

The deformations of Whitham systems and Lagrangian formalism.

A.Ya. Maltsev

L.D.Landau Institute for Theoretical Physics,
119334 ul. Kosygina 2, Moscow, maltsev@itp.ac.ru

Abstract

We consider the Lagrangian formalism of the deformations of Whitham systems having Dubrovin-Zhang form. As an example the case of modulated one-phase solutions of the non-linear "V-Gordon" equation is considered.

1 Introduction.

This paper is a continuation of the paper [72] connected with the deformations of the hyperbolic Whitham systems. The method of deformations of Whitham systems suggested in [72] is connected with the slow modulations of m -phase quasiperiodic solutions

$$\varphi^i(x, t) = \Phi^i(\mathbf{k}(\mathbf{U})x + \boldsymbol{\omega}(\mathbf{U})t + \boldsymbol{\theta}_0, \mathbf{U}) \quad , \quad i = 1, \dots, n \quad (1.1)$$

of some system

$$F^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_t, \boldsymbol{\varphi}_x, \dots) = 0 \quad , \quad i = 1, \dots, n \quad (1.2)$$

Here the functions $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ are 2π -periodic functions w.r.t. each θ^α , $\alpha = 1, \dots, m$, the values $\boldsymbol{\theta}_0 = (\theta_0^1, \dots, \theta_0^m)$ are arbitrary initial phase shifts and the variables $\mathbf{U} = (U^1, \dots, U^N)$ play the role of the parameters of m -phase solutions. The functions $\boldsymbol{\omega}(\mathbf{U}) = (\omega^1(\mathbf{U}), \dots, \omega^m(\mathbf{U}))$ and $\mathbf{k}(\mathbf{U}) = (k^1(\mathbf{U}), \dots, k^m(\mathbf{U}))$ are the "frequencies" and the "wave numbers" of the solution (1.1) such that the functions $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ satisfy the system

$$F^i(\boldsymbol{\Phi}, \omega^\alpha(\mathbf{U})\boldsymbol{\Phi}_{\theta^\alpha}, k^\beta(\mathbf{U})\boldsymbol{\Phi}_{\theta^\beta}, \dots) = 0 \quad (1.3)$$

for every $\boldsymbol{\theta}$ and \mathbf{U} .

In Whitham method the small parameter ϵ is introduced such that $X = \epsilon x$ and $T = \epsilon t$ play the role of the "slow" coordinates in (x, t) -space. The corresponding slowly modulated solutions of (1.1) are represented in the asymptotic form

$$\phi^i(\boldsymbol{\theta}, x, t) = \sum_{k \geq 0} \Psi_{(k)}^i \left(\frac{\mathbf{S}(X, T)}{\epsilon} + \boldsymbol{\theta}_0, X, T \right) \epsilon^k \quad (1.4)$$

where all $\Psi_{(k)}(\boldsymbol{\theta}, X, T)$ are 2π -periodic w.r.t. each θ^α functions. The function $\mathbf{S}(X, T) = (S^1(X, T), \dots, S^m(X, T))$ is the "modulated phase" of the solution (1.4). The function $\Psi_{(0)}(\boldsymbol{\theta}, X, T)$ satisfies the system (1.3) and belongs to the family $\Lambda = \{\Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{U})\}$ at every fixed X and T . We have then

$$\Psi_{(0)}(\boldsymbol{\theta}, X, T) = \Phi(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X, T), \mathbf{U}(X, T)) \quad (1.5)$$

and

$$S_T^\alpha = \omega^\alpha(\mathbf{U}(X, T)) \quad , \quad S_X^\alpha = k^\alpha(\mathbf{U}(X, T))$$

as follows from the substitution of (1.4) in the system (1.2).

The functions $\Psi_{(k)}(\boldsymbol{\theta}, X, T)$ are defined from the linear systems

$$\hat{L}_{j[\mathbf{U}, \boldsymbol{\theta}_0]}^i(X, T) \Psi_{(k)}^j(\boldsymbol{\theta}, X, T) = f_{(k)}^i(\boldsymbol{\theta}, X, T) \quad (1.6)$$

where $\hat{L}_{j[\mathbf{U}, \boldsymbol{\theta}_0]}^i(X, T)$ is a linear operator given by the linearizing of the system (1.3) on the solution (1.5). The resolvability conditions of the system (1.6) can be written as the orthogonality conditions of the functions $\mathbf{f}_{(k)}(\boldsymbol{\theta}, X, T)$ to all the "left eigen vectors" (the eigen vectors of adjoint operator) $\boldsymbol{\kappa}_{[\mathbf{U}(X, T)]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X, T))$ of the operator $\hat{L}_{j[\mathbf{U}, \boldsymbol{\theta}_0]}^i(X, T)$ corresponding to zero eigen-values. The resolvability conditions of (1.6) for $k = 1$ together with

$$k_T^\alpha = \omega_X^\alpha$$

give the Whitham system for m -phase solutions of (1.2) playing the central role in the slow modulations approach.

Like in [72] we will assume here that the parameters $(\mathbf{k}, \boldsymbol{\omega})$ can be considered as the independent parameters on the family Λ such that the full set of independent parameters \mathbf{U} (except initial phases θ_0^α) can be represented in the form $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ where \mathbf{k} and $\boldsymbol{\omega}$ are the wave numbers and the frequencies of the solution and $\mathbf{n} = (n^1, \dots, n^s)$ are some additional parameters (if they exist).

Easy to see that the functions

$$\Phi_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}) \quad , \quad \alpha = 1, \dots, m$$

and

$$\Phi_{n^l}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}) \quad , \quad l = 1, \dots, s$$

give the eigen-vectors of the operator $\hat{L}_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}^i$ corresponding to zero eigen-values.

Let us give also the definition of the full regular family of m -phase solutions of (1.2).¹

Definition 1.1.

We call the family Λ the full regular family of m -phase solutions of (1.2) if

- 1) The functions $\Phi_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$, $\Phi_{n^i}(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ are linearly independent and give (for generic \mathbf{k} and $\boldsymbol{\omega}$) the full basis in the kernel of the operator $\hat{L}_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}^i$;
- 2) The operator $\hat{L}_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}^i$ has (for generic \mathbf{k} and $\boldsymbol{\omega}$) exactly $m+s$ linearly independent "left eigen vectors"

$$\boldsymbol{\kappa}_{[\mathbf{U}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) = \boldsymbol{\kappa}_{[\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0)$$

depending on the parameters \mathbf{U} in a smooth way and corresponding to zero eigen-values.

It can be shown that for full regular family Λ the corresponding Whitham system puts the restrictions only on the functions

$$\mathbf{U}(X, T) = (\mathbf{k}(X, T), \boldsymbol{\omega}(X, T), \mathbf{n}(X, T))$$

having the form

$$\begin{aligned} k_T^\alpha &= \omega_X^\alpha \\ C_\nu^{(q)}(\mathbf{U}) U_T^\nu - D_\nu^{(q)}(\mathbf{U}) U_X^\nu &= 0 \end{aligned} \quad (1.7)$$

($q = 1, \dots, m+s$, $\nu = 1, \dots, N = 2m+s$) and does not include the initial phase shifts $\theta_0^\alpha(X, T)$.

Definition 1.2.

Let us call the Whitham system (1.7) non-degenerate hyperbolic Whitham system if:

- 1) The system (1.7) is resolvable with respect to the time derivatives of parameters U^ν and can be written in the form

$$U_T^\nu = V_\mu^\nu(\mathbf{U}) U_X^\mu, \quad \nu, \mu = 1, \dots, N \quad (1.8)$$

- 2) The matrix $V_\mu^\nu(\mathbf{U})$ has N linearly independent real eigen-vectors with real eigen-values.

