is_2 & \lambda + s_3 + i\delta_1 \end{pmatrix} \begin{pmatrix} \lambda - t_3 + i\delta_2 & t_1 + it_2 \\ t_1 - it_2 & \lambda + t_3 + i\delta_2 \end{pmatrix}. \quad (1.1)$$

Here δ_i are numerical shifts of the spectral parameter λ . Dynamical variables s_i, t_i are coordinates on $so(4) = so(3) \oplus so(3)$ with the following Lie-Poisson brackets

$$\{s_i, s_j\} = \varepsilon_{ijk} s_k, \quad \{s_i, t_j\} = 0, \quad \{t_i, t_j\} = \varepsilon_{ijk} t_k, \quad (1.2)$$

where ε_{ijk} is the totally skew-symmetric tensor.

Matrix $T(\lambda)$ (1.1) defines representation of the Sklyanin algebra

$$\{T^1(\lambda), T^2(\nu)\} = [r(\lambda - \nu), T^1(\lambda)T^2(\nu)], \quad (1.3)$$

on generic symplectic leaves of $so(4)$. Here we use the standard notations $T^1(\lambda) = T(\lambda) \otimes I$, $T^2(\nu) = I \otimes T(\nu)$ and r -matrix has the form

$$r(\lambda - \nu) = \frac{i}{\lambda - \nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.4)$$

Applying the standard machinery [6, 7] to $T(\lambda)$ (1.1) one gets another integrable systems with cubic and quartic additional integrals of motion. As a matter of fact, the construction leads automatically to the separated variables.

2 Cubic integrals of motion

The main property of the Sklyanin algebra (1.3) is that for *any* numerical matrix \mathcal{K} coefficients of the trace of matrix $\mathcal{K}T(\lambda)$ give rise the commutative subalgebra

$$\{\text{tr } \mathcal{K}T(\lambda), \text{tr } \mathcal{K}T(\nu)\} = 0.$$

All the generators of this subalgebra are linear polynomials on coefficients of entries $T_{ij}(\lambda)$, which are interpreted as integrals of motion for integrable system associated with matrix $T(\lambda)$. For instance, representation (1.1) generates one linear and one quadratic integrals of motion in variables s_i, t_i and the corresponding integrable system is equivalent to a special case of Poincaré system [8].

According to [7], we can construct commutative subalgebra generated by *quadratic* polynomials on coefficients of $T_{ij}(\lambda)$. Let us introduce the matrix

$$\tilde{T}(\lambda) = \mathcal{K}_d(\lambda) T(\lambda), \quad (2.5)$$

where $T(\lambda)$ is given by (1.1) and

$$\mathcal{K}_d(\lambda) = \begin{pmatrix} \lambda + \mathcal{A}_0 & a_1\lambda + a_0 \\ b_1\lambda + b_0 & 0 \end{pmatrix}. \quad (2.6)$$

Here a_k, b_k are arbitrary numerical parameters and \mathcal{A}_0 depends on dynamical variables

$$\mathcal{A}_0 = a_1(is_2 + it_2 - t_1 - s_1) - b_1(s_1 + t_1 + is_2 + it_2) - s_3 - t_3.$$

We can say that dynamical matrix \mathcal{K}_d (2.6) describes some *dynamical* boundary conditions.

The trace of $\tilde{T}(\lambda)$

$$\tilde{\tau}(\lambda) = \text{tr } \tilde{T}(\lambda) = \lambda^3 + I_1\lambda + I_2$$

give rise the commutative subalgebra of the Sklyanin brackets (1.3). Quadratic and cubic in variables s_i, t_i polynomials $I_{1,2}$ are integrals of motion for some integrable system on $so(4)$.

To compare this system with the known examples of integrable systems on $so(4)$ we introduce two vectors $\mathbf{J} = (J_1, J_2, J_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ with entries

$$y_i = \varkappa(s_i - t_i), \quad J_i = s_i + t_i, \quad (2.7)$$

which satisfy to the following Lie-Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, y_j\} = \varepsilon_{ijk} y_k, \quad \{y_i, y_j\} = \varkappa^2 \varepsilon_{ijk} J_k. \quad (2.8)$$

Because physical quantities y_k, J_k should be real, parameter \varkappa^2 must be real too and algebra (2.8) is reduced to its two real forms $so(4, \mathbb{R})$ or $so(3, 1, \mathbb{R})$ for positive and negative \varkappa^2 respectfully. The corresponding Casimir elements are equal to

$$C_\varkappa = \varkappa^2 |\mathbf{J}|^2 + |\mathbf{y}|^2, \quad C = (\mathbf{y}, \mathbf{J}). \quad (2.9)$$