Provided that the system (1.7) is satisfied we can find the first correction $\Psi_{(1)}(\boldsymbol{\theta}, X, T)$ in the solution (1.4) modulo the linear combination of the functions $\Phi_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$, $\Phi_{n^i}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$. In general scheme we try to find recursively the higher order corrections $\Psi_{(k)}(\boldsymbol{\theta}, X, T)$ from the linear systems (1.6). The functions $\theta_0^\alpha(X, T)$ and the freedom in the determination of the functions $\Psi_{(k)}(\boldsymbol{\theta}, X, T)$ are used to satisfy the compatibility conditions of the systems (1.6) in higher orders of ϵ , so we get the recursive

¹This definition corresponds to the "weak" definition of full regular family of m -phase solutions given in [72].

restrictions on the corresponding parameters.² The solution of the Whitham system (1.7) (or (1.8)) is considered usually as the central point of the procedure which defines the global properties of the modulated solution. Let us also mention the well known fact that the Whitham systems corresponding to the integrable systems (1.2) possess also the integrability properties.

The first consideration of dispersive corrections to Whitham systems were made in [5] (see also [6]-[7]) where the multi-phase Whitham was also first discussed. As was pointed out in [5] the dispersive corrections can naturally arise in the Whitham method both in one-phase and multi-phase situations.

Here we consider the deformations of Whitham systems (1.8) having the form of Dubrovin-Zhang deformations of Frobenius manifolds ([63, 65]). The problem is thus connected with B.A. Dubrovin problem of deformations of Frobenius manifolds corresponding to Whitham systems of integrable hierarchies.

According to Dubrovin-Zhang approach we call the deformation of the Whitham system (1.7) (or (1.8)) the expression containing the system (1.7) (or (1.8)) as the leading term and the infinite number of "dispersive" corrections containing the higher (T and X) derivatives of the parameters U^ν and polynomial with respect to all derivatives of U^ν . For "non-degenerate hyperbolic" Whitham systems (1.8) it is natural to express recursively all the higher T -derivatives of parameters U^ν in terms of their X -derivatives and represent the deformation of the Whitham system in the evolution (Dubrovin-Zhang) form

$$U_T^\nu = V_\mu^\nu(\mathbf{U}) U_X^\mu + \sum_{k \geq 2} \zeta_{(k)}^\nu(\mathbf{U}, \mathbf{U}_X, \mathbf{U}_{XX}, \dots) \quad (1.9)$$

We require now that all $\zeta_{(k)}^\nu$ satisfy the following conditions

1) All $\zeta_{(k)}^\nu$ are polynomial in derivatives $\mathbf{U}_X, \mathbf{U}_{XX}, \dots$

2) The term $\zeta_{(k)}^\nu$ has degree k according to the following gradation:

All the functions $f(\mathbf{U})$ have degree 0;

The derivatives U_{kX}^ν have degree k ;

The degree of the product of functions having certain degrees is equal to the sum of their degrees.

Let us say that the deformation of the Whitham system having the form (1.9) with all the conditions formulated above is represented in Dubrovin - Zhang form.

Let us describe briefly the main features of deformation procedure of Whitham systems suggested in [72]. We will write here the parameters \mathbf{U} in the form $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$. According to [72] we have to make first the "right renormalization" of the functions $S^\alpha(X, T)$ and

²For systems (1.2) having non-degenerate hyperbolic Whitham systems it was shown in [72] that the asymptotic series (1.4) can be constructed globally in X up to the moment of the breakdown of corresponding solution of Whitham system.

$n^l(X, T)$ arising in Whitham method. Namely, we allow the regular ϵ -dependence of the functions $S^\alpha(X, T)$, $n^l(X, T)$ having the form of the infinite series

$$\begin{aligned} S^\alpha(X, T, \epsilon) &= \sum_{k \geq 0} \epsilon^k S_{(k)}^\alpha(X, T) \\ n^l(X, T, \epsilon) &= \sum_{k \geq 0} \epsilon^k n_{(k)}^l(X, T) \end{aligned}$$

However, we require now that the function

$$\phi_{(0)}(\boldsymbol{\theta}, X, T, \epsilon) = \Phi \left(\frac{\mathbf{S}(X, T, \epsilon)}{\epsilon} + \boldsymbol{\theta}, \mathbf{S}_X(X, T, \epsilon), \mathbf{S}_T(X, T, \epsilon), \mathbf{n}(X, T, \epsilon) \right) \quad (1.10)$$

gives the "best possible" approximation to the full asymptotic solution (1.4). More precisely, we require that the function (1.10) satisfies the following conditions

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{i=1}^n \phi_{(0)\theta^\alpha}^i(\boldsymbol{\theta}, X, T, \epsilon) \phi^i(\boldsymbol{\theta}, X, T, \epsilon) \frac{d^m \theta}{(2\pi)^m} \equiv \quad (1.11)$$

$$\equiv \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{i=1}^n \phi_{(0)\theta^\alpha}^i(\boldsymbol{\theta}, X, T, \epsilon) \phi_{(0)}^i(\boldsymbol{\theta}, X, T, \epsilon) \frac{d^m \theta}{(2\pi)^m} \equiv 0 \quad , \quad \alpha = 1, \dots, m$$

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{i=1}^n \phi_{(0)n^l}^i(\boldsymbol{\theta}, X, T, \epsilon) \phi^i(\boldsymbol{\theta}, X, T, \epsilon) \frac{d^m \theta}{(2\pi)^m} \equiv \quad (1.12)$$

$$\equiv \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{i=1}^n \phi_{(0)n^l}^i(\boldsymbol{\theta}, X, T, \epsilon) \phi_{(0)}^i(\boldsymbol{\theta}, X, T, \epsilon) \frac{d^m \theta}{(2\pi)^m} \quad , \quad l = 1, \dots, s$$

where $\phi(\boldsymbol{\theta}, X, T, \epsilon)$ is the asymptotic solution given by (1.4).

After that we try to make a "re-expansion" of the asymptotic series (1.4) using the higher derivatives of the "renormalized" functions $\mathbf{S}(X, T, \epsilon)$, $\mathbf{n}(X, T, \epsilon)$ instead of the parameter ϵ in the expansion. As was shown in [72] in the case of non-degenerate hyperbolic Whitham system (1.7) it is possible to use only the X -derivatives of parameters $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ in this expansion and put the conditions of the form (1.9) on the "renormalized" functions $\mathbf{k}(X, T, \epsilon)$, $\boldsymbol{\omega}(X, T, \epsilon)$, $\mathbf{n}(X, T, \epsilon)$. The asymptotic solution (1.4) is represented then as the new asymptotic expansion with respect to higher X -derivatives of parameters $\mathbf{U} = (\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ according to the gradation introduced above.

After the "re-expansion" we can forget in fact about the ϵ -dependence of the functions $\mathbf{S}(X, T, \epsilon)$, $\mathbf{n}(X, T, \epsilon)$ and consider the "concrete formal solutions" of the system (1.2) without the additional one-parametric ϵ -families. The X -derivatives of the slow functions $\mathbf{k}(X, T)$, $\boldsymbol{\omega}(X, T)$, $\mathbf{n}(X, T)$ play now the role of the small parameters in the expansion with the gradation introduced above. We keep now the notations X and T for spatial and time variables just to emphasize that we consider the slow functions of x and t .

The formal solution of (1.2) will be written now in the form

$$\phi^i(\boldsymbol{\theta}, X, T) = \Phi^i(\mathbf{S}(X, T) + \boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T, \mathbf{n}) + \sum_{k \geq 1} \Phi_{(k)}^i(\mathbf{S}(X, T) + \boldsymbol{\theta}, X, T) \quad (1.13)$$

where all $\Phi_{(k)}^i(\boldsymbol{\theta}, X, T)$ are local expressions depending on $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots)$ polynomial in derivatives $(\mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots)$ and having degree k .

All $\Phi_{(k)}^i(\boldsymbol{\theta}, X, T)$, $k \geq 1$ satisfy the normalization conditions

$$\int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \Phi_{\theta^\alpha}^i(\boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T, \mathbf{n}) \Phi_{(k)}^i(\boldsymbol{\theta}, X, T) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0 \quad , \quad \alpha = 1, \dots, m \quad (1.14)$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \Phi_{n^l}^i(\boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T, \mathbf{n}) \Phi_{(k)}^i(\boldsymbol{\theta}, X, T) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0 \quad , \quad l = 1, \dots, s \quad (1.15)$$

according to normalization (1.11)-(1.12).

The functions $\Phi_{(k)}^i(\boldsymbol{\theta}, X, T)$ satisfy the linear systems

$$\hat{L}_{j[\mathbf{S}_X, \mathbf{S}_T, \mathbf{n}]}^i(X, T) \Phi_{(k)}^i(\boldsymbol{\theta}, X, T) = \tilde{f}_{(k)}^i(\boldsymbol{\theta}, X, T) \quad (1.16)$$

analogous to (1.6). The systems (1.16) represent now all the terms having gradation k after the substitution of (1.13) in (1.2).

Let us say that the parameters $\theta_0^\alpha(X, T)$ and other additional parameters arising in Whitham method do not appear in this approach being completely "absorbed" by the "renormalized" functions $\mathbf{S}(X, T)$, $\mathbf{n}(X, T)$.

The functions $\zeta_{(k)}^\nu(\mathbf{U}, \mathbf{U}_X, \dots)$ arising in (1.9) are defined now from the compatibility conditions of the systems (1.16) in the k -th order. The functions $\Phi_{(k)}^i(\boldsymbol{\theta}, X, T)$ are uniquely defined then from (1.16) view the conditions (1.14)-(1.15). The first term of the system (1.9) coincides with the right-hand part of the corresponding Whitham system (1.8).

The full system (1.9) in parameters $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ can be written in the form

$$\begin{aligned} k_T^\alpha &= \omega_X^\alpha \\ \omega_T^\alpha &= \sum_{k \geq 1} \sigma_{(k)}^\alpha(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots) \\ n_T^l &= \sum_{k \geq 1} \eta_{(k)}^l(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots) \end{aligned} \quad (1.17)$$

where all $\sigma_{(k)}^\alpha, \eta_{(k)}^l$ are local expressions in $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots)$ polynomial in derivatives and having degree k .

It was shown in [72] that the expansions (1.13) represent all the "particular" formal solutions (1.4) in the case of non-degenerate hyperbolic Whitham system (1.8).

We can consider now the family of slowly-modulated formal solutions of (1.2) parameterized by the functions $\mathbf{k}(X, T)$, $\boldsymbol{\omega}(X, T)$, $\mathbf{n}(X, T)$ (and the general initial phase $\boldsymbol{\theta}_0$) satisfying the system (1.17).

At the end of this Chapter let us make one more remark about the representation of slowly-modulated solutions of (1.2) in the form (1.13). Namely, we can see that the representation (1.13) depends on the choice of the functions $\Phi^i(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$ having "zero initial phase shifts" at every $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$. In particular, the natural change

$$\Phi^i(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}) = \Phi^i(\boldsymbol{\theta} + \delta(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}), \mathbf{k}, \boldsymbol{\omega}, \mathbf{n}) \quad (1.18)$$

of the functions Φ^i is possible in our situation. The change (1.18) of the function Φ^i gives then another representation of the same family of formal solutions parameterized by other functions $\mathbf{k}'(X, T)$, $\boldsymbol{\omega}'(X, T)$, $\mathbf{n}'(X, T)$. In general, the form of the system (1.17) will also depend on the choice of the functions $\Phi^i(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{n})$.

For Dubrovin - Zhang approach to the classification of integrable hierarchies the following statement plays important role ([72]):

The systems (1.17) written for two sets of functions Φ and Φ' connected by the transformation (1.18) are connected by the "trivial transformation" ([63, 65]) i.e.

There exists a change of coordinates

$$\begin{aligned} k'^\alpha &= k^\alpha + \sum_{k \geq 1} K_{(k)}^\alpha(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots) \\ \omega'^\alpha &= \omega^\alpha + \sum_{k \geq 1} \Omega_{(k)}^\alpha(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots) \\ n^l &= n^l + \sum_{k \geq 1} N_{(k)}^l(\mathbf{k}, \boldsymbol{\omega}, \mathbf{n}, \mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots) \end{aligned}$$

where all $K_{(k)}^\alpha$, $\Omega_{(k)}^\alpha$, $N_{(k)}^l$ are polynomial in derivatives $(\mathbf{k}_X, \boldsymbol{\omega}_X, \mathbf{n}_X, \dots)$ (and having degree k) which transforms the corresponding systems (1.17) one into another.

In this paper we will investigate the Lagrangian properties of the systems (1.9) in the case when the initial system (1.2) can be written in "local" Lagrangian form

$$\delta \int \int \mathcal{L}(\varphi, \varphi_t, \varphi_x, \dots) dx dt = 0 \quad (1.19)$$

We do not assume here the integrability of the system (1.2) so the corresponding statements will be valid for both integrable and non-integrable cases. We will show here that the existence of Lagrangian formalism (1.19) gives in fact the convenient procedure of construction of the deformed system (1.9). As the example we will consider the case of one-phase modulated solutions of "V-Gordon" equation

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0$$

and describe the Lagrangian formalism for the corresponding deformation of it's Whitham system up to the first nontrivial correction.

2 Lagrangian formalism.

We will assume now that the initial system (1.2) can be written in the Lagrangian form (1.19) where the Lagrangian density $\mathcal{L}(\varphi, \varphi_t, \varphi_x, \varphi_{tt}, \varphi_{xx}, \dots)$ is the local function of all variables. We will consider here the Lagrangian form of the system (1.9). Let us say that the Lagrangian formalism for the Whitham system (1.7) was constructed by G. Whitham ([3]) who suggested the "averaged" Lagrangian formalism

$$\frac{\delta}{\delta \mathbf{S}(X, T)} \int \int \bar{\mathcal{L}}(\mathbf{S}_{X'}, \mathbf{S}_{T'}) dX' dT' = 0 \quad (2.1)$$

for (1.7). It was also pointed out by G. Whitham that the parameters \mathbf{n} on the family Λ can be described with the aid of specific "pseudo-phases" in the Lagrangian approach such that the system (1.7) can be written in the form (2.1) in the general case. The Lagrangian formalism (2.1) gives then the equations (1.7) written in the terms of phases and "pseudo-phases" as well as the conservation laws for the system (1.7).

Let us omit also here the parameters (n^1, \dots, n^s) and assume that the m -phase solutions of (1.2) are parameterized by the values $(k^1, \dots, k^m), (\omega^1, \dots, \omega^m)$ and the initial phase shifts $(\theta_0^1, \dots, \theta_0^m)$. The corresponding considerations can then be easily generalized to the more general case by the introduction of "pseudo-phases" in Whitham's way.

We will assume then that the system (1.2) has a full regular family Λ of m -phase solutions with parameters $k^\alpha = S_X^\alpha, \omega^\alpha = S_T^\alpha, \theta_0^\alpha, \alpha = 1, \dots, m$. Besides that, we will assume that the corresponding Whitham system (1.7) is non-degenerate hyperbolic system which makes the form (1.9) of the deformation of Whitham system the most natural. The functions $\Phi_{\theta^\alpha}(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega})$ give now the basis (for generic \mathbf{k} and $\boldsymbol{\omega}$) in the kernel of the operator $\hat{L}_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}]}^i$ introduced above. We require also that $\hat{L}_{j[\boldsymbol{\theta}_0, \mathbf{k}, \boldsymbol{\omega}]}^i$ has exactly m (for generic \mathbf{k} and $\boldsymbol{\omega}$) linearly independent "left eigen vectors" $\boldsymbol{\kappa}_{[\mathbf{k}, \boldsymbol{\omega}]}^{(q)}(\boldsymbol{\theta} + \boldsymbol{\theta}_0), q = 1, \dots, m$ corresponding to zero eigen-values. The Whitham system (1.7) can be written then in the form

$$A_\alpha^{(q)}(\mathbf{S}_X, \mathbf{S}_T) S_{TT}^\alpha + B_\alpha^{(q)}(\mathbf{S}_X, \mathbf{S}_T) S_{XT}^\alpha + C_\alpha^{(q)}(\mathbf{S}_X, \mathbf{S}_T) S_{XX}^\alpha = 0 \quad (2.2)$$

$\alpha = 1, \dots, m, q = 1, \dots, m$, or, equivalently

$$\begin{aligned} k_T^\alpha &= \omega_X^\alpha \\ A_\alpha^{(q)}(\mathbf{k}, \boldsymbol{\omega}) \omega_T^\alpha &= -B_\alpha^{(q)}(\mathbf{k}, \boldsymbol{\omega}) \omega_X^\alpha - C_\alpha^{(q)}(\mathbf{k}, \boldsymbol{\omega}) k_X^\alpha \end{aligned} \quad (2.3)$$

in the variables $(\mathbf{k}, \boldsymbol{\omega})$.