Let (\mathbf{y}, \mathbf{J}) and $\mathbf{y} \times \mathbf{J}$ stand for the inner vector product and for the vector cross product respectively. In variables (2.7) the Hamilton function is equal to I_1 up to constants

$$\begin{aligned} H &= 2I_1 + \frac{C_\varkappa}{2\varkappa^2} + 2\delta_1\delta_2 = \\ &= |\mathbf{J}|^2 + (\mathbf{a}, \mathbf{J})(\mathbf{b}, \mathbf{J}) + \varkappa^{-1}(\mathbf{b}, \mathbf{y} \times \mathbf{J}) - \left(\mathbf{b}, (\delta_1 - \delta_2)\varkappa^{-1}\mathbf{y} + (\delta_1 + \delta_2)\mathbf{J} \right) + 2(\mathbf{c}, \mathbf{J}), \end{aligned} \quad (2.10)$$

where numerical vectors are

$$\mathbf{a} = (0, 0, 2i), \quad \mathbf{b} = (i(a_1 + b_1), a_1 - b_1, i), \quad \mathbf{c} = (a_0 + b_0, -i(a_0 - b_0), 0).$$

Additional integrals of motion K looks like

$$\begin{aligned} K &= -4iI_2 = (\mathbf{b}, \mathbf{J}) \left[2|\mathbf{J}|^2 + \varkappa^{-1} \left(\mathbf{a}, \mathbf{y} \times \mathbf{J} - (\delta_1 - \delta_2)\mathbf{y} + \varkappa(\delta_1 + \delta_2)\mathbf{J} \right) - \varkappa^{-2}C_\varkappa - 4\delta_1\delta_2 \right] \\ &- 2\varkappa^{-1}(\delta_1 - \delta_2)(\mathbf{c}, \mathbf{y}) + 2\varkappa^{-1}(\mathbf{c}, \mathbf{y} \times \mathbf{J}) + 2(\delta_1 + \delta_2)(\mathbf{c}, \mathbf{J}) \end{aligned} \quad (2.11)$$

Integrals of motion H and K are defined up to canonical transformations.

Suppose that Hamiltonian function has to depend on the third component of the vector $\mathbf{y} \times \mathbf{J}$ only. The Hamiltonian (2.10) has such form after rotation $\mathbf{y} \rightarrow U\mathbf{y}$ and $\mathbf{J} \rightarrow U\mathbf{J}$ on the following Euler angles

$$\phi = \frac{\pi}{2} - i \frac{\ln a_1 - \ln b_1}{2}, \quad \psi = 0, \quad \theta = \frac{\pi}{2} - i \ln(-i + 2\sqrt{a_1 b_1}) - \frac{i}{2} \ln(4a_1 b_1 + 1).$$

This rotation acts on the numerical vectors \mathbf{a} , \mathbf{b} and \mathbf{c} in the following way

$$\tilde{\mathbf{a}} = U^{-1}\mathbf{a} = (0, 2\sqrt{1-c^2}, 2ic), \quad \tilde{\mathbf{b}} = U^{-1}\mathbf{b} = (0, 0, -ic^{-1}),$$

$$\tilde{\mathbf{c}} = U^{-1}\mathbf{c} = \frac{1}{2c} \left(\alpha, \beta, -\frac{i\sqrt{1-c^2}}{c}\beta \right),$$

where

$$c = \frac{1}{\sqrt{4a_1 b_1 + 1}}, \quad \alpha = \frac{2i(a_1 b_0 - a_0 b_1)}{\sqrt{a_1 b_1(4a_1 b_1 + 1)}}, \quad \beta = \frac{-2(a_1 b_0 + a_0 b_1)}{\sqrt{a_1 b_1(4a_1 b_1 + 1)}}.$$

Substituting $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$ instead vectors \mathbf{a} , \mathbf{b} and \mathbf{c} into (2.10) and (2.11) one gets integrals of motion after rotation. The Hamilton function H (2.10) after rotation and renormalization

$$\hat{H} = \frac{H}{\sqrt{4a_1 b_1 + 1}} = c(J_1^2 + J_2^2 - J_3^2) - 2\sqrt{1-c^2}J_2J_3 + \frac{1}{i\varkappa}(y_2J_1 - y_1J_2) + \alpha J_1 + \beta J_2 + \gamma J_3 + \delta y_3. \quad (2.12)$$

depends on five essential parameters c , α , β and

$$\gamma = -i(\delta_1 + \delta_2) + 4\frac{a_1 b_0 + a_0 b_1}{4a_1 b_1 + 1}, \quad \delta = \frac{\delta_2 - \delta_1}{i\varkappa}.$$

It is a real function on $so(3, 1, \mathbb{R})$ with negative \varkappa^2 only.

According to [5, 7], the separated coordinates $q_{1,2}$ for (2.12) are zeros of the polynomial

$$T_{11}(\lambda) = (\lambda - q_1)(\lambda - q_2) = 0, \quad (2.13)$$

whereas the conjugated momenta are equal to

$$p_k = -i \ln T_{21}(q_k) - \ln(a_1 q_k + a_0). \quad (2.14)$$

We can prove that q_k, p_k are Darboux variables using (2.13-2.14) and brackets (1.3).

By definition the generating function of integrals of motion is

$$\begin{aligned} \tilde{\tau}(\lambda) &= \text{trace } \tilde{T}(\lambda) = \lambda^3 + I_1 \lambda - I_2 = \\ &= (\lambda + \mathcal{A}_0) T_{11}(\lambda) + (a_1 \lambda + a_0) T_{21}(\lambda) + (b_1 \lambda + b_0) T_{12}(\lambda) \end{aligned}$$

Substituting $\lambda = q_k$ into this equation one gets two separated equations

$$q_k^3 + I_1 q_k + I_2 = \exp(ip_k) + \det \tilde{T}(q_k) \exp(-ip_k), \quad k = 1, 2.$$

Here we took into account that $T_{11}(q_k) = 0$ and $T_{12}(q_k) = \det T(q_k) T_{21}^{-1}(q_k)$.

3 Quartic integrals of motion

According to [6], we can construct another commutative subalgebra generated by *quadratic* polynomials on coefficients of $T_{ij}(\lambda)$, which are integrals of motion for another integrable system associated with the same matrix $T(\lambda)$ (1.1).

Let $\mathcal{K}_{\pm}(\lambda)$ be generic numerical solutions of the reflection equation [9]

$$\mathcal{K}_+ = \begin{pmatrix} b_3 \lambda + \alpha & (b_1 + ib_2) \lambda \\ (b_1 - ib_2) \lambda & -b_3 \lambda + \alpha \end{pmatrix}, \quad \mathcal{K}_- = \begin{pmatrix} a_3 \lambda + \beta & (a_1 + ia_2) \lambda \\ (a_1 - ia_2) \lambda & -a_3 \lambda + \beta \end{pmatrix}. \quad (3.15)$$

The Lax matrix for the two-site XXX Heisenberg magnet with boundaries has the form

$$L(\lambda) = \mathcal{K}_+(\lambda) T(\lambda) \mathcal{K}_-(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t(-\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.16)$$

Trace of $L(\lambda)$

$$\tau(\lambda) = \text{tr } L(\lambda) = -2(\mathbf{a}, \mathbf{b}) \lambda^6 - I_1 \lambda^4 - I_2 \lambda^2 - I_3. \quad (3.17)$$

give rise to the commutative subalgebra of the Sklyanin brackets (1.3). Integrals of motion I_1, I_2 and I_3 are second, fourth and sixth order polynomials in variables s_i, t_i .

In variables (2.7) the Hamilton function H (3.18) is equal to I_1 up to constants

$$\begin{aligned} H &= I_1 + (\mathbf{a}, \mathbf{b}) \left(\varkappa^{-2} C_{\varkappa} - 2(\delta_1^2 + \delta_2^2) \right) - 2\alpha\beta = \\ &= (\mathbf{J}, \mathbf{A}\mathbf{J}) + \varkappa^{-1}(\mathbf{a} \times \mathbf{b}, \mathbf{y} \times \mathbf{J}) + 2(\alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{J}) \\ &- \varkappa^{-1}(\delta_1 - \delta_2)(\mathbf{a} \times \mathbf{b}, \mathbf{y}) - (\delta_1 + \delta_2)(\mathbf{a} \times \mathbf{b}, \mathbf{J}), \end{aligned} \quad (3.18)$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ are numerical vectors and

$$A = \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}, \quad A_{ij} = a_i b_j + a_j b_i. \quad (3.19)$$