We want to get the deformation of the system (2.3) in the form

$$\begin{aligned} k_T^\alpha &= \omega_X^\alpha \\ \omega_T^\alpha &= \sum_{k \geq 1} \sigma_{(k)}^\alpha(\mathbf{k}, \boldsymbol{\omega}, \mathbf{k}_X, \boldsymbol{\omega}_X, \dots) \end{aligned} \quad (2.4)$$

where

$$\sigma_{(1)}^\alpha = -\|A^{-1}\|_\beta^\alpha C_\gamma^\beta k_X^\gamma - \|A^{-1}\|_\beta^\alpha B_\gamma^\beta \omega_X^\gamma$$

and all $\sigma_{(k)}^\alpha$ are polynomial in X -derivatives of \mathbf{k} and $\boldsymbol{\omega}$ and have degree k according to the gradation introduced above.

We are trying to find a solution of (1.2) in the form

$$\phi^i(\boldsymbol{\theta}, X, T) = \Phi^i(\mathbf{S}(X, T) + \boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T) + \sum_{k \geq 1} \Phi_{(k)}^i(\mathbf{S}(X, T) + \boldsymbol{\theta}, X, T) \quad (2.5)$$

where all $\phi_{(k)}^i(\boldsymbol{\theta}, X, T)$ are 2π -periodic w.r.t. each θ^α functions which are local functionals of $\mathbf{S}_X, \mathbf{S}_T, \mathbf{S}_{XX}, \mathbf{S}_{XT}, \mathbf{S}_{XXX}, \mathbf{S}_{XXT}, \dots$, polynomial in $\mathbf{S}_{XX}, \mathbf{S}_{XT}, \mathbf{S}_{XXX}, \mathbf{S}_{XXT}, \dots$, and having degree k .

All $\Phi_{(k)}$ satisfy the normalization conditions

$$\int_0^{2\pi} \dots \int_0^{2\pi} \sum_{i=1}^n \Phi_{(k)}^i(\boldsymbol{\theta}, X, T) \Phi_{\theta^\alpha}^i(\boldsymbol{\theta}, \mathbf{S}_X, \mathbf{S}_T) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} \equiv 0 \quad (2.6)$$

$\alpha = 1, \dots, m, k \geq 1$.

The system (2.4) provides the existence of all terms $\Phi_{(k)}(\boldsymbol{\theta}, X, T)$ of solution (2.5) satisfying (2.6) ([72]) which are represented as the local functionals of $(\mathbf{k}, \boldsymbol{\omega}, \mathbf{k}_X, \boldsymbol{\omega}_X, \dots)$ and are defined by the functions $\{k^\alpha(X), \omega^\alpha(X)\}$ at every T .

We can say then that the functional representation (2.5) gives a mapping τ

$$\tau : \{\mathbf{S}(X), \mathbf{S}_T(X)\} \rightarrow \{\boldsymbol{\varphi}(\boldsymbol{\theta}, X)\}$$

from the "loop space" $\{\mathbf{S}(X), \mathbf{S}_T(X)\}$ to the functional space $\{\boldsymbol{\varphi}(\boldsymbol{\theta}, X)\}$. In this approach we can define a "sub-manifold" $Im \tau$ in the functional space $\{\boldsymbol{\varphi}(\boldsymbol{\theta}, X)\}$ parameterized by the functional parameters $\{\mathbf{S}(X), \mathbf{S}_T(X)\}$.

The system (2.4) generates a dynamical flow on the "sub-manifold" $Im \tau$ corresponding to the system (1.2). In this approach the system (2.4) becomes the condition that the formal series (2.5) (with known functionals $\Phi_{(k)}^i(\boldsymbol{\theta}, \mathbf{k}, \boldsymbol{\omega}, \mathbf{k}_X, \boldsymbol{\omega}_X, \dots)$) represents a solution of (1.2).

We can suggest now the Lagrangian form for the system (2.4) using the Lagrangian formalism (1.19) for the system (1.2). We know that the function $\phi(\boldsymbol{\theta}, X, T)$ defined by (2.5) should satisfy the Euler - Lagrange equation

$$\frac{\delta}{\delta \phi^i(\boldsymbol{\theta}, X, T)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} \dots \int_0^{2\pi} \mathcal{L}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_T, \boldsymbol{\varphi}_X, \dots) \frac{d^m \boldsymbol{\theta}}{(2\pi)^m} dX dT \quad (2.7)$$

where \mathcal{L} is the Lagrangian density defined in (1.19).

After the substitution of (2.5) in \mathcal{L} (for known mapping τ) we obtain the Lagrangian functional depending on the functions $S^\alpha(X, T)$. Since the solutions (2.5) satisfy the system (2.7) the Lagrangian functional thus defined has an extremal on the functions $(S^\alpha(X, T))$ satisfying dynamical system (2.4). We have then a "Lagrangian form" for the

flow (2.4), however, the form of the functional dependence of $\Phi_{(k)}$ on the values $\mathbf{S}_X, \mathbf{S}_T, \mathbf{S}_{XX}, \mathbf{S}_{XT}, \dots$, is supposed to be known in this approach.

The "averaged" Lagrangian density $\bar{\mathcal{L}}[\mathbf{S}](X, T)$ contains an infinite number of terms with the increasing numbers of X (and T) differentiations.

The system (2.4) can then be written in the Lagrangian form

$$\frac{\delta}{\delta S^\alpha(X, T)} \int \int \sum_{k \geq 0} \bar{\mathcal{L}}_{(k)}(\mathbf{S}_{X'}, \mathbf{S}_{T'}, \dots) dX' dT' = 0 \quad (2.8)$$

which gives an infinite formal expression containing the higher derivatives of the function $S(X, T)$.

Let us say, however, that the form (2.8) does not coincide exactly with (2.4) since it contains the higher T -derivatives of $\mathbf{S}(X, T)$ like $\mathbf{S}_{TTX}, \mathbf{S}_{TTXX}, \mathbf{S}_{TTTX}, \mathbf{S}_{TTXXX}, \dots$, in the formal expansion. To get the system (2.4) we have to resolve the system (2.8) w.r.t. derivatives S_{TT}^α and then to "remove recursively" all higher T -derivatives $\mathbf{k}_{TT}, \boldsymbol{\omega}_{TT}, \mathbf{k}_{TTX}, \boldsymbol{\omega}_{TTX}, \mathbf{k}_{TTT}, \dots$, of the parameters \mathbf{k} and $\boldsymbol{\omega}$ from the right-hand part.³ After this procedure we will obtain the infinite number of terms having certain degrees in the right-hand part of our system which will coincide with the right-hand part of the system (2.4). We can say then that the procedure described above gives the "Lagrangian formalism" for the system (2.4).