Additional integral of motion

$$K = 2\kappa^2 I_2 - (\mathbf{a}, \mathbf{b}) \left(C^2 + \frac{C_\kappa^2}{4\kappa^2} \right) + 2\alpha\beta C_\kappa \quad (3.20)$$

is a third order polynomial in momenta \mathbf{J} . For brevity we present K at $\delta_1 = \delta_2 = 0$ only

$$\begin{aligned} K &= \left(\kappa^2 |\mathbf{J}|^2 - |\mathbf{y}|^2 \right) \left[\kappa^{-1} (\mathbf{a} \times \mathbf{b}, \mathbf{y} \times \mathbf{J}) + 2(\alpha \mathbf{a} + \beta \mathbf{b}, \mathbf{J}) \right] \\ &+ |\mathbf{J}|^2 \left[8\kappa^2 \alpha \beta - 2(\mathbf{a}, \mathbf{b}) |\mathbf{y}|^2 + 4\kappa (\alpha \mathbf{a} - \beta \mathbf{b}, \mathbf{y}) \right] + (\mathbf{y} \times \mathbf{J}, A \mathbf{y} \times \mathbf{J}) \\ &- 4\kappa (\mathbf{y}, \mathbf{J}) (\alpha \mathbf{a} - \beta \mathbf{b}, \mathbf{J}). \end{aligned} \quad (3.21)$$

The third coefficient I_3 is a constant

$$I_3 = \left(\frac{C_\kappa}{4\kappa^2} + \frac{C}{2\kappa} + \delta_1^2 \right) \left(\frac{C_\kappa}{4\kappa^2} - \frac{C}{2\kappa} + \delta_2^2 \right). \quad (3.22)$$

Integrals of motion (3.18) and (3.21) depend on ten numerical parameters $a_i, b_i, \alpha, \beta, \delta_1, \delta_2$ and they are defined up to canonical transformations.

Suppose that Hamiltonian function has to depend on the third component of the vector $\mathbf{y} \times \mathbf{J}$ only. The Hamiltonian (3.18) has such form after rotation $\mathbf{J} \rightarrow U\mathbf{J}$ and $\mathbf{y} \rightarrow U\mathbf{y}$ with orthogonal matrix U defined by

$$\tilde{\mathbf{a}} = U^{-1}\mathbf{a} = \left(\sqrt{\frac{e_1}{2}}, i\sqrt{\frac{e_2}{2}}, 0 \right), \quad \tilde{\mathbf{b}} = U^{-1}\mathbf{b} = \left(\sqrt{\frac{e_1}{2}}, -i\sqrt{\frac{e_2}{2}}, 0 \right), \quad (3.23)$$

where e_i are eigenvalues of the matrix A (3.19). Substituting $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$ instead vectors \mathbf{a}, \mathbf{b} into (3.18) and (3.21) one gets integrals of motion after rotation.

If \mathbf{a} and \mathbf{b} are linearly dependent vectors, then $\mathbf{a} \times \mathbf{b} = 0$ and matrix A (3.19) has only one non-zero value $e_1 \simeq |\mathbf{a}|^2$. In this case the Hamilton function (3.18)

$$\tilde{H} = J_1^2 + cJ_1, \quad c \in \mathbb{R}$$

determines degenerate or superintegrable system with a noncommutative family of additional integrals of motion. For instance, there is the following quadratic integral

$$\tilde{K} = c_1 J_1^2 + J_2^2 + J_3^2 + c_2 y_1^2 + c_3 (y_2^2 + y_3^2) + c_4 y_1 J_1 + c_5 (y_2 J_3 - y_3 J_2).$$

It is special case of the Poincaré model [8]. For more details see [10].

If $\mathbf{a} \times \mathbf{b} \neq 0$, the matrix A (3.19) has two non-zero eigenvalues

$$e_{1,2} = (\mathbf{a}, \mathbf{b}) \pm |\mathbf{a}||\mathbf{b}|, \quad e_3 = 0.$$

In this case after rotation and renormalization the Hamiltonian H (3.18) is equal to

$$\begin{aligned} \tilde{H} &= \frac{H}{i\sqrt{e_1 e_2}} = cJ_1^2 - c^{-1}J_2^2 + \varkappa^{-1}(y_1 J_2 - y_2 J_1) \\ &+ \tilde{\alpha}J_1 + \tilde{\beta}J_2 - (\delta_1 + \delta_2)J_3 - \varkappa^{-1}(\delta_1 - \delta_2)y_3 \end{aligned} \quad (3.24)$$

where

$$c = -i\sqrt{e_1 e_2^{-1}}, \quad \tilde{\alpha} = c_1 = -i\sqrt{2e_2^{-1}}(\alpha + \beta), \quad \tilde{\beta} = \sqrt{2e_1^{-1}}(\alpha - \beta).$$