3 The deformation of the Whitham system for the non-linear "V-Gordon" equation.

Let us consider the one-phase modulated solutions of the nonlinear "V-Gordon" equation

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0 \quad (3.1)$$

The one-phase solutions of (3.1) satisfy the equation

$$(S_T^2 - S_X^2) \Phi_{\theta\theta} + V'(\Phi) = 0$$

which gives the well-known representation of one-phase periodic solutions for the function $\Phi(\theta)$

$$\theta + \theta_0 = \sqrt{\omega^2 - k^2} \int \frac{d\Phi}{\sqrt{2(E - V(\Phi))}}$$

where the parameter $E(X, T)$ is connected with $\omega(X, T), k(X, T)$ by the formula

$$\oint \frac{d\Phi}{\sqrt{2(E - V(\Phi))}} = \frac{2\pi}{\sqrt{\omega^2 - k^2}}$$

³Let us note that we assume that the function $S_{mT, kX}^\alpha$ ($m \geq 2$) is represented by an infinite sum of terms having degrees $\geq m + k - 1$.

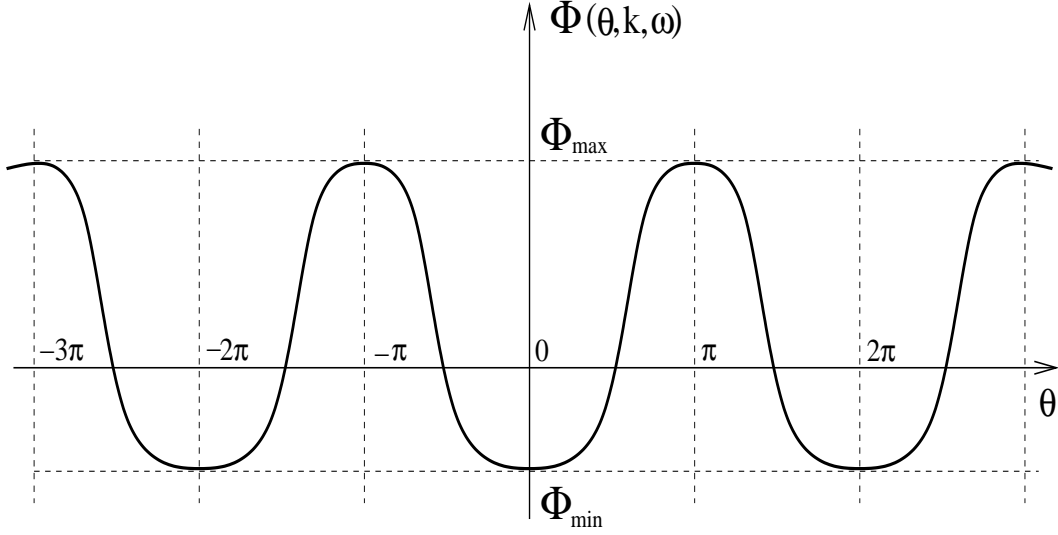


Figure 1: The function $\Phi(\theta, k, \omega)$ having zero initial phase shift.

Let us choose now the functions $\Phi(\theta, k, \omega)$ (having zero initial phase shifts) such that $\Phi(\theta, k, \omega)$ has a local minimum at the point $\theta = 0$ for all k and ω (Fig. 1).

We take now

$$\Phi_{(0)}(\theta, X, T) = \Phi(\theta, S_X, S_T)$$

for the zero approximation in (2.5).

Let us say that the functions S_{TT} , $\Phi_{\theta T}$, Φ_{TT} do not have certain degrees in our approach since the time derivatives ω_T , ω_{TT} are represented by the infinite series given by (2.4). However, we can claim that the functions S_{TT} , $\Phi_{\theta T}$ are given by the sums of terms having degrees ≥ 1 . In the same way the function Φ_{TT} is represented by the sum of terms all having degrees ≥ 2 .

Let us use now the following notation:

Namely, we will denote by the symbols like $S_{TT}^{[k]}$, $\Phi_{\theta T}^{[k]}$, $\Phi_{TT}^{[k]}$, \dots , the terms of the degree k in the infinite expansions of corresponding expressions.

We can write then

$$(S_T^2 - S_X^2) \Phi_{(1)\theta\theta} + V''(\Phi_{(0)}) \Phi_{(1)} = \tilde{f}_{(1)}(\theta, X, T)$$

where

$$\tilde{f}_{(1)}(\theta, X, T) = 2S_X \Phi_{(0)\theta X} - 2S_T \Phi_{(0)\theta T}^{[1]} + (S_{XX} - S_{TT}^{[1]}) \Phi_{(0)\theta} \quad (3.2)$$

for the first approximation in (2.5).

The orthogonality condition

$$\int_0^{2\pi} \tilde{f}_{(1)} \Phi_{(0)\theta} \frac{d\theta}{2\pi} = 0$$

gives the first term (Whitham system) of the system (2.4) having the form

$$\left(S_{TT}^{[1]} - S_{XX} \right) \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} + S_T \left[\int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_T^{[1]} - S_X \left[\int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_X = 0$$

or, equivalently

$$\left[S_T \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_T^{[1]} = \left[S_X \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_X \quad (3.3)$$

We have then

$$\left[S_T \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_{S_T} S_{TT}^{[1]} + \left[S_T \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_{S_X} S_{XT} = \left[S_X \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_X$$

where $S_{TT}^{[1]} = \sigma_{(1)}(S_X, S_T, S_{XX}, S_{XT})$.

Finally we get

$$\sigma_{(1)} = \left(\left[S_X \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_X - \left[S_T \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_{S_X} S_{XT} \right) / \left[S_T \int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} \right]_{S_T} \quad (3.4)$$

where

$$\int_0^{2\pi} \Phi_{(0)\theta}^2 \frac{d\theta}{2\pi} = \oint \frac{\sqrt{2(E - V(\Phi))}}{\sqrt{\omega^2 - k^2}} \frac{d\Phi}{2\pi}$$

The higher systems (1.16) can be written in analogous form

$$(S_T^2 - S_X^2) \Phi_{(k)\theta\theta} + V''(\Phi_{(0)}) \Phi_{(k)} = \tilde{f}_{(k)}(\theta, X, T) \quad (3.5)$$

where the orthogonality conditions

$$\int_0^{2\pi} \tilde{f}_{(k)} \Phi_{(0)\theta} \frac{d\theta}{2\pi} = 0 \quad (3.6)$$

are imposed for all $k \geq 1$.

We impose also the normalization conditions

$$\int_0^{2\pi} \Phi_{(k)} \Phi_{(0)\theta} \frac{d\theta}{2\pi} = 0 \quad (3.7)$$

for all $k \geq 1$.

Let us look for a solution of (3.5) in the form (see also [4, 5])

$$\Phi_{(k)}(\theta, X, T) = \alpha_{(k)}(\theta, X, T) \Phi_{(0)\theta}(\theta, X, T)$$

where $\alpha_{(k)}(\theta, X, T)$ is the function 2π -periodic in θ . We have

$$(S_T^2 - S_X^2) \alpha_{(k)\theta\theta} \Phi_{(0)\theta} + 2(S_T^2 - S_X^2) \alpha_{(k)\theta} \Phi_{(0)\theta\theta} = \tilde{f}_{(k)}$$

or

$$(S_T^2 - S_X^2) \alpha_{(k)\theta} (\Phi_{(0)\theta})^2 = \int^\theta \Phi_{(0)\theta'} \tilde{f}_{(k)}(\theta') d\theta' + \xi_1$$

where ξ_1 is arbitrary constant.

We have then

$$\Phi_{(k)} = \frac{\Phi_{(0)\theta}}{S_T^2 - S_X^2} \int^\theta \frac{d\theta'}{(\Phi_{(0)\theta'})^2} \int^{\theta'} \Phi_{(0)\theta''} \tilde{f}_{(k)}(\theta'') d\theta'' + \frac{\xi_1 \Phi_{(0)\theta}}{S_T^2 - S_X^2} \int^\theta \frac{d\theta'}{(\Phi_{(0)\theta'})^2} + \frac{\xi_2 \Phi_{(0)\theta}}{S_T^2 - S_X^2} \quad (3.8)$$

However, the formula (3.8) has a local character and we have to investigate the solution (3.8) on the whole axis $-\infty < \theta < +\infty$. Let us remind that we assume that the conditions (3.6) are satisfied. Easy to see that the expression $1/(\Phi_{(0)\theta})^2$ has singularities at the points $\theta_n = \pi n$, $n \in \mathbb{Z}$, so the integration should be made carefully in the formula (3.8). Let us consider now two important cases arising:

- I) The function $\tilde{f}_{(k)}(\theta)$ is anti-symmetric in θ : $\tilde{f}_{(k)}(-\theta) = -\tilde{f}_{(k)}(\theta)$;
- II) The function $\tilde{f}_{(k)}(\theta)$ is symmetric in θ : $\tilde{f}_{(k)}(-\theta) = \tilde{f}_{(k)}(\theta)$.

Let us start with the case (I). In the case (I) we have in fact from (3.6)

$$\int_0^\pi \tilde{f}_{(k)} \Phi_{(0)\theta} \frac{d\theta}{2\pi} = \int_{-\pi}^0 \tilde{f}_{(k)} \Phi_{(0)\theta} \frac{d\theta}{2\pi} = \int_{\pi n}^{\pi(n+1)} \tilde{f}_{(k)} \Phi_{(0)\theta} \frac{d\theta}{2\pi} = 0$$

view the anti-symmetry (and periodicity) of the function $\Phi_{(0)}(\theta)$.

It is easy to see then that the expression

$$\frac{1}{(\Phi_{(0)\theta'})^2} \int_0^{\theta'} \Phi_{(0)\theta''} \tilde{f}_{(k)}(\theta'') d\theta'' \quad (3.9)$$

has in fact no singularities at the points $\theta'_n = \pi n$. Moreover, the expression (3.9) defines an anti-symmetric periodic function of θ' so we have

$$\int_0^{2\pi} \frac{d\theta'}{(\Phi_{(0)\theta'})^2} \int_0^{\theta'} \Phi_{(0)\theta''} \tilde{f}_{(k)}(\theta'') d\theta'' = 0$$

We can see then that the expression

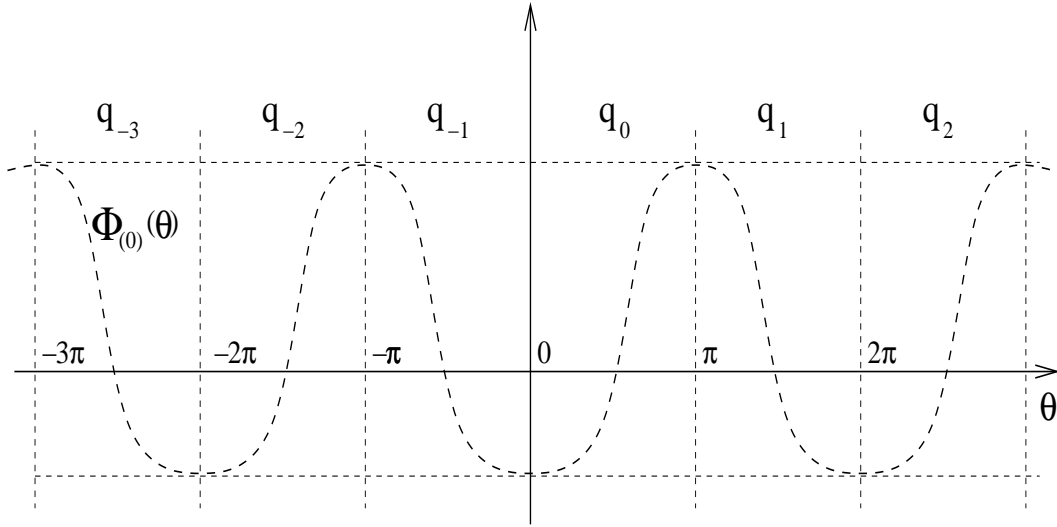


Figure 2: The "odd" and "even" intervals q_n on the axis $-\infty < \theta < +\infty$.

$$\frac{\Phi_{(0)\theta}}{S_T^2 - S_X^2} \int_0^\theta \frac{d\theta'}{(\Phi_{(0)\theta'})^2} \int_0^{\theta'} \Phi_{(0)\theta''} \tilde{f}_{(k)}(\theta'') d\theta''$$

gives a smooth periodic anti-symmetric solution of (3.5). We can omit now the local parameters ξ_1, ξ_2 and use the global freedom in the periodic solution of (3.5) defined modulo the (anti-symmetric) function $a_{(k)}(X, T)\Phi_{(0)\theta}(\theta, X, T)$ to satisfy the normalization condition (3.7). We can formulate now the following Proposition:

Proposition 3.1.

For a smooth periodic anti-symmetric discrepancy function $\tilde{f}_{(k)}(\theta)$ the solution $\Phi_{(k)}(\theta)$ of (3.5) satisfying the normalization conditions (3.7) is a smooth periodic anti-symmetric function $\Phi_{(k)}(-\theta) = -\Phi_{(k)}(\theta)$.

Let us consider now the case (II).

Let us divide the axis $-\infty < \theta < +\infty$ into "even" ($2\pi l < \theta < 2\pi l + \pi$) and "odd" ($2\pi l - \pi < \theta < 2\pi l$) intervals q_n (Fig. 2).

Let us define the "piecewise" solution $\Phi_{(k)}$ defined on any interval q_n by the formula

$$\begin{aligned} \Phi_{(k),\{n\}} &= \frac{\Phi_{(0)\theta}}{S_T^2 - S_X^2} \int_{\pi n + \pi/2}^\theta \frac{d\theta'}{(\Phi_{(0)\theta'})^2} \int_{\pi n + \pi/2}^{\theta'} \Phi_{(0)\theta''} \tilde{f}_{(k)}(\theta'') d\theta'' + \\ &+ \frac{\xi_{1,\{n\}} \Phi_{(0)\theta}}{S_T^2 - S_X^2} \int_{\pi n + \pi/2}^\theta \frac{d\theta'}{(\Phi_{(0)\theta'})^2} + \frac{\xi_{2,\{n\}} \Phi_{(0)\theta}}{S_T^2 - S_X^2} \end{aligned}$$

where the coefficients $\xi_{1,\{n\}}, \xi_{2,\{n\}}$ depend on the interval q_n . Let us put now

$$\xi_{1,\{2l\}} = \xi_{1,\{2l+1\}} = \xi_1, \quad \xi_{2,\{2l\}} = -\xi_{2,\{2l+1\}} = \xi_2, \quad l \in \mathbb{Z}$$

It's not difficult to see then that the total solution $\Phi_{(k)}$ given by all $\Phi_{(k),\{n\}}$ is a symmetric continuous periodic function of θ . Using the parameters ξ_1, ξ_2 we can provide also the C_1 smoothness both on "odd" ($\theta = 2\pi l + \pi$) and "even" ($\theta = 2\pi l$) sides of the intervals q_n . It appears then that the function $\Phi_{(k)}$ is smooth for the smooth $\tilde{f}_{(k)}$ since it satisfies the second-order equation (3.5). The conditions (3.7) are automatically satisfied here view the symmetry of the function $\Phi_{(k)}$. We can formulate the Proposition:

Proposition 3.2.

For a smooth periodic symmetric discrepancy function $\tilde{f}_{(k)}(\theta)$ the solution $\Phi_{(k)}(\theta)$ of (3.5) satisfying the normalization conditions (3.7) is a smooth periodic symmetric function $\Phi_{(k)}(-\theta) = \Phi_{(k)}(\theta)$.