The Hamilton function \tilde{H} (3.24) depends on five parameters instead of ten parameters in the initial Hamiltonian (3.18). It allows us to impose constraints on the vectors \mathbf{a} and \mathbf{b} and to use triangular boundary matrices \mathcal{K}_\pm [6] instead of the general ones (3.15). The similar facts hold true for the BC_n Toda lattices [7] and for the Kowalevski-Goryachev-Chaplygin gyrostat [11], which are also related with the reflection equations.

Thus we can consider low triangular solution of the reflection equations \mathcal{K}_+ (3.15)

$$b_1 - ib_2 = 0 \quad (3.25)$$

without loss of generality. According to the Sklyanin method [5] the separated variables may be defined by entries of the following intermediate matrix

$$\mathcal{T}(\lambda) = T(\lambda) \mathcal{K}_-(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t(-\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.26)$$

which satisfies to the reflection equation [6]

$$\{\overset{1}{\mathcal{T}}(\lambda), \overset{2}{\mathcal{T}}(\nu)\} = \left[r(\lambda - \nu), \overset{1}{\mathcal{T}}(\lambda) \overset{2}{\mathcal{T}}(\nu) \right] + \overset{1}{\mathcal{T}}(\lambda) r(\lambda + \nu) \overset{2}{\mathcal{T}}(\nu) - \overset{2}{\mathcal{T}}(\nu) r(\lambda + \nu) \overset{1}{\mathcal{T}}(\lambda). \quad (3.27)$$

The separated coordinates $q_{1,2}$ are non-trivial zeros of the polynomial

$$\mathcal{T}_{12}(\lambda) = \lambda(\lambda^2 - q_1^2)(\lambda^2 - q_2^2) = 0, \quad (3.28)$$

whereas the conjugated momenta are equal to

$$p_k = -i \ln \mathcal{T}_{11}(q_k) - \ln(b_3 q_k + \alpha). \quad (3.29)$$

We can prove that q_k, p_k are Darboux variables using (3.28-3.29) and the reflection equation (3.27).

Such as \mathcal{K}_+ is a low triangular matrix the generating function of integrals is

$$\begin{aligned}\tau(\lambda) &= \text{trace } L(\lambda) = -2(\mathbf{a}, \mathbf{b})\lambda^6 - I_1\lambda^4 - I_2\lambda^2 - I_3 = \\ &= \text{trace } \mathcal{K}_+ \mathcal{T}(\lambda) = (b_3\lambda + \alpha)\mathcal{T}_{11}(\lambda) + (b_1 + ib_2)\lambda\mathcal{T}_{12}(\lambda) - (b_3\lambda - \alpha)\mathcal{T}_{22}(\lambda)\end{aligned}$$

Substituting $\lambda = q_k$ into this equation one gets two separated equations

$$2(\mathbf{a}, \mathbf{b})q_k^6 + I_1q_k^4 + I_2q_k^2 + I_3 = \exp(ip_k) + \det L(q_k) \exp(-ip_k), \quad k = 1, 2.$$

Here we took into account that $\mathcal{T}_{12}(q_k) = 0$ and $\mathcal{T}_{22}(q_k) = \det \mathcal{T}(q_k) \mathcal{T}_{11}^{-1}(q_k)$.

According to [11] we can introduce another separated variables related with the proposed separated variables by canonical transformation and by flip of parameters. Existence of the different separated variables is associated with the invariance of the Sklyanin brackets with respect to a matrix transposition $T \rightarrow T^t$.

4 Conclusion

The Hamiltonians (2.12) and (3.18) belongs to the following class of the Hamiltonians

$$H = (\mathbf{J}, \mathbf{A}\mathbf{J}) + (\mathbf{a}, \mathbf{y} \times \mathbf{J}) + (\mathbf{b}, \mathbf{J}) + (\mathbf{c}, \mathbf{y}), \quad (4.30)$$

possessing additional integrals of third and fourth degree in momenta. V.V. Sokolov kindly informed us that there exist three different such integrable cases only [12].

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