Thus we have for the first correction $\Phi_{(1)}$ in (2.5):

$$\begin{aligned} \tilde{f}_{(1)}(\theta, X, T) &= 2S_X \Phi_{(0)\theta X} - 2S_T [\Phi_{(0)\theta T}]^{[1]} + (S_{XX} - S_{TT}^{[1]}) \Phi_{(0)\theta} = \\ &= 2k\Phi_{(0)\theta k} k_X + 2k\Phi_{(0)\theta \omega} \omega_X - 2S_T \Phi_{(0)\theta k} \omega_X - \\ &\quad - 2S_T \Phi_{(0)\theta \omega} \sigma_{(1)}(k, \omega, k_X, \omega_X) + \Phi_{(0)\theta} k_X - \Phi_{(0)\theta} \sigma_{(1)}(k, \omega, k_X, \omega_X) \end{aligned} \quad (3.10)$$

where the function $\sigma_{(1)}$ is defined by (3.4).

Easy to see that the function $\tilde{f}_{(1)}$ is anti-symmetric $\tilde{f}_{(1)}(-\theta) = -\tilde{f}_{(1)}(\theta)$. We obtain then that the function $\Phi_{(1)}(\theta)$ is also anti-symmetric in θ : $\Phi_{(1)}(-\theta) = -\Phi_{(1)}(\theta)$.

We have then for $\Phi_{(2)}(\theta, X, T)$

$$(S_T^2 - S_X^2) \Phi_{(2)\theta\theta} + V''(\Phi_{(0)}) \Phi_{(2)} = \tilde{f}_{(2)}(\theta, X, T)$$

where

$$\begin{aligned} \tilde{f}_{(2)}(\theta, X, T) &= -2S_T [\Phi_{(0)\theta T}]^{[2]} - S_{TT}^{[2]} \Phi_{(0)\theta} - S_{TT}^{[1]} \Phi_{(1)\theta} - \frac{1}{2} V'''(\Phi_{(0)}) \Phi_{(1)}^2 + 2S_X \Phi_{(1)\theta X} - \\ &\quad - 2S_T [\Phi_{(1)\theta T}]^{[2]} + S_{XX} \Phi_{(1)\theta} - [\Phi_{(0)TT}]^{[2]} + \Phi_{(0)XX} = \\ &= -(2\omega \Phi_{(0)\theta \omega} + \Phi_{(0)\theta}) \sigma_{(2)}(k, \omega, k_X, \omega_X, k_{XX}, \omega_{XX}) + \\ &\quad - \Phi_{(1)\theta} \sigma_{(1)}(k, \omega, k_X, \omega_X) - \frac{1}{2} V'''(\Phi_{(0)}) \Phi_{(1)}^2 + 2k \Phi_{(1)\theta X} - \\ &\quad - 2\omega \int \frac{\delta \Phi_{(1)\theta}(\theta, X)}{\delta k(Z)} \omega_Z dZ - 2\omega \int \frac{\delta \Phi_{(1)\theta}(\theta, X)}{\delta \omega(Z)} \sigma_{(1)}(k, \omega, k_Z, \omega_Z) dZ + k_X \Phi_{(1)\theta} - \end{aligned}$$

$$\begin{aligned}
& - \Phi_{(0)kk}(\omega_X)^2 - 2\Phi_{(0)k\omega} \omega_X \sigma_{(1)}(k, \omega, k_X, \omega_X) - \Phi_{(0)k} \frac{d}{dX} \sigma_{(1)}(k, \omega, k_X, \omega_X) - \\
& - \Phi_{(0)\omega\omega} \sigma_{(1)}^2(k, \omega, k_X, \omega_X) - \Phi_{(0)\omega} \int \frac{\delta \sigma_{(1)}(k, \omega, k_X, \omega_X)}{\delta k(Z)} \omega_Z dZ - \\
& - \Phi_{(0)\omega} \int \frac{\delta \sigma_{(1)}(k, \omega, k_X, \omega_X)}{\delta \omega(Z)} \sigma_{(1)}(k, \omega, k_Z, \omega_Z) dZ + \Phi_{(0)XX}
\end{aligned}$$

Easy to see that only the first two terms in $\tilde{f}_{(2)}$ (containing $\sigma_{(2)}$) are anti-symmetric in θ and all the other terms are symmetric. Using the orthogonality conditions (3.6) we obtain then

$$\sigma_{(2)}(k, \omega, k_X, \omega_X, k_{XX}, \omega_{XX}) \equiv 0 \quad (3.11)$$

for the second term of the deformation of Whitham system (2.4).

Using (3.11) we see then that the discrepancy function $\tilde{f}_{(2)}$ becomes now a symmetric function of θ , so we can state that the second approximation $\Phi_{(2)}(\theta, X, T)$ in (2.5) is a symmetric function of θ : $\Phi_{(2)}(-\theta) = \Phi_{(2)}(\theta)$. Using the simple induction we can claim in fact that all the discrepancy functions $f_{(k)}$, $k \geq 1$ can be represented in the form

$$\tilde{f}_{(k)}(\theta, X, T) = - (2\omega\Phi_{(0)\theta\omega} + \Phi_{(0)\theta}) \sigma_{(k)}(k, \omega, \dots) + \tilde{f}'_{(k)}(\theta, X, T)$$

where $\tilde{f}'_{(k)}$ does not contain the function $\sigma_{(k)}(k, \omega, \dots)$. Moreover, all the "even" functions $\tilde{f}'_{(2l)}$ will be symmetric in θ and all the "odd" functions $\tilde{f}'_{(2l+1)}$ will be anti-symmetric in θ . The same is true also for the functions $\tilde{f}_{(2l)}$, $\tilde{f}_{(2l+1)}$ after the imposing of the conditions (3.6) and for the corresponding solutions of (3.5) $\Phi_{(2l)}$, $\Phi_{(2l+1)}$.⁴ We can formulate then the following Lemma:

Lemma 3.1.

For the "unified" choice of the functions $\Phi(\theta, k, \omega)$ corresponding to Fig. 1 the following statements are true:

- 1) All the even terms $\sigma_{(2l)}(k, \omega, \dots)$ in the deformation of Whitham system (2.4) are identically zero: $\sigma_{(2l)} \equiv 0$;
- 2) All the odd corrections $\Phi_{(2l+1)}(\theta, X, T)$, $l \geq 0$ in (2.5) are anti-symmetric in θ : $\Phi_{(2l+1)}(-\theta) = -\Phi_{(2l+1)}(\theta)$;
- 3) All the even corrections $\Phi_{(2l)}(\theta, X, T)$, $l \geq 1$ in (2.5) are symmetric in θ : $\Phi_{(2l)}(-\theta) = \Phi_{(2l)}(\theta)$.

The system (2.4) for the given choice of functions $\Phi(\theta, k, \omega)$ can be rewritten now in the form

$$k_T = \omega_X$$

⁴The similar facts were pointed out in [5] for "V"-Gordon equation. However the functions $\Phi_{(k)}$, $\tilde{f}_{(k)}$ are different here from those appeared in [5].

$$\omega_T = \sum_{l \geq 0} \sigma_{(2l+1)}(k, \omega, k_X, \omega_X, \dots) \quad (3.12)$$

where all $\sigma_{(2l+1)}$ are polynomial in derivatives k_X, ω_X, \dots , having degree $(2l + 1)$.

Let us say here that the form of deformations of Hydrodynamic hierarchies containing only odd dispersion terms was considered first by B.A. Dubrovin and Y. Zhang ([63, 65]) as the convenient representative in the class of "equivalent" deformations.

Finally, let us calculate the next (σ_3) non-vanishing term in (3.12). It is convenient to use now the Lagrangian formalism of the system (3.1)

$$\delta \int \int \left[-\frac{1}{2} \varphi_t^2 + \frac{1}{2} \varphi_x^2 + V(\varphi) \right] dx dt = 0$$

We have to add the variable θ and introduce the action functional

$$\Sigma[\varphi] = \int \int \int_0^{2\pi} \left[-\frac{1}{2} \varphi_T^2 + \frac{1}{2} \varphi_X^2 + V(\varphi) \right] \frac{d\theta}{2\pi} dX dT \quad (3.13)$$

Let us introduce the function $\Phi^{(tot)}(\theta, X, T)$ by the formula

$$\Phi^{(tot)}(\theta, X, T) = \sum_{k \geq 0} \Phi_{(k)}(\theta, X, T) = \phi(\theta - S(X, T), X, T)$$

(where $\Phi_{(0)}(\theta, X, T) = \Phi(\theta, S_X, S_T)$).

After the substitution of (2.5) into the functional (3.13) we can write the action functional in the form

$$\begin{aligned} \Sigma &= \int \int \int_0^{2\pi} \left[-\frac{1}{2} S_T^2 \left(\Phi_\theta^{(tot)} \right)^2 + \frac{1}{2} S_X^2 \left(\Phi_\theta^{(tot)} \right)^2 + V \left(\Phi^{(tot)} \right) \right] \frac{d\theta}{2\pi} dX dT + \\ &+ \int \int \int_0^{2\pi} \left[-S_T \Phi_\theta^{(tot)} \Phi_T^{(tot)} + S_X \Phi_\theta^{(tot)} \Phi_X^{(tot)} \right] \frac{d\theta}{2\pi} dX dT + \\ &+ \int \int \int_0^{2\pi} \frac{1}{2} \left[- \left(\Phi_T^{(tot)} \right)^2 + \left(\Phi_X^{(tot)} \right)^2 \right] \frac{d\theta}{2\pi} dX dT = \\ &= \int \int \int_0^{2\pi} (\mathcal{L}' + \mathcal{L}'' + \mathcal{L}''') \frac{d\theta}{2\pi} dX dT \end{aligned}$$

The function

$$\bar{\mathcal{L}}[S] = \int_0^{2\pi} (\mathcal{L}'[S] + \mathcal{L}''[S] + \mathcal{L}'''[S]) \frac{d\theta}{2\pi}$$

gives a Lagrangian density for the "averaged" Lagrangian formalism corresponding to the system (3.12).

The main term $\bar{\mathcal{L}}_{(0)}$ of the averaged Lagrangian density $\bar{\mathcal{L}}$ has the form

$$\bar{\mathcal{L}}_{(0)}(S_X, S_T) = \int_0^{2\pi} \left[\frac{1}{2} (-S_T^2 + S_X^2) \Phi_{(0)\theta}^2 + V(\Phi_{(0)}) \right] \frac{d\theta}{2\pi}$$

The equation

$$\left[\frac{\delta}{\delta S(X, T)} \int \int \bar{\mathcal{L}}_{(0)}(S_X, S_T) dX dT \right]^{[1]} = 0$$

gives the Whitham's Lagrangian formalism for the Whitham system (3.3). (We use as previously the notation ^[1] for the terms of degree 1 in the "gradated" expansion of corresponding expression).

For the determination of the first non-trivial term $\sigma_{(3)}$ in the deformation (3.12) it is enough to calculate the correction $\bar{\mathcal{L}}_{(2)}$ to the main term $\bar{\mathcal{L}}_{(0)}$ given by the expression

$$\begin{aligned} & \int \cdots \int \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{2} \frac{\delta^2 \mathcal{L}'(\theta, X, T)}{\delta \Phi(\theta', X', T') \delta \Phi(\theta'', X'', T'')} \Big|_{\Phi_{(0)}} \Phi_{(1)}(\theta', X', T') \Phi_{(1)}(\theta'', X'', T'') \times \\ & \qquad \qquad \qquad \times d\theta' dX' dT' d\theta'' dX'' dT'' \frac{d\theta}{2\pi} + \\ & + \int \cdots \int \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\delta \mathcal{L}''(\theta, X, T)}{\delta \Phi(\theta', X', T')} \Big|_{\Phi_{(0)}} \Phi_{(1)}(\theta', X', T') d\theta' dX' dT' \frac{d\theta}{2\pi} + \\ & \qquad \qquad \qquad + \int_0^{2\pi} \mathcal{L}'''(\theta, X, T) \Big|_{\Phi_{(0)}} \frac{d\theta}{2\pi} \end{aligned}$$

Using the normalization condition (3.7) we can write the correction $\Phi_{(1)}(\theta, X, T)$ as a local functional of $(S_X, S_T, S_{XX}, S_{XT})$ having the form

$$\begin{aligned} \Phi_{(1)}(\theta, X, T) &= \frac{\Phi_\theta}{S_T^2 - S_X^2} \int_0^\theta \frac{d\theta'}{(\Phi_{\theta'})^2} \int_0^{\theta'} \Phi_{\theta''} \tilde{f}_{(1)}(\theta'') d\theta'' - \\ &- \Phi_\theta \int_0^{2\pi} \Phi_{\theta'}^2 d\theta' \int_0^{\theta'} \frac{d\theta''}{(\Phi_{\theta''})^2} \int_0^{\theta''} \Phi_{\theta'''} \tilde{f}_{(1)}(\theta''') d\theta''' \Big/ (S_T^2 - S_X^2) \int_0^{2\pi} \Phi_{\theta'}^2 d\theta' \end{aligned}$$

where $\tilde{f}_{(1)}(\theta, X, T)$ is given by formula (3.10).

We have then

$$\begin{aligned} \Sigma_{(2)} &= \frac{1}{2} \int \int \int_0^{2\pi} ([-S_T^2 + S_X^2] \Phi_{(1)\theta}^2 + V''(\Phi) \Phi_{(1)}^2) \frac{d\theta}{2\pi} dX dT + \\ &+ \int \int \int_0^{2\pi} (S_{TT} \Phi_\theta + 2S_T \Phi_{\theta T} - S_{XX} \Phi_\theta - 2S_X \Phi_{\theta X}) \Phi_{(1)} \frac{d\theta}{2\pi} dX dT - \\ &- \frac{1}{2} \int \int \int_0^{2\pi} (\Phi_k(\theta, S_X, S_T) S_{TX} + \Phi_\omega(\theta, S_X, S_T) S_{TT})^2 \frac{d\theta}{2\pi} dX dT + \end{aligned}$$

$$+ \frac{1}{2} \iint \int_0^{2\pi} (\Phi_k(\theta, S_X, S_T) S_{XX} + \Phi_\omega(\theta, S_X, S_T) S_{TX})^2 \frac{d\theta}{2\pi} dX dT$$

Let us consider now the equations

$$\left[\frac{\delta}{\delta S(X, T)} (\Sigma_{(0)}[S] + \Sigma_{(2)}[S]) \right]^{[1]} = 0$$

and

$$\left[\frac{\delta}{\delta S(X, T)} (\Sigma_{(0)}[S] + \Sigma_{(2)}[S]) \right]^{[3]} = 0$$

The first equation gives again the Whitham system (3.3). The second equation defines the first non-trivial correction $\sigma_{(3)}$ of the system (3.12). We have

$$\left[\frac{\delta}{\delta S(X, T)} \Sigma_{(0)}[S] \right]^{[3]} = \left[S_T \int_0^{2\pi} \Phi_\theta^2 \frac{d\theta}{2\pi} \right]_{S_T} S_{TT}^{[3]}$$

where $S_{TT}^{[3]} = \sigma_{(3)}(S_X, S_T, S_{XX}, S_{XT}, S_{XXX}, S_{XXT}, S_{XXX}, S_{XXXT})$.

Now finally we obtain

$$\sigma_{(3)} = - \left[\frac{\delta}{\delta S(X, T)} \Sigma_{(2)}[S] \right]^{[3]} \Big/ \left[S_T \int_0^{2\pi} \Phi_\theta^2 \frac{d\theta}{2\pi} \right]_{S_T}$$

It's not difficult to see that all the higher T -derivatives $S_{TT}, S_{TTX}, S_{TTT}, \dots$, arising in the right-hand part can be replaced just by their main values defined by the Whitham system (3.3) in this approximation.

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